E C O N O M I C S   B U L L E T I N

Conjectural variations and general oligopoly equilibrium in pure exchange economies

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Abstract

In this note, we introduce conjectural variations in a simple general oligopoly equilibrium model of a pure exchange economy. Three results are obtained. First, the price and utility levels generally increase with the value taken by conjectures. Second, when the conjectural variation takes its minimum value the competitive equilibrium is reached, independently of the number of agents acting in each sector. Third, the Cournot-Walras equilibrium may be viewed as a case where the conjectural variations in one sector take a zero value.

This note is a piece of a more general research devoted to some micro-foundations of macroeconomic models of coordination failures.
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1. Introduction

In imperfectly competitive economies each agent when making a decision generally does note the effect of his/her action on the market (Bowley (1924)). The conjectural approach takes into account the perceptions by individuals of their market environment and intends to study price formation without an auctioneer by attempting a general equilibrium analysis of imperfect competition (Gale (1978), Hahn (1977)). The role played by (consistent) conjectures has been mainly developed in the context of production economies (Bresnahan (1981), Figuières and *alii* (2004) or Perry (1982)).

In this note, we propose to study the role of conjectural variations in pure exchange economies. We focus on strategic multilateral exchange in general oligopoly equilibrium. We thus refer to the framework initially developed by Gabszewicz and Vial (1972) in exchange economies with production and later pursued by Codognato-Gabszewicz (1991), (1993) in pure exchange economies. These authors study the relationship between market power and the number of agents. Different concepts of oligopoly equilibria can be developed depending on the way strategic behavior is introduced (Gabszewicz and Michel (1997)). In these general equilibrium Nashian perspectives, the oligopoly equilibria can be identified to the competitive equilibrium for large economies (under a replication procedure or an asymptotic identification).

We show, through a simple example of a pure exchange economy that the introduction of conjectural variations can lead to the same results, independently of a replicating procedure or an asymptotic identification. Moreover, we study the effects of conjectures on prices and indirect utility at the symmetric oligopoly equilibrium. We show that the price and the utility levels generally increase with the value of the conjectural variations for agents who form it, but decrease with the conjecture formed by others.

The paper is organized as follows. The basic economy is described in section 2. In section 3, we study the relation between the values taken by conjectures and the type of equilibrium, and then present the several results obtained.

2. The basic economy

Consider a pure exchange economy with two consumption goods (1 and 2) and $m+n$ consumers. Preferences are represented by the following utility function:

$$U_i = x_{i1}^{\alpha} x_{i2}^{1-\alpha} \quad , 0 < \alpha < 1 \quad , \forall i \, . \tag{1}$$

The structure of the initial endowments is assumed to be the same as in the case of the homogeneous oligopoly developed by Gabszewicz-Michel (1997):

$$\omega_i = \left(\frac{1}{m}, 0 \right) \quad , i = 1,2,\dots,m$$

$$\omega_i = \left(0, \frac{1}{n} \right) \quad , i = m+1,\dots,m+n \, . \tag{2}$$

It is assumed that good 2 is taken as the *numéraire*, so p is the price of good 1 as expressed in units of good 2.

We consider that each agent is an oligopolist and behaves strategically. Each agent *i* manipulates the price by contracting his/her supply, i.e. the quantity of good 1 or 2 he/she offers. We denote s_{i1} the pure strategy of agents $i = 1,2,...,m$, with $s_{i1} \in [0,1/m]$, and s_{i2} the pure strategy of agents $i = m+1,...,m+n$, with $s_{i1} \in [0,1/n]$.

Finally, let us assume the agents who have an endowment in good 1 form a conjectural variation about the combined strategic supplies of good 1 response of other consumers to a unit of change in their own output level:

$$\frac{\partial \sum_{-i \neq i} s_{-i1}}{\partial s_{i1}} = \gamma \quad \text{, where } \gamma \in [-1, m-1] \qquad (3)$$

We thus assume γ to be the same for all consumers and independent of both the supply of the other agents and the number of agents[1]. Equivalently, we define the conjectural variation for consumers who initially own quantities of good 2:

$$\frac{\partial \sum_{-i \neq i} s_{-i2}}{\partial s_{i2}} = \varepsilon \quad \text{, where } \varepsilon \in [-1, n-1] \qquad (4)$$

