E C O N O M I C S   B U L L E T I N

A riemannian metric on the equilibrium manifold

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Abstract

Under the assumption that the utility function is real analytic, we construct a complete metric on the equilibrium manifold with fixed total resources such that a minimal geodesic joining any two regular equilibria intersects the set of critical equilibria in a finite number of points.

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1 Introduction

In a recent work (Loi and Matta 2004), a complete metric on the equilibrium manifold was constructed satisfying the property that a minimal geodesic joining two regular equilibria intersects the codimension one stratum, S_1, of critical equilibria, E_c, in a finite number of points. This geodesic represents an *efficient* path joining two regular equilibria, (p, ω) and (p', ω'), where efficiency means that the distance between (p, ω) and (p', ω') is minimized and that this path intersects the set of critical equilibria S_1 transversally, i.e., in a measure zero set. The economic meaning of this geodesic is evident if we consider that there exist infinite paths joining two arbitrary equilibria because of the arconnectedness property of the equilibrium manifold (Balasko 1988). Each of these paths identify a redistribution policy of resources and a metric is constructed in order to choose the path (the geodesic) which enjoys the desirable economic properties of minimizing distance and discontinuities of prices originated in the set of critical equilibria.

Two complete Riemannian metrics were proposed: g_Φ and g. The metric g_Φ was constructed on the natural parametrization Φ of the equilibrium manifold (see Balasko 1988 and Section 2 below). It represented a partial solution to the problem of choosing an efficient path. In fact one had to slightly deform its geodesic in order to intersect S_1 transversally. Conversely, the metric g represented a complete solution to the problem. It was obtained by deforming g_Φ in a neighborhood of S_1. It is worth noting that the attention of the authors was concentrated on S_1, being the only stratum of E_c which can disconnect the equilibrium manifold.

In this paper, under the assumption that the utility function is real analytic, we show that g_Φ is indeed a complete solution to the problem of choosing an efficient path.

A justification for this assumption can be found in Kannai (1974), where he showed that monotone convex preferences orders can be approximated by strictly monotone preferences orders and by analytic ones. The idea, roughly speaking, is that the space of real analytic preferences is dense in the space of continuous preferences, once a topology is constructed on the space of preferences in order to reflect, in that space, the concept of closeness. We can observe that with similar approximation arguments one can show that a continuous consumer can be approximated arbitrarily close by a smooth one (see Mas-Colell 1974 and Ch. 2 in Mas-Colell 1985) and this leads us to standard smooth consumer theory, where utility function is assumed to be smooth in order to get a smooth demand function.

This paper is organized as follows. Section 2 recalls the economic setting. Section 3 contains the main results. Section 4 sums up our result and addresses further direction of research.

2 Preliminaries

We consider a pure exchange economy with l goods and m consumers. Let $S = \{p = (p_1, \dots p_l) | p_j > 0, j = 1, \dots l, p_l = 1\}$ be the set of normalized prices. Let $\Omega = (\mathbb{R}^l)^m$ denote the space of endowments $\omega = (\omega_1, \dots, \omega_m)$, $\omega_i \in \mathbb{R}^l$. We assume that the standard assumptions of smooth consumer's theory are satisfied (see Balasko 1988). The problem of maximizing the smooth utility function $u_i : \mathbb{R}^l \to \mathbb{R}$ subject to the budget constraint $p \cdot \omega_i = w_i$ gives the unique solution $f_i(p, w_i)$, i.e., consumer's i demand. Let E be the closed set of pairs $(p, \omega) \in S \times \Omega$ satisfying the following equations:

$$\sum_i^m f_i(p, p \cdot \omega_i) = \sum_i^m \omega_i.$$

The set E is a smooth submanifold of $S \times \Omega$ globally diffeomorphic to $(\mathbb{R}^l)^m$. An explicit global parametrization of E is given through the map defined by:

$$\Phi : E \to S \times \mathbb{R}^m \times \mathbb{R}^{(l-1)(m-1)} \cong \mathbb{R}^{lm}$$

$$(p, \omega_1, \dots \omega_m) \mapsto (p, p \cdot \omega_1, \dots, p \cdot \omega_m, \bar{\omega}_1, \dots, \bar{\omega}_{m-1}), \tag{1}$$

where $\bar{\omega}_i$ denotes the vector of $\mathbb{R}^{l-1}$ defined by the first $(l-1)$ -coordinates of ω_i. Let $F_{(p,w)}$ be the fiber associated with $(p, w) = (p, w_1, \dots w_m) \in S \times \mathbb{R}^m$, namely the subset of E consisting of pairs (p, ω) such that $p \cdot \omega_i = w_i$. It is a linear manifold of dimension $(l-1)(m-1)$ embedded in E. Let $\pi : E \to \Omega$ be the *natural projection*, i.e. the smooth map defined by the restriction to E of $(p, \omega) \mapsto \omega$. Let E_c be the set of critical equilibria, namely the pairs $(p, \omega) \in E$ such that the derivative of π at (p, ω) is not onto. Let us assume that total resources are fixed. Let $r \in \mathbb{R}^l$ denote the vector representing the total resources of the economy and $\Omega(r)$ denote the space of economies associated with the fixed total resources, i.e., $\Omega(r) = \{\omega \in (\mathbb{R}^l)^m | \sum_i \omega_i = r\}$. Define

