E C O N O M I C S   B U L L E T I N

A canonical first passage time model to pricing nature−linked bonds

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Abstract

This paper shows that pricing catastrophe bonds boils down to computing first−passage time distributions of jump−diffusion processes. It derives a generic valuation expression by assuming that the jump risk is not systematic and then performs simulations, which can stress the sensitivity of insurance bond values to changes in underlying parameters.

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1. Introduction

Catastrophe bonds (or catbonds) are examples of specialized securities that increase the ability of insurers to provide insurance protection while transferring the risk to investors around the world. Their proceeds are contingent upon occurrences of natural catastrophes or insurance claims from these events. The first successful catastrophe bonds were issued by USAA to cover its US east cost hurricane exposure (US$ 477 in June 1997) and by Swiss Re to cover California earthquake losses (US$ 137 in July 1997). Transactions on the market for catastrophe risk are documented in Froot (2001) and The Alternative Risk Transfer Portal (www.artemis.bm).

Catastrophe bonds are typically engineered as follows. The hedger (e.g. a firm in the energy sector or an insurance company) pays a premium in exchange for a pre-specified coverage if a natural hazard materializes (generally a reinsurance policy for regulatory and tax reasons); and investors purchase a catbond for cash. The total amount (premium + cash proceeds) is directed to a tailor-made fund, called a special-purpose company (SPC), which issues the catbond to investors and purchases Treasury bonds. Therefore, investors hold insurance-linked assets whose cash flows – coupons and / or principal – are contingent on the risk occurrence. If the covered events take place during the risk-exposure period, the SPC compensates the firm and there is full or partial forgiveness of the repayment of principal and / or interest. Otherwise, the investors receive their principal plus interest equal to the risk free rate plus a risk-premium.

On the one hand, actuarial-science-based calculus cannot be applied to catastrophe bonds, as it tends to price insurance contingent claims as random sums where weights are taken according to the historical probability. Indeed, the law of large numbers breaks down when confronted by infrequent and severe catastrophe risks. On the other hand, there is little academic research devoted to the pricing of insurance-linked securities. Examples within an arbitrage-free framework are: Cummins and Geman (1995) who provide formulas for CBOT-traded derivatives (catastrophe insurance futures and call spreads), whereas the present contribution deals with OTC derivatives; Briys (1998) who derives a simple formula for non-catastrophic insurance-linked bonds. Lee and Yu (2002) who price catastrophe bonds, but are primarily interested in analyzing default risk, and therefore, specialize their paper to the case where catbonds are directly issued by insurers.

The present paper evaluates catastrophe bonds within an arbitrage-free framework. It shows that pricing catbonds boils down to computing first-passage time distributions of jump-diffusion processes, as bondholders are short on digital options based on risk-tracking indices. It vindicates a well-defined arbitrage price, notwithstanding discontinuities in underlying processes. It then reports fairly accurate simulations and further conducts a comparative static analysis that stresses the sensitivity of prices of catbonds to exposure to nature risk.

The remainder of the paper is organized as follows. Section 2 presents the valuation framework. Section 3 performs simulations. Section 4 summarizes the article. For ease of exposition, most proofs are in an appendix.

2. Method

In this section, we establish the binary structure of insurance-linked securities and we extend and adapt the jump-diffusion model of Merton (1976) to develop a valuation framework that allows for catastrophic events, interest rate uncertainty, and non-traded underlying state variables.

2.1. *Product structure*

Catbond payoffs can be thought of as corporate bonds with insurance risk instead of default risk. The bondholder accepts to lose interest payments or a fraction of the principal if an index standing for natural risk or associated insurance claims, whose value at date t is denoted I_t, hits a pre-specified threshold K. More specifically, if the index does not reach the threshold during a risk exposure period T, the bondholder is paid the face value F. Otherwise, he receives the face value minus a write-down coefficient in percentage w. We allow the bond maturity T' to be longer than the risk exposure period T to account for possible lags in the risk-index assessment at expiration. We specialize the paper to the case where the risk index starts below the barrier. This boundary is supposed to be fixed.

Formally, bondholders receive at T':

$$F\,\mathbf{1}_{T_{I,\,K} > T} + (1-w)\,F\,\mathbf{1}_{T_{I,\,K} \le T} = F - w\,F\,\mathbf{1}_{T_{I,\,K} \le T} \qquad (1)$$

where $\mathbf{1}_A$ stands for the indicator function of set A, and $T_{I,\,K}$ is the first passage time of I through K. Therefore, bondholders have a short position on a one-touch digital up-and-in option on the risk-tracking index I.

