E C O N O M I C S B U L L E T I N

The coincidence of the core and the dominance core on multi−choice games

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game

Abstract

We propose a necessary and sufficient condition for the existence of dominance core and a necessary and sufficient condition for coincidence of the core and the dominance core to the setting of multi−choice games.

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1 Introduction

There are two different definitions of the core of TU games. Gillies (1959) defined the core in terms of the binary relation-domination. The other definition of the core is defined as the solution of a system of linear inequalities. We will call the former dominance core and the latter core. Chang (2000) proposed a necessary and sufficient condition for the existence of dominance core, and a necessary and sufficient condition for coincidence of the core and dominance core to the setting of TU games.

A multi-choice game, was introduced by Hsiao and Raghavan (1993), is a game in which each player has a certain number of activity levels at which he or she can choose to play. This is formalized as follows. Let $N = \{1, \ldots, n\}$ be a set of players ($n \in \mathbb{N}$) and suppose each player $i \in N$ has $m_i + 1 \in \mathbb{N}$ activity levels at which he can play. We set $M_i = \{0, 1, \ldots, m_i\}$ as the action space of player $i \in N$, where the action 0 means not participating, and the zero vector $(0, \ldots, 0)$ will be denoted by θ. A function $v : \prod_{i \in N} M_i \to \mathbb{R}$ with $v(\theta) = 0$ gives for each coalition $s = (s_1, \ldots, s_n) \in \prod_{i \in N} M_i$ the worth that the players can obtain when each player i plays at level $s_i \in M_i$. van den Nouweland et al. (1995) extended the core and dominance core to the setting of multi-choice games, and introduced a notion of *balancedness* to generalize the Theorem of Bondareva (1963) and Shapley (1967) to the class of multi-choice games. In this note, we will generalize Chang's (2000) results to the setting of multi-choice games.

2 Definitions, Notations and Facts

A multi-choice game is a triple (N, m, v), where N is the set of players, $m \in (\mathbb{N} \cup \{0\})^N$ is the vector describing the number of activity levels for all players, and $v : \prod_{i \in N} M_i \to \mathbb{R}$ is the *characteristic function* with $v(\theta) = 0$. We will consider that $m_i \geq 1$ for each player $i \in N$ and if there can be no confusion we will denote a game (N, m, v) by v. We denote the set of all multi-choice games with player set N by MC^N.

A multi-choice game v is called *zero-normalized* if the players cannot gain anything by working alone, i.e., $v(je^i) = 0$ for all $i \in N$ and $j \in M_i \setminus \{0\}$. For an arbitrary multi-choice game v, the *zero-normalization* game v_0 of v is defined by $v_0(s) = v(s) - \sum_{i \in N} a(s_i e^i)$ for all $s \in \prod_{i \in N} M_i$ where $a(je^i) = v(je^i)$ for all $i \in N$ and $j \in M_i \setminus \{0\}$.

Let $(N, m, v) \in MC^N$. We define $M = \{(i, j) : i \in N, j \in M_i\}$. A *(level)* payoff vector for the game v is a function $x : M \to \mathbb{R}$, where, for all $i \in N$ and $j \in M_i \setminus \{0\}$, x_{ij} denotes the increase in payoff to player i corresponding to a change of activity from level $j - 1$ to level j by this player and $x_{i0} = 0$ for all $i \in N$. Let $S \subseteq N$. By e^S we denote the vector in $\mathbb{R}^N$ satisfying $e^S_i = 0$ if $i \notin S$ and $e^S_i = 1$ if $i \in S$.

A payoff vector is called *efficient* if $\sum_{i \in N} \sum_{j=1}^{m_i} x_{ij} = v(m)$ and it is called *level increase rational* if, for all $i \in N$ and level $j \in M_i \setminus \{0\}$, $x_{ij} \geq v(je^i) - v((j-1)e^i)$.

Definition 2.1 *A payoff vector is an* imputation *of v if it is efficient and level increase rational.*

We denote the set of imputations of the game v by $I(v)$. It is easily seen that

$$I(v) \neq \emptyset \Longleftrightarrow v(m) \geq \sum_{i \in N} v(m_i e^i)$$

Now let x be a payoff vector for the game v. If a player i works at his jth level ($j \in M_i$), then he obtains, according to x, the amount $\sum_{k=0}^{j} x_{ik}$. It will often be more natural to look at these accumulated payoffs. For $i \in N$ and $j \in M_i$ we denote $X_{ij} = \sum_{k=0}^{j} x_{ik}$. The members of a coalition $s \in \prod_{i \in N} M_i$ obtain $X(s) = \sum_{i \in N} X_{is_i}$. Using this, we come to the following