These conjectures indicate the way that each consumer of one sector think the other consumers supply choice will vary according the first agents vary their own supply choice. Certain values taken by conjectures in (3) and (4) are of particular interest in the context of production economies (Perry (1982)). When $\gamma = \varepsilon = -1$, each agent acts as a price taker and the equilibrium is competitive: each agent expects the other agents in their sector to absorb exactly its supply expansion by a corresponding supply reduction, here respectively $\partial s_{-i1} / \partial s_{i1} = -1/(m-1)$ and $\partial s_{-i2} / \partial s_{i2} = -1/(n-1)$. When $\gamma = \varepsilon = 0$, each agent ignores the consequence of his/her action on the others' actions: this is the Cournot equilibrium. Finally, when $\gamma = m-1$ and $\varepsilon = n-1$, the equilibrium is collusive: agents behave in each sector so as to maximize joint payment. We consider that the conjectures are consistent[2]. We verify through a simple example that these results obtained in economies with production hold in pure exchange general oligopoly equilibrium.

3. Cournot equilibria and conjectural variations

We define a *symmetric oligopoly equilibrium* as a $(m+n)$-tuple of strategies $(\tilde{s}_{11},...,\tilde{s}_{m1},\tilde{s}_{m+12},...,\tilde{s}_{m+n2})$, with $\tilde{s}_{i1} \in [0,1/m]$ for $h = 1,2,...,m$ and $\tilde{s}_{i2} \in [0,1/n]$ for $h = m+1,...,m+n$, and an allocation $(\tilde{x}_1,...,\tilde{x}_m,\tilde{x}_{m+1},...,\tilde{x}_{m+n}) \in IR_+^{2(m+n)}$ such that (i) $\tilde{x}_i = x_i(\tilde{s}_{i1},\tilde{s}_{-i1})$ and $U_i(x_i(\tilde{s}_{i1},\tilde{s}_{-i1})) \geq U_i(x_i(s_{i1},\tilde{s}_{-i1}))$ for $i = 1,2,...,m$ and (ii) $\tilde{x}_i = x_i(\tilde{s}_{i2},\tilde{s}_{-i2})$ and $U_i(x_i(\tilde{s}_{i2},\tilde{s}_{-i2})) \geq U_i(x_i(s_{i2},\tilde{s}_{-i2}))$ for $i = m+1,...,m+n$. We assume that there exists symmetric oligopoly equilibrium and that it is unique.

[1] An advantage of a sector conjectural variation over individual conjectural variations is that there is no inconsistency in defining the equilibrium of the rest of the sector in terms of both s_{i1} and γ.

[2] See Bresnahan (1981). In our context of a pure exchange economy, this means that conjectures and reactions are the same. Each consumer's conjectures about other consumer's reactions are perfectly correct (not only about the levels of one another consumer action but also about the functions according to which they are reacting).

The market clearing condition implies that the price must be $p=\frac{\sum_{i=m+1}^{i=m+n} s_{i2}}{\sum_{i=1}^{i=m} s_{i1}}=\frac{s_2}{s_1}$. The non-cooperative equilibrium is associated with the resolution of the simultaneous strategic programs:

$$\underset{\{\tilde{s}_{i1}\}}{Arg\max}\left(\frac{1}{m}-s_{i1}\right)^{\alpha}\left(\frac{s_2}{s_1}s_{i1}\right)^{1-\alpha} \quad ,i=1,2,...,m \tag{5}$$

$$\underset{\{\tilde{s}_{i2}\}}{Arg\max}\left(\frac{s_1}{s_2}s_{i2}\right)^{\alpha}\left(\frac{1}{n}-s_{i2}\right)^{1-\alpha} \quad ,i=m+1,...,m+n\,. \tag{6}$$

The *(m+n)* conditions of optimality $\partial U_i/\partial s_{i1}=0$ for $i=1,2,...,m$, and $\partial U_i/\partial s_{i2}=0$ for $i=m+1,...,m+n$, yield to the following optimal strategies:

$$\tilde{s}_{i1}=\frac{(1-\alpha)[m-(1+\gamma)]}{m[m-(1-\alpha)(1+\gamma)]} \quad ,i=1,2,...,m \tag{7}$$

$$\tilde{s}_{i2}=\frac{\alpha[n-(1+\varepsilon)]}{n(n-\alpha(1+\varepsilon)]} \quad ,i=m+1,...,m+n\,. \tag{8}$$

