E C O N O M I C S   B U L L E T I N

A note on an equilibrium in the great fish war game

Andrzej Nowak
Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Gora

Abstract

The great fish war game by Levhari and Mirman is studied under the limiting average utility criterion. It turns out that a stationary equilibrium in this game has a turnpike property, leads to higher steady state compared with those of discounted games but gives a higher steady state consumption. The convergence of the equilibrium functions in the finite horizon games to an equilibrium function in the infinite horizon discounted game is also proved.

Citation: Nowak, Andrzej, (2006) "A note on an equilibrium in the great fish war game." *Economics Bulletin,* Vol. 17, No. 2 pp. 1−10
Submitted: December 27, 2005. **Accepted:** March 23, 2006.
URL: http://www.economicsbulletin.com/2006/volume17/EB−05Q20009A.pdf

A Note on an Equilibrium in the Great Fish War Game

Andrzej S. Nowak

Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Podgórna 50, 65-246 Zielona Góra, Poland; e-mail: a.nowak@wmie.uz.zgora.pl, nowak@im.pwr.wroc.pl

Abstract

The great fish war game by Levhari and Mirman (1980) is studied under the limiting average utility criterion. It turns out that a stationary equilibrium in this game has a turnpike property, leads to a higher steady state compared with those of discounted games but gives a higher steady state consumption. The convergence of the equilibrium functions in the finite horizon games to an equilibrium function in the infinite horizon discounted game is also proved.
Keywords: Dynamic resource extraction game; Fishery game; Perfect equilibrium; Tragedy of the commons
JEL classification: C73, Q20

1. Introduction

It is commonly believed that free noncooperative exploitation of a common-property resource leads to the *tragedy of the commons*. This phenomenon simply means overexploitation and overconsumption. The great fish war game by Levhari and Mirman (1980) illustrates this issue in a very clear and appealing manner. It turns out that overconsumption increases when the agents (the owners of the resource) accept discounting as a time preference and the discount factor tends to zero (or equivalently, the discount rate grows up). Some arguments against discounting in social planning were presented long back by Pigou (1920) and Rawls (1971). They argued that positive discounting reflects some bias against future generations. The results of Levhari and Mirman (1980) confirm this opinion to some extent. Dutta and Sundaram (1993) show that the owners can avoid the tragedy of the commons by reducing consumption in some states. They study discontinuous (piece-wise linear) equilibrium strategies which look unrealistic. According to Dutta and Sundaram (1993) the players should drastically reduce consumption in some good states.

In this note, we examine Levhari and Mirman's example from some new viewpoints. Our main objective is to study the limiting stationary equilibrium as the discount factor tends to one. It turns out that this strategy profile constitutes an equilibrium in the game with the limiting average utilities and has a kind of turnpike property. An interesting feature of this limiting equilibrium is that it leads to a higher steady state compared with the steady states in all discounted games, and at the same time, so-called "permanent" consumption is higher as well. Therefore, it seems that accepting the undiscounted (limiting average) utility leads to an equilibrium that is good for the present and next generations.

Our second objective is to give a precise description of symmetric Markov equilibria and corresponding utilities in the finite horizon games and prove their convergence as the horizon tends to infinity. The extrapolation method of Levhari and Mirman (1980) used to solve the infinite horizon discounted game is in such a way clarified and justified.

Various extensions or modifications of Levhari and Mirman (1980) can be found in Fischer and Mirman (1992), Datta and Mirman (1999) and their references. A related dynamic game was studied by Sundaram (1989) using a fixed point theorem.

Further and more detailed comments are given in the concluding remarks.