Definition 2.2 *The core $C(v)$ of the game v consists of all $x \in I(v)$ that satisfy $X(s) \geq v(s)$ for all $s \in \prod_{i \in N} M_i$, i.e.,*

$$C(v) = \{x \in I(v) : X(s) \geq v(s) \text{ for all } s \in \prod_{i \in N} M_i\}.$$

Remark 2.3 *Let v be a zero-normalized game and let*

$$\mathcal{C} = \{z \in I\!R^N_+ : \sum_{i \in N} z_i = v(m) \text{ and } \sum_{i \in A(s)} z_i \geq v(s), \text{ for all } s \in \prod_{i \in N} M_i\}.$$

If x is a payoff vector in $C(v)$, we can define a vector $z \in I\!R^N_+$ by $z_i = \sum_{j=1}^{m_i} x_{ij}$ for all $i \in N$ such that $z \in \mathcal{C}$. On the other hand, let a vector $z \in \mathcal{C}$, we can also define a payoff vector $x : M \to I\!R$ such that $x \in C(v)$ by

$$x_{ij} = \begin{cases} z_i & \text{if } i \in N \text{ and } j = 1 \\ 0 & o.w, \end{cases}$$

That is, $C(v) = \{x \in I(v) : \sum_{i \in A(s)} \sum_{j=1}^{s_i} x_{ij} \geq v(s), \text{ for all } s \in \prod_{i \in N} M_i\} \neq \emptyset$ if and only if $\mathcal{C} = \{z \in I\!R^N_+ : \sum_{i \in N} z_i = v(m) \text{ and } \sum_{i \in A(s)} z_i \geq v(s), \text{ for all } s \in \prod_{i \in N} M_i\} \neq \emptyset$.

Let $s \in \prod_{i \in N} M_i$ and x, $y \in I(v)$. The imputation y dominates the imputation x via coalition s, denote y dom_s x, if $Y(s) \leq v(s)$ and $Y_{is_i} > X_{is_i}$ for all $i \in A(s)$, where $A(s) = \{i \in N : s_i > 0, s \in \prod_{i \in N} M_i\}$ is the set of players who participate in s. We say that the imputation y dominates the imputation x if there exists a coalition $s \in \prod_{i \in N} M_i$ such that y dom_s x.

Definition 2.4 *The dominance core $DC(v)$ of the game v consists of all $x \in I(v)$ for which there exists no $y \in I(v)$ such that y dominates x, i.e.,*

$$DC(v) = \{x \in I(v) : \nexists y \in I(v) \text{ such that } y \text{ dominates } x\}.$$

The following two Lemmas were studied by van den Nouweland et al. (1995,p.292,293).

Lemma 2.5 *For each game v the core $C(v)$ is a subset of the dominance core $DC(v)$.*

Lemma 2.6 *Let v be an arbitrary game and v_0 its zero-normalization. Let x be a payoff vector for this game. Define $y : M \to I\!R$ by $y_{ij} = x_{ij} - v(je^i) + v((j-1)e^i)$ for all $i \in N$ and $j \in M_i \setminus \{0\}$. Then we have*

(1) $x \in I(v) \Longleftrightarrow y \in I(v_0)$

(2) $x \in C(v) \Longleftrightarrow y \in C(v_0)$

(3) $x \in DC(v) \Longleftrightarrow y \in DC(v_0)$.

A notion of balancedness to the setting of multi-choice games was introduced by van den Nouweland et al. (1995) as follows.

Definition 2.7 *A multi-choice game v is called balanced if for all maps $\lambda : \prod_{i\in N} M_i \to I\!R_+$ satisfying*

$$\sum_{s\in\prod_{i\in N} M_i} \lambda(s)e^{A(s)} = e^N$$

it holds that

$$\sum_{s\in\prod_{i\in N} M_i} \lambda(s)v_0(s) \leq v_0(m),$$

where v_0 is the zero-normalization of v.

The next Theorem is an extension of the Theorem of Bondareva (1963) and Shapley (1967) to the setting of multi-choice games and gives a necessary and sufficient condition for the nonemptiness of the core of a game by van den Nouweland et al. (1995,p.297).

Theorem 2.8 *Let v be a multi-choice game. Then the core $C(v)$ of v is non-empty if and only if v is balanced.*

To end this section, we give two examples to explain that why we define such balancedness, corresponding to zero-normalization, on multi-choice games. One is that we provide a multi-choice game v with nonempty core but it does not satisfy

$$\sum_{s\in\prod_{i\in N} M_i} \lambda(s)v(s) \leq v(m) \text{ whenever } \sum_{s\in\prod_{i\in N} M_i} \lambda(s)e^{A(s)} = e^N. \tag{2.1}$$

The other is that a multi-choice game v satisfies the condition (2.1) but it has empty core.