$$E(r) = \{(p, \omega) \in S \times \Omega(r) | \sum_i f_i(p, p \cdot \omega_i) = r\},$$

denote by $\pi : E(r) \to \Omega(r)$ the restriction of the natural projection to $E(r)$ and by $E_c(r)$ the set of critical points of π. Let

$$B(r) = \{(p, w) \in S \times \mathbb{R}^m | \sum_i f_i(p, w_i) = r)\}.$$

For $(p, w) \in B(r)$, let $F_{(p,w)}(r)$ be the fiber over (p, w), i.e., the pairs $(p, \omega) \in E(r)$ such that $p \cdot \omega_i = w_i$. The restriction of the map (1) to $E(r)$ defines a diffeomorphism, still denoted by Φ,

$$\Phi : E(r) \to B(r) \times \mathbb{R}^{(l-1)(m-1)}. \tag{2}$$

Balasko (1988) showed that $E(r)$ is a smooth manifold of dimension $l(m-1)$, $F_{(p,w)}(r)$ is a smooth submanifold of $E(r)$ of dimension $\mathbb{R}^{(l-1)(m-1)}$ and $B(r)$ is a smooth manifold diffeomorphic to $\mathbb{R}^{m-1}$. The set of critical equilibria $E_c(r)$ is the disjoint union of closed smooth submanifolds $\mathcal{S}_i, i = 1, \ldots, \inf(l-1, m-1)$ of $E(r)$. The manifold $\mathcal{S}_i$ has dimension $l(m-1) - i^2$ and $\mathcal{S}_i = \emptyset$ for $i > \inf(l-1, m-1)$ (see Balasko 1992 for further properties of the set of critical equilibria).

3 Main results

We consider a finite pure exchange economy as described in section 2 and we introduce the further assumption that the utility function $u : \mathbb{R}^l \to \mathbb{R}$ is real analytic. The following lemma and its corollary are implications of this assumption.

Lemma 3.1 *If the utility function $u : \mathbb{R}^l \to \mathbb{R}$ is real analytic, then the demand function $f : S \times \Omega \to \mathbb{R}^l$ is real analytic.*

Proof: Balasko (1988) showed that under the standard assumptions of smooth consumer theory the demand function f is a diffeomorphism. Let us consider its inverse $g : \mathbb{R}^l \to S \times \mathbb{R}$, defined by the formula $g(x) = (grad_n u(x), x \cdot grad_n u(x))$. It is evident that it is real analytic because the utility function is assumed to be real analytic. Hence the demand function is real analytic being the inverse of a real analytic diffeomorphism. □

Corollary 3.2 *If the demand function $f : S \times \Omega \to \mathbb{R}^l$ is real analytic, then the equilibrium manifold $E(r)$ is real analytic. Moreover, the set of critical equilibria $E_c(r)$ is the disjoint union of closed real analytic submanifolds S_i of $E(r)$.*

Proof: Recall that the equilibrium manifold $E(r)$ is the closed submanifold of pairs $(p, \omega) \in S \times \Omega$ satisfying the equations $\sum_i^m f_i(p, p \cdot \omega_i) - r = 0$. Hence $E(r)$ is real analytic being zeros of real analytic equations. The second part follows from the definition of critical equilibria (see Balasko 1988, Ch.4). □

We now consider the Riemannian metric constructed on the natural parametrization defined by formula (1).

We recall that a Riemannian metric g on a n-dimensional smooth manifold X is a correspondence which associates to each point p of X an inner product $<,>_p$ on the tangent space T_pX such that for every local parametrization $x : U \subset \mathbb{R}^n \to X$, with $x(x_1, \ldots, x_n) = q \in x(U)$ and $\frac{\partial}{\partial x_i}(q) = dx_q(0, \ldots, 1, \ldots, 0)$,

$$g_{ij}(x_1, \ldots, x_n) =< \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) >_q$$

is a smooth function on U. A smooth manifold with a Riemannian metric g is called a *Riemannian manifold*.

Any Riemannian manifold can be equipped with the structure of *metric space*. Given any two points $x, y \in X$, the distance between x and y, denoted by $d(x, y)$, is defined as the infimum of the lengths of all curves γ_{xy}, where γ_{xy} is a piecewise differentiable curve joining x and y. Let(X, g), $X \subset \mathbb{R}^n$, be a Riemannian manifold and let $I \subset \mathbb{R}$ be an open interval. A smooth curve $\gamma : I \to X$ is a *geodesic* if its acceleration vector $\gamma''(t_0)$ is orthogonal to $T_{\gamma(t_0)}X$ for all $t_0 \in I$.