2. The model

Suppose that there are m agents (the owners) each of whom can extract a renewable resource, e.g., fish. The resource stock develops over time according to a (biological) growth rule given by

$$x_{t+1} = x_t^{\alpha}, \quad 0 < \alpha < 1.$$

We assume that $x_t \in X := [0,1]$ called the *state space*. The boundary point $x = 1$ is the *stable steady state* of the resource population after a normalization when there is no extraction. We shall consider the *symmetric case* where the instantaneous utility function of every agent i is $u^i(c) := \log c$, $c \in X$. Let c_t^i be a resource consumption of agent i in period t. The aim of this agent is to maximize his utility $U^i(c_1^i, c_2^i, ...)$ subject to

$$x_{t+1} = \left(x_t - \sum_{j=1}^{m} c_t^j\right)^{\alpha}. \tag{1}$$

A *strategy* for agent i specifies a planned consumption level for i after each possible history of the game. Let Π^i denote the set of all possible strategies for agent i. In our further analysis some special subclasses of Π^i are important. Let F be the set of all functions $f : X \mapsto X$ such that $f(x) \leq x$ for each $x \in X$. A *Markov strategy* for agent i is a sequence $\pi^i = (f_1^i, f_2^i, ...)$ where $f_t^i \in F$ for each t. Thus, a Markov strategy for agent i specifies a consumption level of i in every period t as a function of the stock x_t at the beginning of this period. If $\pi^i = (f^i, f^i, ...)$ with $f^i \in F$, then π^i is called a *stationary strategy* and is identified with f^i.

Any profile $\pi = (\pi^1, \pi^2, ..., \pi^m)$ of strategies and an initial stock x_1 uniquely determine a sequence $(c_1^i, c_2^i, ...)$ of consumption levels for agent i. In this paper, we shall consider three utility functions:
(a) the *finite horizon discounted utility*

$$U_h^i(\pi)(x_1) := \sum_{t=1}^{h} \beta^{t-1} \log c_t^i$$

where $\beta \in (0, 1)$ is a fixed *discount factor* and $h < \infty$ is a finite horizon [1] of the game,
(b) the *infinite horizon discounted utility*

$$U_\beta^i(\pi)(x_1) := \sum_{t=1}^{\infty} \beta^{t-1} \log c_t^i,$$

(c) the *undiscounted* or *limiting average utility*

$$U^i(\pi)(x_1) := \liminf_{h\to\infty} \frac{1}{h} \sum_{t=1}^{h} \log c_t^i.$$

The utility functions defined above have an interesting common feature. If player i consumes nothing in some period his utility is $-\infty$. Therefore, the players cannot extract everything during the play if the game has to be continued. Whenever they do that everybody's utility is $-\infty$.

For any $\pi = (\pi^1, \pi^2, ..., \pi^m)$ and $\sigma^i \in \Pi^i$, we write (π^{-i}, σ^i) to denote π with π^i replaced by σ^i.

A profile of strategies $\pi_* = (\pi_*^1, \pi_*^2, ..., \pi_*^m)$ is a *Nash equilibrium* in the finite horizon game if and only if $U_h^i(\pi_*)(x_1) \geq U_h^i((\pi_*^{-i}, \sigma^i))(x_1)$ for every agent i and $\sigma^i \in \Pi^i$, $x_1 \in X$.

Nash equilibria are similarly defined for the infinite horizon games. Since the game is symmetric, we shall study symmetric Nash equilibria where every agent uses the same

[1] Clearly, in the finite horizon case a Markov strategy π^i of agent i is a finite sequence of functions from the set F.

strategy.

3. The equilibrium strategies

First, we give a full closed-loop solution for the finite horizon games.

Proposition 1. *The finite k-step game has a symmetric Nash equilibrium $\pi_*(k) = (f_*(k), f_*(k), ..., f_*(k))$ where $f_*(k) = (c_1, c_2, ..., c_k)$ and*

$$c_t(x) = \frac{x}{m + \alpha\beta + \cdots + \alpha^{k-t}\beta^{k-t}}, \quad t = 1, ..., k-1, \quad c_k(x) = \frac{x}{m}, \quad x \in X. \tag{2}$$

