E C O N O M I C S B U L L E T I N

On autoregressive errors in singular systems of equations

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Abstract

Dhrymes (1994, Econometric Theory, 10, 254−285) demonstrates the arising identification and estimation problems in singular equation systems when the error vector obeys an autoregressive scheme, as an extension of restricted least squares. Unfortunately, his main theorem concerning the identification of such systems, does not hold in general, though.

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Dhrymes (1994) considers a system of m general linear models

$$y_{t\cdot} = x_{t\cdot}B + u_{t\cdot}, \qquad t = 1, 2, \ldots, T, \tag{1}$$

with a singular error covariance matrix $\Omega^* = \text{cov}(u'_{t\cdot})$ subject to adding-up restrictions on the dependent variables $y_{t\cdot}e = x_{tG}$. In addition, the system errors follow a first-order vector autoregressive scheme

$$u_{t\cdot} = u_{t-1\cdot}H + \epsilon_{t\cdot}, \qquad t = 1, 2, \ldots, T. \tag{2}$$

From the seminal work of Berndt and Savin (1975), above setup leads to parameter restrictions both on the model parameters B and on the autoregressive parameters H. Moreover, the singularity of $\Sigma = \text{cov}(\epsilon'_{t\cdot})$ arises as He lies in the null space of Σ. Combining (1) and (2) leads to

$$y_{t\cdot} = x_{t\cdot}B + y_{t-1\cdot}H - x_{t-1\cdot}BH + \epsilon_{t\cdot}, \qquad t = 1, 2, \ldots, T. \tag{3}$$

We then minimize $\sum_t (y_{t\cdot} - w_{t\cdot})\Sigma_g(y_{t\cdot} - w_{t\cdot})'$ subject to the constraints mentioned above on the coefficient matrices B, H and possibly other a priori restrictions (see Haupt and Oberhofer, 2002), where $w_{t\cdot} = x_{t\cdot}B + y_{t-1\cdot}H - x_{t-1\cdot}BH$ and Σ_g is the g-inverse of Σ.

The resulting normal equations

$$g(\beta) = 0 \tag{4}$$

are nonlinear in the parameters $\beta = (\text{vec}(B)', \text{vec}(H)')'$ and Dhrymes (1994, equation (32)) writes them in the form

$$S_\beta \beta = s_\beta, \tag{5}$$

where the index indicates that S_β and s_β depend on β, and S_β has the form

$$\begin{pmatrix} P_\beta & R'_\beta \\ R_\beta & 0 \end{pmatrix}. \tag{6}$$

In Remark 8 following his main Theorem 1, Dhrymes (1994) states that the non-singularity of P_β implies the identifiability of β, or, in other words that the nonlinear normal equations (4) can only be solved, if this applies to (5) for fixed S_β and s_β. The parameter vector β, however, can be unique even in the case of a singular S_β (see Lemma 1 in Haupt and Oberhofer, 2002, who provide a discussion of the corresponding very mild and usual assumptions).

Due to the vital importance of this issue, we will explicitly state two arguments against the correctness of Dhrymes' (1994) arguments:

(i) It is well known (e.g., Rao, 1965) that linear systems with a coefficient matrix (6) have a solution even if the matrix P_β therein is singular.

(ii) Equation (5) is just one of the many possible ways to represent the first order conditions in (4). Thus the estimability of the model parameters should not hinge on the non-singularity of the coefficient matrix, as it is quite likely that even for the singular case, the highly nonlinear system (5) (or the first order conditions in (4)) still has a unique solution. For example suppose that a scalar parameter θ satisfies the first order conditions $g(\theta) = \theta^2 + 2\theta - 3 = 0$ for $\theta \in \Theta = [0, \infty)$. Then, the only admissible solution in the parameter space Θ is $\theta = 1$. Now suppose that we rewrite the first order conditions as $G(\theta)\theta = g(\theta)$, that is $(\theta - 2)\theta = 3 - 4\theta$. Then, the coefficient $G(\theta) = \theta - 2$ is not always non-singular in the parameter space Θ, but this does not mean that the nonlinear first order condition does not have a solution.

As a consequence Dhrymes' (1994) system (32) can have a unique solution in general, even if the coefficient matrix (and the matrix P_β therein) is singular for admissible values of the parameters. It is further questionable whether the fact that nonlinear normal equations have a unique solution, implies identification of parameters. Let us illustrate this claim with an example where we consider a nonlinear regression model $y_t = h(x_t, b_0) + u_t$, $t = 1, 2, \dots, T$, with $\hat{b}_T = \arg\min_b [G_T(y_1, \dots, y_T, b)]$, where $G_T(y_1, \dots, y_T, b) \to G(b)$ and $b_0 = \arg\min_b G(b)$. Then, the convexity of G_T implies $\operatorname{plim} \hat{b}_T = b_0$. The latter result, however, does not follow if G_T is not convex.

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