We deduce the price $\tilde{p}=\frac{\sum_{i=m+1}^{i=m+n}\tilde{s}_{i2}}{\sum_{i=1}^{i=m}\tilde{s}_{i1}}$, which is:

$$\tilde{p}=\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{n-(1+\varepsilon)}{n-\alpha(1+\varepsilon)}\right)\left(\frac{m-(1-\alpha)(1+\gamma)}{m-(1+\gamma)}\right). \tag{9}$$

The individual allocations are:

$$(\tilde{x}_{i1},\tilde{x}_{i2})=\left(\frac{\alpha}{m-(1-\alpha)(1+\gamma)},\frac{\alpha[n-(1+\varepsilon)]}{m[n-\alpha(1+\varepsilon)]}\right) \quad ,i=1,2,...,m \tag{10}$$

$$(\tilde{x}_{i1},\tilde{x}_{i2})=\left(\frac{(1-\alpha)(m-(1+\gamma)]}{n[m-(1-\alpha)(1+\gamma)]},\frac{1-\alpha}{n-\alpha(1+\varepsilon)}\right) \quad ,i=m+1,...,m+n \tag{11}$$

The utility levels reached by each type of consumers are respectively:

$$\tilde{U}_i=\frac{\alpha}{m}\left[\frac{m}{m-(1-\alpha)(1+\gamma)}\right]^{\alpha}\left(\frac{n-(1+\varepsilon)}{n-\alpha(1+\varepsilon)}\right)^{1-\alpha} \quad ,\ i=1,2,...,m \tag{12}$$

$$\tilde{U}_i=\left(\frac{1-\alpha}{n}\right)\left[\frac{m-(1+\gamma)}{m-(1-\alpha)(1+\gamma)}\right]^{\alpha}\left(\frac{n}{n-\alpha(1+\varepsilon)}\right)^{1-\alpha} \quad ,i=m+1,...,m+n\,. \tag{13}$$

Result 1. *When* $\gamma\in[-1,m-1)$ *, we have* $\partial\tilde{p}/\partial\gamma>0$ *,* $\partial\tilde{U}_h/\partial\gamma>0$ *and* $\partial\tilde{p}/\partial\varepsilon<0$ *,* $\partial\tilde{U}_h/\partial\varepsilon<0$ *for* $i=1,2,...,m$ *. The symmetric conclusions hold for all agents* $i=m+1,...,m+n$ *when* $\varepsilon\in[-1,n-1)$ *.*

Proof. From (9), we have $\frac{\partial\tilde{p}}{\partial\gamma}=\frac{\alpha^2 m[m-(1+\varepsilon)]}{(1-\alpha)[n-\alpha(1+\varepsilon)][m-(1+\gamma)]^2}>0$ and $\frac{\partial\tilde{p}}{\partial\varepsilon}=\frac{-\alpha m[m-(1-\alpha)(1+\varepsilon)]}{[m-(1+\gamma)][m-\alpha(1+\varepsilon)]^2}<0$. Moreover, from (12) we have

$$\frac{\partial \widetilde{U}_i}{\partial \gamma} = \frac{\alpha(1-\alpha)}{m-(1-\alpha)(1+\gamma)} \widetilde{U}_i > 0 \quad \text{and} \quad \frac{\partial \widetilde{U}_i}{\partial \varepsilon} = \frac{mn(1-\alpha)^2}{\alpha[n-(1+\varepsilon)][n-\alpha(1+\varepsilon)]} \widetilde{U}_i < 0 \quad .$$

When $\varepsilon = n-1$ we verify that $\frac{\partial \widetilde{U}_i}{\partial \gamma} = 0$. When $\gamma = m-1$ and $\varepsilon = n-1$ we have $\frac{\partial \widetilde{U}_i}{\partial \varepsilon} = 0$. This completes the proof.

We remark that the variation of the oligopoly equilibrium price is indeterminate in the presence of collusion in both sectors, whereas such a variation is virtually infinite when there is collusion in sector 1 whatever happens in sector 2.

Result 2. *When $\gamma = \varepsilon = -1$, the symmetric oligopoly equilibrium identifies to the Walrasian equilibrium, independently on the number of agents.*

Proof. We first compute the symmetric oligopoly equilibrium for $\gamma = \varepsilon = -1$. Second, we determine the competitive allocation and compare it with the results found in the first step[3].