Example 2.9 *Let (N, m, v) be a multi-choice game where $N = \{1, 2\}$, $m = (2, 1)$ and $v((0, 1)) = v((1, 1)) = v((2, 1)) = 0$, $v((1, 0)) = 1$ and $v((2, 0)) = -1$. Then the payoff vector x with $x_{11} = 1$, $x_{12} = -1$ and $x_{21} = 0$ is in $C(v)$. For this game, we find a collection $\beta = \{(1, 0), (0, 1)\}$ and $\lambda((1, 0)) = 1$, $\lambda((0, 1)) = 1$ such that $\sum_{s\in\beta} \lambda(s)v(s) = 1 > 0 = v((2, 1))$.*

Example 2.10 *Let (N, m, v) be a multi-choice game where $N = \{1, 2\}$, $m = (2, 1)$ and $v((0, 1)) = v((1, 0)) = v((1, 1)) = -1$, $v((2, 0)) = 1$ and $v((2, 1)) = 0$. Then v clearly satisfies the condition (2.1). To verify that it has empty core, consider the zero-normalization v_0 of v with $v_0((0, 1)) = v_0((1, 0)) = v_0((2, 0)) = v_0((2, 1)) = 0$, and $v_0((1, 1)) = 1$. It is easy to see that $v_0((2, 1)) = 0 < 1 = \sum_{s\in\beta} \lambda(s)v_0(s)$ for $\beta = \{(1, 1)\}$ and $\lambda((1, 1)) = 1$, thus $C(v_0) = \emptyset$.*

3 Main Results

In this section we will extend Chang's (2000) results from TU games to multi-choice games. It is known that the core and the dominance core are invariant under strategic equivalence by Lemma 2.6. Hence, w.l.o.g., we assume that all multi-choice games are zero-normalized. Besides, we will assume that $v(m) \geq 0$ and thus $I(v) \neq \emptyset$.

Let (N, m, v) be a game. We define a new game by $v'(s) = \min\{v(s), v(m)\}$ for all $s \in \prod_{i\in N} M_i$. Then $v'(m) = v(m)$ and $v'(je^i) = v(je^i) = 0$ for all $i \in N$ and $j \in M_i \setminus \{0\}$. Hence (N, m, v') is also with $v'(m) \geq 0$ and $v'(je^i) = 0$ for all $i \in N$ and $j \in M_i$. And it is easy to see that $I(v) = I(v')$.

Lemma 3.1 *Let $s \in \prod_{i\in N} M_i$, $s \neq \theta$, and let x, $y \in I(v) = I(v')$. Then x dom_s y in v' if and only if x dom_s y in v.*

proof: Let $s \in \prod_{i\in N} M_i$, $s \neq \theta$, and let x, $y \in I(v) = I(v')$. If x dom_s y in v', then $X(s) \leq v'(s)$ and $X_{is_i} > Y_{is_i}$ for all $i \in A(s)$. Therefore $X(s) \leq v(s)$ and x dom_s y in v. On the other hand, if x dom_s y in v, then $X(s) \leq v(s)$ and $X_{is_i} > Y_{is_i}$ for all $i \in A(s)$. Since $x \in I(v)$, $X(s) = \sum_{i\in N}\sum_{j=1}^{m_i} x_{ij} - \sum_{i\in N}\sum_{j=s_i+1}^{m_i} x_{ij} \leq v(m)$. These imply that $X(s) \leq v'(s)$ and x dom_s y in v'. *Q.E.D.*

Lemma 3.2 *For any game* $(N, m, v) \in MC^N$, $DC(v) = DC(v')$.

proof: It follows from Lemma 3.1. *Q.E.D.*

Lemma 3.3 *For any game* $(N, m, v) \in MC^N$, $C(v') = DC(v')$.

proof: According to Lemma 2.5, we know that $C(v') \subseteq DC(v')$. If $DC(v') = \emptyset$, it is easy to see that $C(v') = DC(v')$. If $DC(v') \neq \emptyset$, it remains to show that $DC(v') \subseteq C(v')$. Let $x \in DC(v')$ and suppose that $x \notin C(v')$. Then there exists a coalition $s \in \prod_{i\in N} M_i$ such that $X(s) < v'(s)$. Since $v'(t) \leq v'(m)$ for all $t \in \prod_{i\in N} M_i$, we can define a payoff vector $y : M \to I\!R$ by

$$y_{ij} = \begin{cases} x_{ij} + \frac{v'(s)-X(s)}{\sum_{k\in N} s_k} & \text{if } i \in N, j \in \{1, 2, \ldots, s_i\} \\ \frac{v'(m)-v'(s)}{\sum_{k\in N}(m_k - s_k)} & \text{if } i \in N, j \in \{s_i + 1, \ldots, m_i\}. \end{cases}$$

