E C O N O M I C S   B U L L E T I N

Existence of pure strategy equilibria among geographically dispersed firms

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Abstract

This paper gives a proof of existence of price equilibria under certain parameters for a model of product differentiation commonly used in the empirical geographic differentiation literature. This proof is needed because the assumptions of Caplin and Nalebuff (1991) are not generally satisfied once data is introduced to geographic models. The theorem also has implications for existence in mixed−logit demand models.

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1. Introduction

There have been many recent empirical studies of geographic competition that estimate theoretical models of product differentiation (e.g., Davis (2001), Manuszak (2000), Thomadsen (2004)). These studies use discrete-choice models where demand is generally derived from a utility function that includes a distaste for spending money, a distaste for travel, and an additive residual utility term that specifies how much each consumer enjoys the product of each individual firm, which is distributed i.i.d. type I extreme value. Most of these models also specify that the firms with constant marginal costs compete through static Bertrand price competition.

While this stylized model is popular in the empirical literature, there has been no proof that static price equilibria exist in these models.[1] Most product differentiation papers rely on the proof of existence exposited in Caplin and Nalebuff (1991). While the assumptions in Caplin and Nalebuff hold for many models of product differentiation, they do not generically hold for those with geographic differentiation. In particular, there is no way to ensure that the probability density of consumers' utilities will be ρ-concave if real distributions of consumers are used. For example, it is possible for a market to have pockets of areas where consumers live, and other areas interspersed in between these where no consumers are present.

A proof of existence in these models is therefore needed. Ultimately, this field needs a generic proof covering markets with any distribution of consumer and firm locations, along with any potential ownership structures. This paper is less ambitious: I prove that pure-strategy equilibria exist under certain (sufficient) parameter restrictions for any distribution of consumers and firms when the firms are all operating under independent ownership.

2. Geographic Models

Recent empirical papers with geographic differentiation have modeled demand using a discrete-choice framework, with consumers choosing either to purchase one unit of the good at firm j or to consume an outside good. Consumers are spread across the market area, but have the same utility function except for differences due to their demographics and to their unobserved tastes for each good.

The utility consumer i derives from consuming one unit of the good from firm j is

$$U_{i,j} = X'_j\beta - P_j\gamma - D_{i,j}\delta + \eta_{i,j} \tag{1}$$

where X_j is a vector of firm j's product attributes, P_j is the price at firm j, $D_{i,j}$ is the distance between consumer i and firm j, and $\eta_{i,j}$ is the idiosyncratic portion of utility for individual i at firm j. β, γ, and δ are parameters, with γ and δ assumed to be positive. These parameters are not necessary if the characteristics of the goods are measured in appropriate units, but I include them in the model because characteristics tend to be measured in units that lead to non-one coefficients (e.g., miles or dollars) in empirical

[1] Anderson, de Palma and Thisse (1992) prove existence under some parameter values for a similar model (see p. 366). However, the parameter restrictions and logic of the proof reply on the attributes of the model that differ from the empirical models. Therefore, these proofs can not be applied to empirical models. Similarly, Anderson, de Palma and Thisse restrict travel disutility from being too large, while the parameter restriction presented below in Section 3 does not constrain travel costs.

papers. In this case, the parameters are also implicitly normalized by the variance of η.

The consumers can also choose to consume only the outside good, in which case their utility will be

$$U_{i,0} = \beta_0 + \pi M + \eta_{i,o}, \tag{2}$$

where M is a vector of the consumer's demographic characteristics.

The consumer purchases from the firm that delivers the highest utility if that utility is greater than the utility of the outside good; otherwise they consume only the outside good. Because adding a constant to the utility for every option does not affect consumers' choices, β_0 and the coefficient on one of demographics in M are normalized to be zero. I assume that the smallest coefficient of π is the one normalized to be zero.

Consumers are spread across the market. Instead of integrating over a geographic space, the consumers' locations are usually approximated as a discrete set of points. However, changing the sums to integrals in the proof that follow changes nothing. Each firm's demand is then calculated in two steps. First, integrate over the unobserved component of utility to get the percentage of consumers at a given location and demographic who patronize each firm as a function of the utility parameters. Then aggregate these choices across the different locations and demographics to get the total demand for each firm.