When $\gamma = \varepsilon = -1$, the equilibrium price, the supply of each good and the associated allocation for each type of agent become respectively: $\widetilde{p} = \alpha/(1-\alpha)$, $\widetilde{s}_{i1} = (1-\alpha)/m$ and $(\widetilde{x}_{i1}, \widetilde{x}_{i2}) = (\alpha/m, \alpha/m)$ for $i = 1,2,...,m$, and $\widetilde{s}_{i2} = \alpha/n$ and $(\widetilde{x}_{i1}, \widetilde{x}_{i2}) = ((1-\alpha)/n, (1-\alpha)/n)$ for $i = m+1,...,m+n$. The utility levels reached are then $\widetilde{U}_i = \alpha/m$, $i = 1,2,...,m$ and $\widetilde{U}_i = (1-\alpha)/n$, $i = m+1,...,m+n$.

When the behavior of each agent is competitive, the individual plans come from a non-strategic maximization of the utility subject to the budget constraint. We determine the competitive equilibrium price, the competitive supply of each good and the associated allocation for each type of agent, which are respectively: $p^* = \alpha/(1-\alpha)$, $s^*_{i1} = (1-\alpha)/m$ and $(x^*_{i1}, x^*_{i2}) = (\alpha/m, \alpha/m)$ for $i = 1,2,...,m$ and $s^*_{i2} = \alpha/n$ and $(x^*_{i1}, x^*_{i2}) = ((1-\alpha)/n, (1-\alpha)/n)$ for $i = m+1,...,m+n$. The utility levels reached are $U^*_i = \alpha/m$ for $i = 1,2,...,m$ and $U^*_i = (1-\alpha)/n$ for $i = m+1,...,m+n$. Thus $\widetilde{p}_{|\gamma=\varepsilon=-1} = p^*$, $\widetilde{s}_{i1|\gamma=\varepsilon=-1} = s^*_{i1}$ and $(\widetilde{x}_{i1}, \widetilde{x}_{i2})_{|\gamma=\varepsilon=-1} = (x^*_{i1}, x^*_{i2})$ for $i = 1,2,...,m$, and $\widetilde{s}_{i2|\gamma=\varepsilon=-1} = s^*_{i2}$ and $(\widetilde{x}_{i1}, \widetilde{x}_{i2})_{|\gamma=\varepsilon=-1} = (x^*_{i1}, x^*_{i2})$ for $i = m+1,...,m+n$. Moreover $\widetilde{U}_{i|\gamma=\varepsilon=-1} = U^*_i$ for $i = 1,2,...,m$ and $\widetilde{U}_{i|\gamma=\varepsilon=-1} = U^*_i$ for $i = m+1,...,m+n$. This completes the proof.

Result 3. *When $\gamma = 0$ (resp. $\varepsilon = 0$) and $\varepsilon = -1$ (resp. $\gamma = -1$), the symmetric oligopoly equilibrium identifies to the asymmetric Cournot-Walras equilibrium[4].*

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[3] The symmetric oligopoly equilibrium tends to the competitive one when *m* and *n* become large.

[4] A *Cournot-Walras equilibrium* is given by a m-uple of strategies $(\hat{s}_{11}, \hat{s}_{21}, ..., \hat{s}_{m1})$, with $\hat{s}_{i1} \in [0, 1/m]$, $i = 1,2,...,m$, and an allocation $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_m) \in IR^{2m}_+$ such that (i) $\hat{x}_i = x_i(\hat{s}_{i1}, \hat{s}_{-i1})$ and (ii) $U_i(x_i(\hat{s}_{i1}, \hat{s}_{-i1})) \geq U_i(x_i(s_{i1}, \hat{s}_{-i1}))$, $i = 1,2,...,m$, where s_{-i1} denotes the strategy of every agent who owns a quantity of good 1 and who is different from agent *i*.

In the first case, for which $\gamma = 0$ and $\varepsilon = -1$, the agents who have an endowment in good 1 adopt a strategic behaviour in manipulating the price by means of the quantity of good 1 they offer, whereas agents who have an endowment in good 2 behave competitively. In the second case, for which $\varepsilon = 0$ and $\gamma = -1$, the situation is reversed: the strategists are the last *n* agents, the first *m* agents behaving as price-takers.

Proof. It suffices to establish the proof for the first case. We have to determine the equilibrium price, the individual allocations and the indirect utilities which are associated to $\gamma = 0$ and $\varepsilon = -1$. Hence, we determine the Cournot-Walras equilibrium for this economy and compare it with the computations previously obtained.