If $x = x_1$ is the initial stock and $V_k(x) := U_k^i(\pi_*(k))(x)$ is the equilibrium function of any agent i corresponding to the symmetric equilibrium $\pi_*(k)$ in the k-period game, then $V_1(x) = \log(x/m)$ and

$$V_{n+1}(x) = (1 + \alpha\beta + \cdots + \alpha^n\beta^n)\log x + \log B_{n+1} \tag{3}$$

where $B_1 = 1/m$ and

$$B_{n+1} := \frac{B_n^\beta(\alpha\beta + \cdots + \alpha^n\beta^n)^{\alpha\beta+\cdots+\alpha^n\beta^n}}{(m + \alpha\beta + \cdots + \alpha^n\beta^n)^{1+\alpha\beta+\cdots+\alpha^n\beta^n}}. \tag{4}$$

Moreover,

$$\begin{aligned} V_{n+1}(x) &= \log c_1(x) + \beta V_n((x - mc_1(x))^\alpha) \\ &= \max_{0\le c\le x}[\log c + \beta V_n((x-(m-1)c_1(x) - c)^\alpha)]. \end{aligned} \tag{5}$$

Proof. This result follows by backward induction. □

In the sequel, we shall use the following notation. If $f \in F$ is a stationary strategy of any agent, then $\bar{f} := (f, f, ..., f)$ is a stationary multistrategy of the agents.

Proposition 2. *The symmetric stationary strategy profile $\bar{f}_* = (f_*, f_*, ..., f_*)$ where*

$$f_*(x) = \frac{(1-\alpha\beta)x}{m + (1-m)\alpha\beta}, \quad x \in X,$$

constitutes an equilibrium in the β-discounted infinite horizon game. Moreover, for any $x \in (0, 1]$, the equilibrium utilities $U_\beta^i(\bar{f}_*)(x)$ are the same, and if we denote them by $V_\beta(x)$, then

$$V_\beta(x) = \lim_{n\to\infty} V_{n+1}(\pi_*(n+1))(x) = \inf_n V_{n+1}(\pi_*(n+1))(x).$$

Proof. *Step 1.* First we prove that the sequence $\{B_n\}$ is decreasing. For this purpose consider the function $\phi(y) := (y/(m+y))^y$, $y \in (0,1)$. Note that

$$(\log \phi(y))' = \frac{\phi'(y)}{\phi(y)} = \log\left(\frac{y}{m+y}\right) - \frac{y}{m+y} + 1 < 0, \quad y \in (0,1).$$

Hence, ϕ is decreasing, and also $\psi(y) := \phi(y)/(m+y)$ is a decreasing function. Define

$$y_k := \frac{(\alpha\beta + \cdots + \alpha^k\beta^k)^{\alpha\beta+\cdots+\alpha^k\beta^k}}{(m+\alpha\beta+\cdots+\alpha^k\beta^k)^{1+\alpha\beta+\cdots+\alpha^k\beta^k}}.$$

Since ψ is decreasing, it follows that $y_n > y_{n+1}$ for every n. We now show that $\{B_n\}$ is decreasing by induction. Clearly,

$$B_2 = \frac{1}{m^\beta(m+\alpha\beta)}\left(\frac{\alpha\beta}{m+\alpha\beta}\right)^{\alpha\beta} < B_1 = \frac{1}{m}.$$

Suppose that $B_{n+1} < B_n$ for some n. Since $y_{n+1} < y_n$, we obtain $B_{n+2} = B_{n+1}^\beta y_{n+1} < B_n^\beta y_n = B_{n+1}$, which completes the induction step. We shall prove that the sequence $\{B_n\}$ is bounded below by some positive constant. Note that

$$B_{n+1}^{1-\beta} = B_{n+1}/B_{n+1}^\beta > B_{n+1}/B_n^\beta = y_n > \lim_{k\to\infty} y_k > 0.$$

Therefore, $l := \lim_{n\to\infty} B_{n+1}$ exists and $l = \inf_n B_{n+1} > 0$. Now from $B_{n+1} = B_n^\beta y_n$ and (4), it follows that

$$l = \frac{l_1}{l_2}, \quad \text{where} \quad l_1 = \left(\frac{\alpha\beta}{1-\alpha\beta}\right)^{\alpha\beta/(1-\alpha\beta)}, \quad l_2 = \left(\frac{m+(1-m)\alpha\beta}{1-\alpha\beta}\right)^{1/(1-\alpha\beta)}. \tag{6}$$