Let $(\Omega, \mathfrak{I}, P)$ define a probability space, where Ω is the set of states of the world, $\mathfrak{I}$ is a σ-algebra of subsets of Ω and P is a probability measure on $\mathfrak{I}$. Processes are defined on this probability space and on a trading horizon $[0, T']$. $\{W_{1t}: 0 \le t \le T'\}$ and $\{W_{2t}: 0 \le t \le T'\}$ are two standard Brownian motions. $\{N_t: 0 \le t \le T'\}$ is a Poisson process with an intensity parameter λ_p. $\{U_j: j \ge 1\}$ is a sequence of identically and independently distributed (i.i.d.) random variables with values in $]0; +\infty[$. U_j occurs at time τ_j defined by (N_t), that is, $\tau_j = \inf\{0 \le t \le T', N_t = j\}$. (W_{1t}), (W_{2t}), (N_t), and (U_j) are independent. For all t in $[0, T']$, let F_t be the σ-field generated by the random variables W_{1s}, W_{2s}, N_s for $s \le t$, and $U_j\,\mathbf{1}_{\{j \le N_t\}}$ for $j \ge 1$. The filtration $\{F_t: 0 \le t \le T'\}$ represents the information flow reaching market players. (F_t) is further augmented to encompass all P-null events. The four sources of randomness (W_{1t}), (N_t), (U_j), and (W_{2t}) account for non-catastrophic nature risk, the occurrences of catastrophes, the size of catastrophes, and the uncertainty of interest rates, respectively.

2.2. *Assumptions*

We assume that financial markets are frictionless: there are no transaction costs or differential taxes, trading takes place continuously in time, borrowing and short selling are allowed without restriction and with full proceeds available, and borrowing and lending rates are equal. In addition:

Assumption 1: *Interest rates obey an Ornstein-Uhlenbeck process*

The risk-free spot interest rate r follows, under the historical probability P:

$$dr(t) = a(b - r(t))\, dt + \sigma_r\, dW_{2t} \qquad (2)$$

where a, b, and σ_r are constants.

Assumption 2: *Risk index dynamics: Poisson jump-diffusion process*

$(I_t)_{t\geq 0}$ is right-continuous and satisfies under the historical probability P:

$$dI_t \,/\, I_t\text{-} = \mu(t)\, dt + \sigma(t)\, dW_{1t} + J_t\, dN_t \qquad (3)$$

where I_t - stands for the underlying value just before t. $\mu(.)$ is the drift parameter and can be stochastic. $\sigma(.)$ is the deterministic volatility parameter of the Brownian component of the process. (N_t) is a Poisson process accounting for the expected number of jumps per time unit. (J_t) depicts the stochastic size of the jumps:

$$J_t = \Sigma_{n=1,\,+\infty}\, U_n\, \mathbf{1}_{]\tau_{n-1},\tau_n]}\,(t) \qquad (4)$$

where (U_j) and (τ_j) were previously defined. Thus, at time τ_j, the jump of I_t is given by: $\Delta I_{\tau_j} = I_{\tau_j} - I_{\tau_j^-} = (I_{\tau_j^-})\, U_j$. Therefore, $J_t\, dN_t$ is a handy notation to designate a compound Poisson process. In addition, the U_j's are log-normally i.i.d..

Poisson jump-diffusion processes are adapted to catastrophe bonds in that they fit in the case of reported insurance losses and may be used as a proxy for some physical indices. As for insurance claims, the Wiener process reflects small catastrophes and randomness in reporting, while the Poisson process represents major catastrophes. The former process is vindicated on the ground that although small catastrophes occur in discrete time, claim reporting by policyholders is continuous. As for physical indices, both catastrophic and non-catastrophic events must be simultaneously accounted for. For instance, abnormal rainfall may raise the level of rivers to non-catastrophic heights or trigger catastrophic land sliding, or the wind speed may vary from a breeze to a hurricane.

Assumption 3: *Investors are neutral toward nature jump risk (A31) and non-catastrophic changes in the risk index can be replicated by existing quoted securities (A32), as well as changes in interest rates*

Assumption (A31) means that investors acknowledge that natural catastrophe risk can be diversified away when it comes to pricing contingent claims upon environmental risk. The underlying rationale is that localized natural catastrophes are barely correlated to global financial markets. Therefore, we follow along the lines drawn by Merton (1976), "jump risk is not systematic". Lee and Yu (2002) take the same stance to price default-risky catbonds. This assumption is important in that risk-neutral pricing could not be applied otherwise, and instead, it would be necessary to introduce equilibrium valuation in line with Lucas (1978) or other techniques, such as the variance-minimizing hedging approach or utility-indifference techniques. As for Assumption (A32), we may for instance argue that continuous changes in the risk index can be mimicked by instruments such as energy and power derivatives, weather derivatives or contingent claims on several commodities.

3. Results

In this section, we derive a generic valuation expression for insurance-linked securities, and then we perform simulations and lay down some comparative static outcomes.

