E C O N O M I C S B U L L E T I N

Extending Xu's results to Arrow's Impossibility Theorem

Herrade Igersheim
CODE, Universitat Autònoma de Barcelona

Abstract

This note shows that results similar to Arrow's Impossibility Theorem can be proved by replacing the weak Pareto principle by a weaker condition called Pareto Neutrality and used by Xu (1990) to state another version of Sen's liberal paradox. Our result strengthens Xu's arguments for taking into account non−welfarist information into the social−choice−theoretic framework.

I am particularly grateful to Claude d'Aspremont and two anonymous referees for their helpful comments. The usual caveat, of course, applies.
Citation: Igersheim, Herrade, (2005) "Extending Xu's results to Arrow's Impossibility Theorem." *Economics Bulletin*, Vol. 4, No. 13 pp. 1−6
Submitted: October 4, 2005. **Accepted:** November 29, 2005.
URL: http://www.economicsbulletin.com/2005/volume4/EB−05D70027A.pdf

1. Introduction

The purpose of this note is to prove another version of Arrow's famous Impossibility Theorem (1963) by using a condition defined by Xu (1990) in a paper on Sen's liberal paradox (1970a,b). Xu notes that the weak Pareto principle consists of two parts: a neutrality factor called the Pareto Neutrality on the one hand, and a unanimity factor called the Pareto Unanimity on the other hand. Xu shows that the Pareto Neutrality is responsible for generating the Impossibility of a Paretian liberal. We attempt here to prove that the Pareto Neutrality is responsible for Arrow's Impossibility Theorem as well.

Our study proceeds from a stream of research focusing on the understanding of analytical connections between Arrow's and Sen's impossibility results, which the very recent work of Saari (1998, 2001) is part of. One of the objectives of this study is to stress conceptual drawbacks of the social-choice-theoretic framework and then to be able to circumvent these negative results in a satisfying way.

In this note, we follow the line of reasoning of Wilson (1972) and Kelsey (1985, 1988). Indeed, Kelsey (1985) proves another version of Sen's liberal paradox by replacing the weak Pareto principle by a condition of non-imposition used by Wilson (1972) within the framework of Arrow's Impossibility Theorem. Beyond formulating a new analytical result, we aim at strengthening Xu's view on flaws of the social choice theory, which tends to challenge Wilson's and Kelsey's conclusions.

The remainder of the paper is organized as follows. In Section 2, we present the basic concepts of our analysis. Section 3 deals with the proof of our theorem and finally, Section 4 concludes the paper.

2. Basic concepts

Let $N = \{1, 2, ..., n\}$ be the finite set of individuals, which constitutes society ($n \geq 2$). X denotes the finite set of all conceivable social states and contains at least three distinct social states. R_i is a preference relation of the individual $i \in N$ on social states. We assume that R_i is a complete pre-ordering on X (complete and transitive binary relation on X). P_i and I_i are the asymmetric and symmetric parts of R_i, respectively. A $n-$list of individual preferences $(R_1, R_2, ..., R_n)$ will be called a profile and designed by d. Let $\mathcal{D}$ be the set of all conceivable profiles. A collective choice rule f specifies a social preference relation for each profile d of $\mathcal{D}$: $R = f(d)$. As for R_i, P and I are the asymmetric and symmetric parts of R, respectively. If R is always a complete pre-ordering, f is a "social welfare function" (SWF) in the sense of Arrow (1963). If R is only complete and acyclic, f is a "social decision function" (SDF) in the sense of Sen (1970b).

Some conditions required by Arrow's and Sen's impossibility theorems must be defined. We state these conditions below.

Condition 1 (U) ***Unrestricted domain*** *The domain of f is the set $\mathcal{D}$.*

Condition 2 (P) ***Weak Pareto principle*** *For any $x, y \in X$, if xP_iy for every $i \in N$, then xPy.*

Condition 3 (I) ***Independence of irrelevant alternatives*** $\forall\, x, y \in X$, $\forall\, d, d' \in \mathcal{D}$, $[\forall\, i,\, xR_iy \iff xR'_iy] \Longrightarrow [xRy \iff xR'y]$.