As in the literature, I assume that the $\eta_{i,j}$'s are distributed i.i.d. type I extreme value, which implies that when there are J firms then the fraction of consumers of demographic type M located in location b who choose to purchase a unit from firm j is:

$$S_{j,b,M}(P) = \frac{e^{\varphi_j}}{e^{\pi M} + \sum_{t=1}^{J} e^{\varphi_t}} \tag{3}$$

where $\varphi_j = X'_j\beta - P_j\,\gamma - D_{b,j}\delta$.[2] The demand for each firm is then the product of the fraction of the consumers of demographic M at each location b who patronize the firm multiplied by the mass of consumers of that demographic at that location, $h(b,M)$:

$$Q_j(P) = \sum_b \sum_M h(b,M) S_{j,b,M}(P)\,. \tag{4}$$

Supply is generally modeled by assuming that each firm sets prices according to a static Bertrand game. Formally, let the firms' costs consist of fixed costs plus a constant marginal cost for each unit. Firm j's profits are then

$$\Pi_j = \left(P_j - C_j\right) Q_j(P) - FC_j \tag{5}$$

where FC_j is firm j's fixed costs, and C_j is the firm-specific marginal cost of firm j. The fixed costs can be ignored for the sake of profit maximization.

3. Caplin and Nalebuff

Before proving that a pure-strategy equilibrium can exist for the models used in the empirical geographic-differentiation papers, I discuss why the assumptions of Caplin and Nalebuff (1991) do not hold for these models.

Caplin and Nalebuff assume that there are m firms, each of which produces a single product at a constant marginal cost. The products can be completely described by

[2] See McFadden (1973) for this derivation. $D_{b,j}$ represents the distance between (all of) the customers at location b and outlet j.

a vector of product characteristics $\chi \in R^w$. Consumers' utilities are then defined as $U(\alpha,\chi,z)$, where $\alpha \in R^n$ denotes the type of the individual, and z represents a numeraire commodity. Consumers choose to consume one unit of the good which maximizes their utility.

Caplin and Nalebuff impose two constraints on consumers' preferences. First, they assume that preferences are linear in type α:

$$U(\alpha,\chi,z) = \sum_{k=1}^{n} \alpha_k t_k(\chi) + g(z)t_{n+1}(\chi) + t_{n+2}(\chi), \qquad (6)$$

where g is a strictly increasing function concave function and $t_{n+1}(\chi) > 0$. Second, they assume that the distribution of α is ρ-concave over some convex support $B \subset R^n$ with positive volume:

$$f(\alpha_\lambda) \geq \left[(1-\lambda)f(\alpha_0)^\rho + \lambda f(\alpha_1)^\rho\right]^{1/\rho}, \qquad 0 \leq \lambda \leq 1, \qquad \rho = -1/(n+1). \qquad (7)$$

While the notation of Caplin and Nalebuff is a bit different to that used for the geographic models, it is possible to map the model of Section 2 into that of Section 3. The $X'_j\beta$ term of equation (1) corresponds to $t_{n+2}(\chi)$ in equation (6). The $-P_j\,\gamma$ term corresponds to $g(z)t_{n+1}(\chi)$. Finally, the $-\,D_{i,j}\delta\,+\,\eta_{i,j}$ terms can be rewritten as $\sum_{k=1}^{n}\alpha_k t_k(\chi)$.[3] The translation of the $\eta_{i,j}$ terms into types is straight-forward, but the translation of the distance term is not. However, the distance term can be accommodated by creating a vector of potential outlet locations, with $t_l(\chi) = 1$ if the outlet is located at location l, $t_l(\chi) = 0$ otherwise. In this case, $\alpha_l = D_{i,j}\delta$.[4]

To see why ρ-concavity does not need to hold in real datasets, consider a simple market that consists of a line segment with endpoints 0 and 1, and suppose that there is only one demographic type M. Suppose, too, that the distribution of η, $t_3(\eta)$, is independent of the consumer's location on the line, which is represented as $h(b)$, where $b \in [0,1]$. Let $\delta = 1$, and let $h(b) = (1-b)^2 + b^2$. In this case $\rho = -\,¼$. Consider the following types: $\alpha_0 = (0,1,0)$ and $\alpha_1 = (1,0,0)$, where the triple refers to the distance to the outlet located at 0, the distance to the outlet located at 1, and the value of η, respectively. In this case $f(\alpha_0) = h(0)t_3(0)$, while $f(\alpha_1) = h(1)t_3(0)$. Let $\lambda = ½$. Then

$$[½f(\alpha_0)^{-¼}+½f(\alpha_1)^{-¼}]^{-4} = t_3(0)[(½)(1)+(½)(1)]^{-4} = t_3(0) > f(\alpha_{½}) = h(½)t_3(0) = t_3(0)(½). \qquad (8)$$

This violates the ρ-concavity condition given in equation (7).

Section 4. Theorem and Proof

Theorem: If consumer utility is as in equations (1) and (2), and firm profits are as represented in equation (5) (with all firms under separate ownership), then there exists a pure strategy in prices whenever $C_j + 2/\gamma > X'_j\beta/\gamma$ for every firm.[5]

[3] If the utility function in equation (1) were modified to include random coefficients, then the random component of these coefficients would appear here, too.

[4] The utility of the outside good, which has none of the attributes of X or distance, can also be represented in this notation. For the outside good, $\sum_{k=1}^{n}\alpha_k t_k(\chi) = \eta_{i,0}$, $g(z)t_{n+1}(\chi) = \pi M$, and $t_{n+2}(\chi) = 0$.