3.1. *Arbitrage price*

Lemma 1. There exists a well-defined arbitrage price for exchange-traded contingent claims upon the risk index. More specifically, let C_I be such a contingent claim. Then,

$$C_I(t) = \boldsymbol{E}^{Q} (D(t, T') \, C_I(T') \parallel F_t) \tag{5}$$

where Q is the risk-adjusted probability measure under which the discounted price process of securities traded in financial markets is a martingale, and this expectation operator is conditional upon the information, F_t, available to investors at Time t. The stochastic discount factor $D(t, T')$ is given by $\exp(-\int_{t, T'} r(u)\, du)$. The dynamics of I and r under Q are described by:

$$dI_t / I_{t^-} = (\mu(t) - \lambda(t)\, \sigma(t))\, dt + \sigma(t)\, dW_{1t}' + J_t\, dN_t \tag{6}$$

and

$$dr(t) = a(b' - r(t))\, dt + \sigma_r\, dW_{2t}' \tag{7}$$

where $\lambda(.)$ is the market price of nature risk, and W_1' and W_2' are independent Q-standard Brownian motions. Last, $b' = b - \lambda_r\, \sigma_r / a$, with $(-\lambda_r)$ the risk premium for riskfree bonds.

Proof: See Appendix *A*. □

Proposition 1. Let $IB(t)$ be the price of a zero-coupon insurance bond at time t and $T_{I,\, K}$ the first passage time of I through K. We suppose that all cash payments are done at date of maturity T'. Then,

$$IB(t) = F\, P(t, T') \, \{1 - w\, \boldsymbol{E}^{Q} (\mathbf{1}_{T_{I,\,K} \leq T} \mid F_t) \} \tag{8}$$

where $P(t, T)$ is given by the Vasicek (1977) formula:

$$P(t, T') = exp[-(T' - t)\, R(T' - t\, , r(t))]$$

with

$$R(\theta, r) = R_\infty - 1/(a\theta)\, [\, (R_\infty - r)\, (1 - e^{-a\theta}) - \sigma_r^{\,2}/(4a^2)\, (1 - e^{-a\theta})^2\,]$$

and

$$R_\infty = b' - \sigma_r^{\,2}/(2a^2)$$

Proof: See Appendix *B*. □

3.2. *Monte Carlo simulations*

Proposition 2. The core subroutine to assess $\boldsymbol{E}^{Q}(\mathbf{1}_{T_{I,K} \leq T})$ is:

$$I_{n+1} = I_n \{1 + [(\mu(T\,n\,/\,N) - \lambda(T\,n\,/\,N)\,\sigma(T\,n\,/\,N)]\,\Delta t$$
$$+ \sigma(T\,n\,/\,N)\,g(0,1)\sqrt{\Delta t} + \Sigma_{j=1,\,N(\lambda_p\,\Delta t)}\,[exp(g_j(kv,\delta))]\} \tag{9}$$

and

$$r_{n+1} = a\,b'\,\Delta t + (1 - a\,\Delta t)\,r_n + \sigma_r\,g_2(0,1)\sqrt{\Delta t} \tag{10}$$

where N is the number of steps, $\Delta t = T\,/\,N$, $kv = \boldsymbol{E}(ln(U_1))$, $\delta^2 = \boldsymbol{var}(ln(U_1))$, and where $g(0,1)$, $g_2(0,1)$, and $g_j(kv,\delta)$ follow independent normal distributions with respective parameters (0,1) and $(kv,\ \delta)$. $N(\lambda_p\ \Delta t)$ is simulated by using that, if N_θ is a Poisson random variable with intensity θ, then,

$$N_\theta = \Sigma_{n\geq 1}\, n\mathbf{1}_{\{UF_1\,UF_2 \ldots UF_n UF_{n+1} \leq e^{-\theta} \leq UF_1\,UF_2\,\ldots\,UF_n\}} \tag{11}$$

where $(UF_i)_{i\geq 1}$ are uniformly i.i.d. on [0,1].

Then, we track if the index hits the barrier during the risk exposure period by incrementing by 1 if I breaks K along a given sample path, and obtain $\boldsymbol{E}^{Q}(\mathbf{1}_{T_{I,K} \leq T})$ by averaging over the number of trajectories.

Proof: The whole scheme straightforwardly comes from the Q-dynamics determined in Lemma 1 and we need not elaborate. □

3.3. *Exposure to nature risk*

Prices of insurance-linked securities decrease when the exposure to nature risk increases, be it through a shorter distance between the actual index level and the barrier, a larger index volatility, a longer risk exposure period, or sharper parameters of the Poisson process, as documented in Table 1. Indeed, in all these circumstances, the probability of hitting the threshold is higher. In addition, prices of catastrophe bonds are increasing in the market price of nature risk, as the risk-adjusted drift decreases.