Before stating the conditions of nondictatorship and of minimal libertarianism imposed by Arrow and Sen on the collective choice rule, another definition, which clarifies the concept of decisiveness, must be introduced:

Definition 1 ***Decisiveness*** *A set of individuals V of N is decisive for x against y if xPy when xP_iy for every $i \in V$.*

Accordingly:

Condition 4 (D) ***Nondictatorship*** *There is no individual i such that: $\forall\, d \in \mathcal{D}, \forall\, x, y \in X : xP_iy \Longrightarrow xPy$.*

Condition 5 (L*) ***Minimal libertarianism*** *There is at least one pair of persons decisive both ways over at least one pair of alternatives each; i.e. for each of them i, there is a pair of alternatives in X, x, y, such that xP_iy implies xPy and yP_ix implies yPx.*

Theorem 1 (Arrow) *There is no SWF satisfying Conditions U, P, I and D.*

Theorem 2 (Sen) *There is no SDF satisfying Conditions U, P and L*.*

Let us examine now how Xu (1990) proposes to decompose the weak Pareto principle and which part of it gives rise to Sen's liberal paradox.

Condition 6 (WPU) ***Weak Pareto Unanimity*** *There exists x and y in X, and a preference profile d^* such that xP^*_iy for every $i \in N$ implies xP^*y.*

Condition 7 (PN) ***Pareto Neutrality*** *For any x, y, a and b in X, and for any preference profiles d and d', if xP_iy and aP'_ib for every $i \in N$, then $aP'b \iff xPy$.*

Lemma 1 (Xu) *Condition P is equivalent to Conditions PN and WPU.*

Condition PN is highly demanding since it enables to "look around" to get information from other pairs. This strong property is sufficient to bring about the paradox of a Paretian liberal:

Theorem 3 (Xu) *There is no SDF satisfying Conditions U, PN and L*.*

The theorem that we shall prove in the next section is thus that Condition PN triggers off another version of Arrow's theorem.

3. Proof of the theorem

We begin by introducing some supplementary definitions and two conditions.

Definition 2 ***Undecisiveness*** *A set of individuals V of N is undecisive for x against y if yRx when xP_iy for every $i \in V$.*

Definition 3 ***Almost Undecisiveness*** *A set of individuals V of N is almost undecisive for x against y if yRx when xP_iy for every $i \in V$ and yP_ix for every $i \in (N - V)$.*

Notationally, let $J = \{j\}$ denote a set which contains only one individual j. Besides, $UD(x, y)$ means that J is undecisive for x against y, and $\overline{UD}(x, y)$ means that J is almost undecisive for x against y. Note that $UD(x, y) \Longrightarrow \overline{UD}(x, y)$.

Condition 8 (AUD) ***Anti-Undecisiveness*** *There is no individual i such that: $\forall\, d \in \mathcal{D}, \forall\, x, y \in X, xP_iy \Longrightarrow yRx$.*

Please note that an individual that would satisfy Condition D could be called a dictator. On the other hand, an individual that would satisfy Condition AUD could be called an anti-dictator[1].

Condition 9 (AP) ***Anti-weak Pareto principle*** *For any $x, y \in X$, if xP_iy for every $i \in N$, then yRx.*

We now turn to two lemmas from which our theorem will be derived.

Lemma 2 (Restatement of Lemma 1) *Under Condition PN, if there exists a preference profile d^* such that uP_i^*v for every $i \in N$ implies uP^*v for at least one pair of alternatives $\{u, v\}$ in X, then Condition P holds.*

Lemma 3 *If there is no preference profile d^* such that uP_i^*v for every $i \in N$ implies uP^*v for at least one pair of alternatives $\{u, v\}$ in X, then Condition AP holds.*

The proof of Lemma 3, obvious, is omitted here.

We are now ready to prove our theorem.

[1]Actually, one could argue that an individual which satisfies Condition AUD could be called a "weak" anti-dictator in the sense of Mas-Colell and Sonnenschein (1972), which define on the other hand a "weak" dictator. Therefore, the case of collective impotence, i.e. $\forall\, x, y \in X : xIy$, is taken into account in our theorem 4 thanks to Condition AUD.