[5] If the coefficients on M are not normalized as specified above then the right-hand side becomes $(X'j\beta - \min(\pi M))/\gamma$. These are sufficient, not necessary, conditions.

Proof:

Vives (1999, p. 16) states the following theorem: Consider a game (A_i, π_i, $i \in N$). If the strategy sets are non-empty, convex and compact subsets of Euclidean space and the payoff to firm i is continuous in the actions of all firms and quasiconcave in its own action, then there is a Nash equilibrium.

One can constrain the strategy set to the non-empty, convex, compact interval [C_j, $C_j + 2/\gamma$]. The lower bound is C_j because firms will earn negative profits at lower prices. It is also the case that a price above $C_j + 2/\gamma$ cannot be profit maximizing. To see this latter point, suppose that $P_j \geq C_j + 2/\gamma$. Then $P_j > X'_j\beta/\gamma$ (because $C_j + 2/\gamma > X'_j\beta/\gamma$). This, in turn, implies that $S_{j,b,M} = \dfrac{e^{X_j'\beta - D_{b,j}\delta - P_j\gamma}}{e^{\pi M} + \sum_{t=1}^{J} e^{X_t'\beta - D_{b,t}\delta - P_t\gamma}} < \dfrac{e^{X_j'\beta - D_{b,j}\delta - P_j\gamma}}{1 + e^{X_j'\beta - D_{b,j}\delta - P_j\gamma}} <$ ½ because the numerator will be less than 1. The firm's first-order condition will then be:

$$\sum_b \sum_M h(b,M) S_{j,b,M}(P)\left[1 - \gamma(P_j - C_j)(1 - S_{j,b,M}(P))\right] \qquad (9)$$

$$\leq \sum_b \sum_M h(b,M) S_{j,b,M}(P)\left[1 - 2\,(1 - S_{j,b,M}(P))\right] < 0. \qquad (10)$$

Thus, a firm's profits are decreasing in prices for all $P_j \geq C_j + 2/\gamma$.

It is clear from equations (4) and (5) that each firm's profit functions are continuous in the prices of all firms. The final requirement is that each firm's profits be quasiconcave in their own price. Quasiconcavity holds if profits first strictly increase, then strictly decrease, in price. Since I have already shown that the first derivative of profits is positive at the lower bound and negative at the upper bound, it is sufficient to show that the second derivative of profits is negative.

The second derivative of profits with respect to price is

$$\sum_b \sum_M h(b,M)\gamma(S_{j,b,M}(P) - S^2_{j,b,M}(P))\left[-2 + \gamma(P_j - C_j)(1 - 2S_{j,b,M}(P))\right]. \qquad (11)$$

Note that this is clearly negative whenever $\gamma(P_j - C_j) < 2$ (i.e., $P_j < C_j + 2/\gamma$), which is the range of prices we were looking at. **QED**.

5. Conclusion

While the theorem provides a limited set of conditions under which there exists a pure strategy equilibrium, the key property of the parameter restrictions is that no outlet receives more than half of the market share from the consumers at any location. This result has implications for existence under mixed-logit demand systems, as would occur if consumers had different preferences over a good's non-price characteristics based on their observable demographic characteristics. In this case, the probability density of consumers' utilities also cannot be assumed to be ρ-concave, as required by Caplin and Nalebuff. However, the logic of the above proof can be used to prove existence if every good's market share will be less than one-half for every demographic group.

While existence is shown for any distribution of consumers and firms, what is lacking are proofs of existence under any ownership structure, as well as proofs of uniqueness under either of these cases. Hopefully the field will soon find these proofs.

REFERENCES

Anderson, S., A. de Palma, and J.-F. Thisse (1992) *Discrete Choice Theory of Product Differentiation*, The MIT Press: Cambridge, MA.

Caplin, A. and B. Nalebuff (1991) “Aggregation and Imperfect Competition: On the Existence of Equilibrium” *Econometrica* **59**, 25-60.

Davis, P. (2001) “Spatial Competition in Retail Markets: Movie Theaters” Mimeo, MIT Sloan.

Manuszak, M. (2000) “Firm Conduct and Product Differentiation in the Hawaiian Retail Gasoline Industry” Mimeo, Carnegie Mellon.

McFadden, R. (1973) “Conditional Logit Analysis of Qualitative Choice Behavior” in *Frontiers in Econometrics* by P. Zarembka, ed., Academic: New York, 105-142.

Thomadsen, R. (2004) “The Effects of Ownership Structure on Prices in Geographically Differentiated Industries” *RAND Journal of Economics*, forthcoming.

Vives, X. (1999) *Oligopoly pricing: old ideas and new tools*, The MIT Press: Cambridge, MA.