Step 2. First note that in the $(n+1)$-step game

$$c_1(x) = \frac{x}{m+\alpha\beta+\cdots+\alpha^n\beta^n}$$

depends on n. Hence $\lim_{n\to\infty} c_1(x) = f_*(x) := (1-\alpha\beta)x/(m+(1-m)\alpha\beta)$. Put $v_\beta(x) := \lim_{n\to\infty} V_{n+1}(x)$. By Step 1, v_β is well-defined (is finite). By letting $n \to \infty$ in (5) and using (2) and (3), we obtain

$$v_\beta(x) = \log f_*(x) + \beta v_\beta((x - mf_*(x))^\alpha) = \max_{0\le c\le x}[\log c + \beta v_\beta((x-(m-1)f_*(x)-c)^\alpha)].$$

This is the well-known Bellman equation for discounted negative dynamic programming, see Strauch (1966) or Stokey *et al.* (1989). By standard iteration arguments, one can show that $v_\beta(x) = V_\beta(x) = U_\beta^i(\bar{f}_*)(x) = \sup_{\sigma^i\in\Pi^i} U_\beta^i((\bar{f}_*^{-i},\sigma^i))(x)$ for any agent i. □

Corollary 1. *From Proposition 1 and (6), one can easily obtain that*

$$V_\beta(x) = \frac{\log x}{1-\alpha\beta} + \frac{\alpha\beta\log(\alpha\beta) + (1-\alpha\beta)\log(1-\alpha\beta) - \log(m+(1-m)\alpha\beta)}{(1-\alpha\beta)(1-\beta)}.$$

Define

$$g_*(x) = px, \quad \bar{g}_* = (g_*, g_*, ..., g_*), \quad p := \frac{1-\alpha}{m+(1-m)\alpha}. \tag{7}$$

Proposition 3. *For each $\epsilon > 0$, there exists $\beta_0 \in (0,1)$ such that for each $\beta \in (\beta_0, 1)$, $x \in X$, and every agent i,*

$$U^i_\beta(\bar{g}_*)(x) + \epsilon \geq \sup_{\sigma^i \in \Pi^i} U^i_\beta((\bar{g}^{-i}_*, \sigma^i))(x). \tag{8}$$

Proof. Levhari and Mirman (1980) observed that if the agents employ linear strategies, then the discounted utility is of the form $A \log x + B$ and A, B can be found using some functional equation [2]. Applying the approach of Levhari and Mirman (1980), one can show that

$$U^i_\beta(\bar{g}_*)(x) = \frac{\log x}{1-\alpha\beta} + \frac{\alpha\beta\log\alpha + (1-\alpha\beta)\log(1-\alpha) - \log(m+(1-m)\alpha)}{(1-\alpha\beta)(1-\beta)} \tag{9}$$

and

$$\sup_{\sigma^i \in \Pi^i} U^i_\beta((\bar{g}^{-i}_*, \sigma^i))(x) = \frac{\log x}{1-\alpha\beta} + \frac{\alpha\beta\log(\alpha\beta) + (1-\alpha\beta)\log(1-\alpha\beta) - \log(m+(1-m)\alpha)}{(1-\alpha\beta)(1-\beta)}.$$

Hence, for any $x \in (0,1]$, it follows that

$$d(\beta) := U^i_\beta(\bar{g}_*)(x) - \sup_{\sigma^i \in \Pi^i} U^i_\beta((\bar{g}^{-i}_*, \sigma^i))(x) = \frac{(1-\alpha\beta)[\log(1-\alpha) - \log(1-\alpha\beta)] - \alpha\beta\log\beta}{(1-\alpha\beta)(1-\beta)}.$$