Then $y_{ij} > x_{ij} \geq 0$ and

$$\begin{aligned} Y(m) &= \sum_{i\in N} \sum_{j=1}^{m_i} y_{ij} \\ &= \sum_{i\in N} \{ \sum_{j=1}^{s_i} y_{ij} + \sum_{j=s_i+1}^{m_i} y_{ij} \} \\ &= (\sum_{i\in N} \sum_{j=1}^{s_i} x_{ij} + \sum_{i\in N} s_i \frac{v'(s)-X(s)}{\sum_{k\in N} s_k}) + \sum_{i\in N} (m_i - s_i) \frac{v'(m)-v'(s)}{\sum_{k\in N}(m_k - s_k)} \\ &= X(s) + v'(s) - X(s) + v'(m) - v'(s) \\ &= v'(m). \end{aligned}$$

Hence $y \in I(v')$. Since $Y_{is_i} > X_{\ is_i}$ and $Y(s) = \sum_{i\in N} \sum_{j=1}^{s_i} y_{ij} = v'(s)$, y *dom*$_s$ x in v'. This contradicts the assumption. Hence $x \in C(v')$ and $DC(v') \subseteq C(v')$. *Q.E.D.*

Lemma 3.4 *For any game* $(N, m, v) \in MC^N$, $DC(v) = C(v')$.

proof: It follows from Lemmas 3.2 and 3.3. *Q.E.D*

Lemma 3.5 *For any game* $(N, m, v) \in MC^N, DC(v) \neq \emptyset$ *if and only if* (N, m, v') *is balanced.*

proof: It follows from Theorem 2.8 and Lemma 3.4. *Q.E.D.*

Lemma 3.6 *For any game* $(N, m, v) \in MC^N$ *with* $C(v) \neq \emptyset$, $C(v) = C(v')$.

proof: Using Lemmas 2.5 and 3.4, we know that $C(v) \subseteq C(v')$. It remains to show that $C(v') \subseteq C(v)$. Let $x \in C(v')$, then $x \in I(v') = I(v)$ and $X(s) \geq v'(s)$ for all $s \in \prod_{i \in N} M_i$. Now we will show that $v(s) \leq v(m)$ for all $s \in \prod_{i \in N} M_i$. Since $C(v) \neq \emptyset$, there exists an $y \in C(v)$ such that

$$Y(s) \geq v(s) \text{ for all } s \in \prod_{i \in N} M_i \qquad \text{and}$$
$$Y(s) = \sum_{i \in N} \sum_{j=1}^{m_i} y_{ij} - \sum_{i \in N} \sum_{j=s_i+1}^{m_i} y_{ij} \leq v(m).$$

Hence $v(s) \leq v(m)$. Therefore $X(s) \geq v'(s) = v(s)$ for all $s \in \prod_{i \in N} M_i$ and $x \in C(v)$. This completes the proof. *Q.E.D.*

Lemma 3.7 *For any game $(N, m, v) \in MC^N$, $C(v) = C(v')$ if and only if (N, m, v) is balanced or (N, m, v') is not balanced.*

proof: For any game $(N, m, v) \in MC^N$. If $C(v) = C(v')$, then either both $C(v)$ and $C(v')$ are empty or both are nonempty. If both $C(v)$ and $C(v')$ are empty, then (N, m, v') is not balanced. If both $C(v)$ and $C(v')$ are nonempty, then (N, m, v) is balanced. On the other hand, if (N, m, v') is not balanced, $C(v) \subseteq C(v') = \emptyset$. This implies $C(v) = C(v')$. If (N, m, v) is balanced, $C(v) \neq \emptyset$. Using Lemma 3.6, we have $C(v) = C(v')$. *Q.E.D.*

Theorem 3.8 *For any game $(N, m, v) \in MC^N$, $C(v) = DC(v)$ if and only if (N, m, v) is balanced or (N, m, v') is not balanced.*

proof: Since we have known that $DC(v) = C(v')$ for any game $(N, m, v) \in MC^N$ by Lemma 3.4, it suffices to show $C(v) = C(v')$ if and only if (N, m, v) is balanced or (N, m, v') is not balanced. Then, using Lemma 3.7, we obtain $C(v) = DC(v)$ if and only if (N, m, v) is balanced or (N, m, v') is not balanced. *Q.E.D.*

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