When $\gamma = 0$, we easily find $\tilde{p} = p^*[m-(1-\alpha)]/(m-1)$, $\tilde{s}_{h1} = (1-\alpha)(m-1)/m[m-(1-\alpha)]$, $i = 1,2,...,m$ and $\tilde{s}_{i2} = \alpha / n$, $i = m+1,...,m+n$. Moreover, $(\tilde{x}_{i1}, \tilde{x}_{i2}) = (\alpha /[m-(1-\alpha)], \alpha / m)$ for $i = 1,2,...,m$ and $(\tilde{x}_{i1}, \hat{x}_{i2}) = ((1-\alpha)(m-1)/n[m-(1-\alpha)], (1-\alpha)/n)$ for $i = m+1,...,m+n$. Finally, $\widehat{U}_i = U_i^* \left(\dfrac{m}{m-(1-\alpha)} \right)^\alpha$, $i = 1,2,...,m$ and $\widehat{U}_i = U_i^* \left\{ \dfrac{m-1}{[m-(1-\alpha)]} \right\}^\alpha$, $i = m+1,...,m+n$.

We now determine the *Cournot-Walras equilibrium*. In this case, the equilibrium price verifies $\sum_{i=1}^{i=m} s_{i1} = \sum_{i=m+1}^{i=m+n} \dfrac{\alpha}{np(s_{11}, s_{21},..., s_{m1})}$, so $p = \dfrac{\alpha}{\sum_{i=1}^{i=m} s_{i1}}$. The non-cooperative equilibrium is associated to the resolution of the simultaneous strategic programs $\underset{\{s_{i1}\}}{Arg \max} \left(\dfrac{1}{m} - s_{i1} \right)^\alpha \left\{ \dfrac{\alpha s_{i1}}{[s_{i1} + (m-1)s_{-i1}]} \right\}^{1-\alpha}$, $i = 1,2,...,m$. All the agents of the same sector being identical, we have $\hat{s}_{i1} = \hat{s}_{-i1}$, so the *m* equilibrium strategies are $\widehat{s}_{i1} = \dfrac{(1-\alpha)(m-1)}{m[m-(1-\alpha)]}$, $i = 1,2,...,m$. We thus have $\widehat{p} = p^* \dfrac{[m-(1-\alpha)]}{(m-1)}$. The individual allocations and the associated utility levels for each type of agents can be written respectively $(\widehat{x}_{i1}, \widehat{x}_{i2}) = \left(\dfrac{\alpha}{m-(1-\alpha)}, \dfrac{\alpha}{m} \right)$ and $\widehat{U}_i = U_i^* \left(\dfrac{m}{m-(1-\alpha)} \right)^\alpha$ for $i = 1,2,...,m$ and $(\widehat{x}_{i1}, \widehat{x}_{i2}) = \left(\dfrac{(1-\alpha)(m-1)}{n[m-(1-\alpha)]}, \dfrac{1-\alpha}{n} \right)$ and $\widehat{U}_i = U_i^* \left\{ \dfrac{m-1}{[m-(1-\alpha)]} \right\}^\alpha$ for $i = m+1,...,m+n$.

Thus $\tilde{p}_{|(\gamma=0,\varepsilon=-1)} = \hat{p}$, $\tilde{s}_{i1|(\gamma=0;\varepsilon=-1)} = \hat{s}_{i1}$ and $(\tilde{x}_{i1}, \tilde{x}_{i2})_{|(\gamma=0,\varepsilon=-1)} = (\hat{x}_{i1}, \hat{x}_{i2})$ for $i = 1,2,...,m$, and $\tilde{s}_{i2(|\gamma=0,\varepsilon=-1)} = \hat{s}_{i2}$ and $(\tilde{x}_{i1}, \tilde{x}_{i2})_{(|\gamma=0,\varepsilon=-1)} = (\hat{x}_{i1}, \hat{x}_{i2})$ for $i = m+1,...,m+n$. Moreover $\tilde{U}_{i(|\gamma=0,\varepsilon=-1)} = \hat{U}_i$ for $i = 1,2,...,m$ and $\tilde{U}_{i|(\gamma=0,\varepsilon=-1)} = \hat{U}_i$ for $i = m+1,...,m+n$. This completes the proof.

4. Concluding remarks

The role of conjectural variations can be transposed in general oligopoly equilibrium for pure exchange economies. The Cournot-Walras equilibria and the competitive equilibrium may be viewed respectively as a case where the conjecture takes a zero value in one sector or in both sectors.

Moreover, market power (market size of consumers) does not depend only on the fundamentals (preferences and structure of endowments) or on the number of agents, but also on the way agents form their conjecture in strategic interactions.

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