Using de L'Hospital's rule, one obtains

$$\lim_{\beta \to 1^-} d(\beta) = \frac{\alpha}{1-\alpha} \lim_{\beta \to 1^-} [\log(1-\alpha) - \log(1-\alpha\beta) + \log\beta] = 0.$$

This easily implies (8). □

Proposition 4. *$\bar{g}_*$ is a stationary equilibrium in the game with the limiting average utilities.*

Proof. Let

$$v_* = \frac{\alpha\log\alpha + (1-\alpha)\log(1-\alpha) - \log(m+(1-m)\alpha)}{1-\alpha}.$$

[2] One can use approximations by finite horizon discounted utilities as we did in Proposition 2, but then the derivations (which are more rigorous) are longer.

From (9), it follows that

$$v_* = \lim_{\beta \to 1^-} (1-\beta) U^i_\beta(\bar{g}_*)(x) \tag{10}$$

for any agent i and initial stock $x \in (0,1]$. In other words, the discounted equilibrium utilities associated with $\bar{g}_*$ are Abel summable. It is known that the Abel summability of a series does not imply (in general) the Cesaro summability, see Appendix H in Filar and Vrieze (1997). Therefore, we shall determine the finite stage undiscounted utilities, say W_k, (same for every agent) associated with the multistrategy $\bar{g}_*$. Let p be as in (7). Then by induction one can show that

$$W_1(x) = \log(px) \quad \text{and} \quad W_{n+1}(x) = (1+\alpha+\cdots+\alpha^n)\log x + \log\left(p^{n+1}(1-mp)^{S_n}\right),$$

where

$$\begin{aligned} S_n &= n\alpha + (n-1)\alpha^2 + \cdots + 2\alpha^{n-1} + \alpha^n \\ &= \frac{\alpha}{1-\alpha}\left(n - \alpha - \alpha^2 - \cdots - \alpha^n\right) = \frac{\alpha}{1-\alpha}\left(n - \frac{\alpha(1-\alpha^n)}{1-\alpha}\right). \end{aligned}$$

Hence

$$\lim_{n\to\infty} \frac{S_n}{n+1} = \frac{\alpha}{1-\alpha},$$

and consequently

$$U^i(\bar{g}_*)(x) = \lim_{n\to\infty} \frac{W_{n+1}}{n+1} = v_* \tag{11}$$

for any agent i and $x \in (0,1]$. Now choose any $\sigma^i \in \Pi^i$. From (11), (10), Proposition 3, and the Hardy-Littlewood theorem, stated as Theorem H2 in Filar and Vrieze (1997), it follows that

$$\begin{aligned} v_* &= U^i(\bar{g}_*)(x) = \lim_{\beta \to 1^-} (1-\beta) U^i_\beta(\bar{g}_*)(x) \\ &\geq \liminf_{\beta \to 1^-} (1-\beta) U^i_\beta((\bar{g}_*^{-i}, \sigma^i))(x) \geq U^i((\bar{g}_*^{-i}, \sigma^i))(x). \end{aligned} \tag{12}$$

Clearly, (12) says that $\bar{g}_*$ is a stationary equilibrium in the limiting average case. Moreover, the equilibrium utilities are independent of the initial stock $x \in (0,1]$. □

4. A look at the steady states and concluding remarks

If every agent uses a linear stationary strategy $\varphi(x) := cx$ $(0 < c < 1)$ in an infinite horizon game, then by (1), $x_{t+1} = x_t^\alpha (1-mc)^\alpha$ and therefore

$$\bar{x}(\bar{\varphi}) := \lim_{t\to\infty} x_t = (1-mc)^{\alpha/(1-\alpha)}.$$

Define

$$\bar{x}_\beta := \bar{x}(\bar{f}_*) = \left(\frac{\alpha\beta}{m+(1-m)\alpha\beta}\right)^{\alpha/(1-\alpha)} \quad \text{and} \quad \bar{x}_{la} := \bar{x}(\bar{g}_*) = \left(\frac{\alpha}{m+(1-m)\alpha}\right)^{\alpha/(1-\alpha)}.$$