A Riemannian manifold (X, g) is said to be *complete* if for every pair of points $x, y \in X$ there exists a geodesic γ joining them such that its length equals $d(x, y)$ (see Chapter 7 in Do Carmo 1992). Observe that a geodesic γ is in general not unique. We call such a geodesic a *minimal geodesic* between the points x and y.

One of the simplest examples of complete Riemannian manifold is the Euclidean space $\mathbb{R}^n$, whose metric, g_{can}, is defined by the standard inner product. In this case the (unique) geodesic passing through a point x with direction $v \in \mathbb{R}^n$ is simply the straight line $x + tv$ with $t \in \mathbb{R}$.

Let us consider the diffeomorphism $\Phi : E(r) \to \mathbb{R}^{l(m-1)}$. Let the linear map $d\Phi_p : T_pE(r) \to T_{\Phi(p)}\mathbb{R}^{l(m-1)}$ be the *differential* of Φ. Given a point $p \in E(r)$ and two vectors $v, w \in T_pE(r)$, we can transfer on $E(r)$, via the diffeomorphism Φ, the metric g_{can} of $\mathbb{R}^{l(m-1)}$ by defining

$$(g_\Phi)_p(v, w) = g_{can}(d\Phi_p(v), d\Phi_p(w)).$$

This means that the pair $(E(r), g_\Phi)$ is a 'picture' of $(\mathbb{R}^{l(m-1)}, g_{can})$ and, then, all the properties satisfied by g_{can} are also satisfied by g_Φ. In particular, g_Φ is complete and there exists a unique minimizing geodesic joining two arbitrary points of $E(r)$. For other interesting properties of the metric g_Φ we refer the reader to Loi and Matta (2004).

The following theorem represents our main result. It is shown that the metric g_Φ enjoys the property that its geodesic intersects the set of critical equilibria in a finite number of points. We observe that g_Φ represents a complete (unique) solution to the problem of choosing an efficient path joining two regular equilibria.

Theorem 3.3 *If the utility function u is real analytic, the metric g_Φ satisfies the property that the geodesic γ joining two arbitrary regular equilibria intersects the set of critical equilibria E_c in a finite number of points.*

Proof: Let x and y be two arbitrary regular points of $E(r)$. Consider the (minimal) geodesic $\gamma : [0,1] \to E(r)$ joining them, i.e., $\gamma(0) = x$ and $\gamma(1) = y$. Observe that, by the definition of g_Φ, γ is simply the preimage of the straight line on $\mathbb{R}^{l(m-1)}$ joining $\Phi(x)$ and $\Phi(y)$ via the (real analytic) map $\Phi : E(r) \to \mathbb{R}^{l(m-1)}$. Hence γ is real analytic. Let us assume, by contradiction, that γ intersects $E_c(r)$ in an infinite number of points. Then there exists at least a $S_j, 1 \leq j \leq inf(l-1, m-1)$ which intersects $\gamma([0,1])$ non transversally. Let $(-\delta, \delta) \subset [0,1]$ be such that $\gamma(-\delta, \delta) \subset S_j$. Let us consider the real analytic functions $F_j : [0,1] \to \mathbb{R}$ defined by

$$F_j(t) = h_j(\gamma(t)),$$

where $h_j : E(r) \to \mathbb{R}$ is a real analytic function such that $S_j = h_j^{-1}(0)$. Then $F_j(-\delta, \delta) = 0$ implies that $F_j(t) = 0$ for all values of $t \in [0,1]$. In particular, $F_j(0) = 0$ (and $F_j(1) = 0$), which gives the desired contradiction being x (and y) a regular equilibrium. □

4 Conclusion

We have showed that, under the assumption that the utility function is real analytic, the metric g_Φ gives a complete solution to the problem of joining two regular equilibria with a path minimizing distance and catastrophes, i.e., discontinuities of prices originated when it crosses the set of critical equilibria. Furthermore g_Φ has the feature of representing a well defined algorithm to determine such a path. In fact in Loi and Matta (2004) a complete solution was found by deforming (in an arbitrary manner) the metric g_Φ in a neighborhood of S_1. In such a way we could construct different geodesics which share the desired property. In contrast, our result does not depend on any deformation. The geodesic here constructed is unique, being the correspondent of the straight line of the isometric Euclidean space $\mathbb{R}^{l(m-1)}$.

Many directions of research still have to be investigated. In particular, we believe that the connection between Riemannian metric and economics deserves further research. In fact the construction of a metric reflects the problem one wants to solve. Here we were concerned with the issue regarding distance and catastrophes, which represent the two arguments of our (non explicitly defined) cost function, and our metric has been constructed to give a solution to the problem of minimizing such a function. Many other issues could be relevant and the metric can represent a useful tool to deal with them.

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