Theorem 4 *There is no SWF satisfying Conditions U, PN, I, D and AUD.*

Proof Two cases have to be distinguished: on the one hand, if there exists a preference profile d^* such that uP_i^*v for every $i \in N$ implies uP^*v for at least one pair of alternatives $\{u, v\}$ in X, then Condition P holds according to Lemma 2 and the proof is similar as Arrow's. Hence, there is a dictator in society, which leads to a contradiction with Condition D.

On the other hand, if there is no preference profile d^* such that uP_i^*v for every $i \in N$ implies uP^*v for at least one pair of alternatives $\{u, v\}$ in X, then Condition AP holds according to Lemma 3 and we must prove that there is no SWF satisfying Conditions U, AP, I and AUD. We follow the same line of reasoning as Arrow's proof (1963, pp. 97-100), but in an opposite way.

(1) Firstly, we prove that if there is some individual j such that $J = \{j\}$ is almost undecisive over any ordered pair of alternatives, then an SWF satisfying Conditions U, AP and I implies that j must be an anti-dictator.

Suppose that $J = \{j\}$ is almost undecisive for some x against some y, i.e. $\exists\, x, y \in X$ such that $\overline{UD}(x, y)$. Let z be another alternative, and let i refer to all individuals other than j. Assume that xP_jy & yP_jz and that yP_ix & yP_iz. By transitivity of the individual preferences, xP_jz. Notice that we have not specified the preferences of persons other than j between x and z. Now, zRy from Condition AP. Further, yRx since J is almost undecisive for x against y. Consequently, zRx by the transitivity of R. This result, zRx, is obtained without any assumption about the preferences of individuals other than j regarding x and z. If the preferences of i over $\{y, x\}$ and $\{y, z\}$ affect the ranking of the pair $\{x, z\}$ in R, then Condition I is violated. Hence, $J = \{j\}$ is undecisive for x against z:

$$\overline{UD}(x, y) \implies UD(x, z).$$

The rest of the proof for (1) is as follows: the combinations of individual orderings are the same as Arrow's, while their consequences for R are opposite. Finally, one obtains that $\overline{UD}(x, y)$ over any pair $\{x, y\}$ in X implies $UD(a, b)$, $\forall\, a, b \in X$. Therefore, individual j is an anti-dictator and (1) is proved.

(2) Secondly, we must prove that there exists an individual j such that $J = \{j\}$ is almost undecisive over at least one pair of alternatives. Hence, according to (1), one can deduce that this individual j is an anti-dictator, which contradicts Condition AUD. We make the contrary supposition, i.e. that there is no such individual j, and show that it leads to an inconsistency.

If Condition AP holds, then for any $x, y \in X$ such that xP_iy for every i in N, we obtain yRx. Therefore, for any pair of alternatives, there is at least one undecisive set, i.e. the set of all individuals. Thus, for any pair of alternatives, there is also at least one almost undecisive set.

Here again, the rest of the proof for (2) uses the same arguments as Arrow's, but with opposite conclusions for R. It leads to a contradiction since there exists at least one almost undecisive set over at least one pair of alternatives, which contains only one individual. The theorem now follows from (1) since such an individual must be an anti-dictator. ■

4. Conclusion

Our theorem 4 means that an SWF which satisfies Conditions U, PN and I is either dictatorial (there is a dictator in society) or anti-dictatorial (the social preference relation goes systematically against preferences of an individual – even weakly: he is an anti-dictator). Please note that this result is close to Wilson's (1972), even though the set of invoked conditions is different.

Actually, Wilson (1972) and Kelsey (1985, 1988) both use a condition of non-imposition to state their theorems, whereas Xu (1990) and Theorem 4 invoke Condition PN. Therefore, the implications one can draw from these two groups of results are opposite. On the one hand, Wilson and Kelsey tend to moderate the responsibility of the weak Pareto principle in the negative results developed into the social-choice-theoretic framework. On the other hand, Xu and Theorem 4 agree with Sen's argument against the weak Pareto principle and plead for the integration of non-welfarist information into individual preferences.

These observations must lead to further investigation by comparing, both on analytical and conceptual levels, these two families of results. It will then be possible to clarify conceptual issues they raise.

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