Then, $\bar{x}_\beta$ ($\bar{x}_{la}$) is the steady state in the β-discounted (limiting average) utility game corresponding with the stationary equilibrium $\bar{f}_*$ ($\bar{g}_*$). Further, we put

$$c_\beta := \frac{f_*(x)}{x} = \frac{1-\alpha\beta}{m+(1-m)\alpha\beta} \quad \text{and} \quad c_{la} := \frac{g_*(x)}{x} = \frac{1-\alpha}{m+(1-m)\alpha}.$$

Proposition 5. *For any $\beta \in (0,1)$, the following inequalities are true:*

$$\bar{x}_{la} > \bar{x}_\beta, \quad c_{la} < c_\beta, \quad \text{and} \quad g_*(\bar{x}_{la}) = c_{la}\bar{x}_{la} > f_*(\bar{x}_\beta) = c_\beta \bar{x}_\beta.$$

Proof. The first two inequalities are quite obvious. We only prove the last (most interesting) one. Let $\xi(\beta) := \log(c_\beta \bar{x}_\beta)$. It is sufficient to show that this function is increasing in $(0,1)$. We have

$$\xi(\beta) = \log(1-\alpha\beta) + \frac{\alpha}{1-\alpha}\log(\alpha\beta) - \frac{1}{1-\alpha}\log(m+(1-m)\alpha\beta).$$

Hence

$$\xi'(\beta) = \frac{\alpha(1-2\alpha\beta+\beta\alpha^2)}{(1-\alpha)\beta(1-\alpha\beta)} + \frac{(m-1)\alpha}{(1-\alpha)(m+(1-m)\alpha\beta)}.$$

Since $1-2\alpha\beta+\beta\alpha^2 > 0$ and $m \geq 2$, we have $\xi'(\beta) > 0$. This completes the proof. □

Remark 1. The *Golden Rule* steady state $\bar{x}_{gr}$ used in economic growth theory (see for example Phelps (1961)), in the case of the Cobb-Douglas production function $f(x) := x^\alpha$, maximizes the consumption per period of the form $x^\alpha - x$, $x \in X$. Therefore, $\bar{x}_{gr} = \alpha^{\frac{1}{1-\alpha}}$. We would like to point out that $\bar{x}_{gr} < \bar{x}_{la}$ if the recreation is good enough, i.e., α is small enough. If $m = 2$, we have $\bar{x}_{gr} < \bar{x}_{la}$ for $0 < \alpha < 0.602269$.

Remark 2. (a) Proposition 1 gives a precise description of the equilibrium utility functions for all finite horizon games. It is basic to prove the convergence in the proof of Proposition 2. The extrapolation method applied in Levhari and Mirman (1980) and other papers, see for example page 277 in Fischer and Mirman (1992) or page 239 in Datta and Mirman (1999), is now better understood.

(b) Proposition 3 is related to turnpike theorems in economic theory and implies Proposition 4. The converse implication fails to hold in many mathematically more general models of dynamic games. Propositions 3 and 4 are new.

(c) The inequalities $\bar{x}_{la} > \bar{x}_\beta$ and $c_{la} < c_\beta$ are not surprising and $\bar{x}_{la} > \bar{x}_\beta$ says that overconsumption, typical for discounted games, is to a certain extent reduced. Much more interesting is the fact that the steady state consumption $c_{la}\bar{x}_{la}$ in the limiting average case is bigger than $c_\beta \bar{x}_\beta$, for any β. It seems that the limiting average utility (although doubtful in many economic considerations) is very natural to studying extraction models. It reflects the interest of the present and next generations.