Table 1. **Catastrophe bond price and nature risk**

Intensity λ_p	0	0.5	1	2
$X = 0.5$	760	555	400	235
$X = 0.8$	370	270	200	140
E(ln U_1) = 0.1	760	555	400	235
E(ln U_1) = 0.2	760	540	390	225
$\sigma = 0.2$	895	595	410	240

$\sigma = 0.5$	760	555	400	235
$T = 0.5$	860	755	635	455
$T = 1$	760	555	400	235
$\lambda = 0.1$	760	555	400	235
$\lambda = 0.2$	785	570	420	245
Riskfree bond	905	905	905	905

This table exhibits the variation of prices of catastrophe bonds according to risk exposure, where the index follows a jump-diffusion process: Poisson intensity, index-barrier distance, jump size, index volatility (diffusion component), and risk exposure period. Default parameters: $F = 1000$, $X = I/K = 0.5$, $w = 0.9$, $T = T' = 1$ *year,* $\sigma = 0.5$, $\mathbf{E}(\ln U_1) = 0.1$, $\delta = 0.2$, $r = 0.1$, $a = 0.1$, $b' = 0.1$, $\sigma_r = 0.03$, $\mu = 0.2$, *and* $\lambda = 0.1$. *Each line corresponds to replacing exactly one of the default parameters. Number of simulations: 5000.*

4. Conclusion

This article simulated catastrophe bonds within an arbitrage-free framework. A well-defined arbitrage price for catastrophe bonds has been vindicated, notwithstanding discontinuities in underlying processes. We showed that their valuation boils down to determining first-passage time distributions, as bondholders have a short position on one-touch digital options on risk-tracking indices that follow Poisson jump-diffusion processes. Then, we resorted to numerical simulations.

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Appendix

A. Proof of Lemma 1.

Let Q be the unique martingale measure, equivalent to the objective probability measure P, with complete financial markets. Making use of Assumption (A32), we introduce W_1' and W_2' the Q-standard Brownian motions that are routinely derived from the P-standard Brownian motions W_1 and W_2 by means of the Girsanov theorem.

To fix ideas, let I' be the process which dynamics are described by the geometric-Brownian component of the jump-diffusion process set by Formula (3). Then, any attainable contingent claim $C_{I'}$ has a well-defined price given by: $C_{I'}(t) = \boldsymbol{E}^{Q}(D(t, T') \, C_{I'}(T') \mid F_t)$. Now, due to Assumption (A31), investors use the same arbitrage price when it comes to contingent claims upon I. Indeed, since they are neutral toward nature jump risk, the risk-adjusted probability measure Q is used to evaluate non-attainable contingent claims C_I upon I: $C_I(t) = \boldsymbol{E}^{Q}(D(t, T') \, C_I(T') \mid F_t)$; which yields Formula (5).

We now only have to account for the index non-tradability to identify the dynamics of I under Q. We introduce the market price of nature risk, λ, associated with I'. Using again Assumption (A31), this approach can be extended to I. Finally, the jump component remains the same under Q due to the independence of (W_{1t}), (W_{2t}), (N_t), and (U_j), and we obtain the dynamics of Formula (6). Formula (7) for interest rates is usual.

B. Proof of Proposition 1.

Using Lemma 1 and Formula (1) gives:

$$\begin{aligned} IB(t) &= \boldsymbol{E}^{Q}(D(t, T') \, F \, (1 - w \, \mathbf{1}_{T_{I,K} \le T}) \mid F_t) \\ &= F \, \boldsymbol{E}^{Q}(D(t, T') \mid F_t)) - w \, F \, \boldsymbol{E}^{Q}(D(t, T') \, \mathbf{1}_{T_{I,K} \le T} \mid F_t) \end{aligned} \qquad \text{(B1)}$$

By definition, $P(t, T') = \boldsymbol{E}^{Q}(D(t, T') / F_t))$. In addition, since W_1' and W_2' are independent under Q, and since r does not appear in the risk-adjusted drift of I, then $D(., T')$ and $\mathbf{1}_{T_{I,K} \le T}$ are independent under Q. So, we have:

$$\boldsymbol{E}^{Q}(D(t, T') \, \mathbf{1}_{T_{I,K} \le T}) \mid F_t) = \boldsymbol{E}^{Q}(D(t, T') \mid F_t) \, \boldsymbol{E}^{Q}(\mathbf{1}_{T_{I,K} \le T} \mid F_t) \qquad \text{(B2)}$$

We obtain:

$$IB(t) = F \, P(t, T') \, \{ 1 - w \, \boldsymbol{E}^{Q}(\mathbf{1}_{T_{I,K} \le T} \mid F_t)\} \qquad \text{(B3)}$$