Remark 3. The log Cobb-Douglas framework used in this note plays an important role in the proofs of Propositions 2 and 3. The only related result showing the convergence of

Nash equilibrium utilities in finite horizon symmetric games (where the transition function is of different type) with growing horizon is given in Balbus and Nowak (2004). An important consequence of the logarithmic framework accepted in this note is the fact that the equilibrium strategies of the agents are *linear functions* of the state variable. This property enables us to obtain the equilibrium functions in the finite horizon games in a quite simple form. In a more abstract setting, related results to Propositions 2-4 seem to be difficult to prove. We would like to point out that more general resource extraction dynamic games with identical utility functions were already studied by Sundaram (1989). He proved that a symmetric Nash equilibrium exists in a class of lower semicontinuous stationary strategies for the agents. His proof is based on a fixed point arguments. We hope that an approximation by finite step games can also be used in the framework of Sundaram (1989), at least under some additional regularity conditions. One can observe that in our example the derivative $V_n'(x)$ of the equilibrium function in the n-step game is increasing in n for every $x \in (0, 1]$. We believe that this property may play an essential role in extending our results to more general dynamic games.

Remark 4. Many economic issues concerning ecology or sustainable growth can be studied using a multigenerational framework or time-inconsistent utilities, see Phelps and Pollak (1968) and Strotz (1956). More recent works are quoted in Nowak (2006a), where a stochastic version of Phelps and Pollak (1968) model is treated using a fixed point technique in an infinite dimensional space. The approach of Phelps and Pollak (1968) has found a lot of different applications during the last decade, see for example Nowak (2006b) and the references cited therein. Alj and Haurie (1983) extended the idea of Phelps and Pollak (1968) by assuming that each generation consists of finitely many players. Such an idea is applied to the multigenerational resource extraction game with the Cobb-Douglas production in Nowak (2006b), under the so-called quasi-geometric (or hyperbolic) discounting.

Acknowledgment. The author wishes to thank the associate editor and an anonymous referee for useful and helpful comments.

References

Alj, A., Haurie, A. (1983). Dynamic equilibria in multigenerational stochastic games. *IEEE Transactions on Automatic Control* **28**, 193-203.

Balbus, Ł., Nowak, A.S. (2004). Construction of Nash equilibria in symmetric stochastic games of capital accumulation. *Mathematical Methods of Operations Research* **60**, 267-277.

Datta, M., Mirman, L.J. (1999). Externalities, market power, and resource extraction. *Journal of Environmental Economics and Management* **37**, 233-255.

Dutta, P.K., Sundaram, R.K. (1993). The tragedy of the commons? *Economic Theory* **3**, 413-426.

Filar, J., Vrieze, K. (1997). *Competitive Markov Decision Processes*. Springer-Verlag, New York.

Fischer, R.D., Mirman, L.J. (1992). Strategic dynamic interaction: Fish wars. *Journal of Economic Dynamics and Control* **16**, 267-287.

Levhari, D., Mirman, L.D. (1980). The great fish war: an example using a dynamic Cournot-Nash solution. *Bell Journal of Economics* **11**, 322-334.

Nowak, A.S. (2006a). On perfect equilibria in stochastic models of growth with intergenerational altruism. *Economic Theory* **28**, 73-83.

Nowak, A.S. (2006b). A multigenerational game of resource extraction. *Mathematical Social Sciences* **51**, Issue 3.

Phelps, E.S. (1961). The golden rule of accumulation: a fable for growthmen. *American Economic Review* **51**, 638-643.

Phelps, E.S., Pollak, R.A. (1968). On second-best national savings and game equilibrium growth. *Review of Economic Studies* **35**, 195-199.

Pigou, A.C. (1920). *The Economics of Welfare*. Macmillan, London.

Rawls, J. (1971). *A Theory of Justice*. Harvard University Press, Cambridge, MA.

Stokey, N.L., Lucas, R.E., Prescott, E. (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge, MA.

Strauch, R. (1966). Negative dynamic programming. *Annals of Mathematical Statistics* **37**, 871-890.

Strotz, R.H. (1956). Myopia and inconsistency in dynamic utility maximization. *Review of Economic Studies* **23**, 165-180.

Sundaram, R.K. (1989). Perfect equilibrium in non-randomized strategies in a class of symmetric dynamic games. *Journal of Economic Theory* **47**, 153-177.