A Riemannian metric on the equilibrium manifold: the smooth case

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Abstract

In a pure exchange smooth economy with fixed total resources, we construct a Riemannian metric on the equilibrium manifold such that the minimal geodesic connecting two (sufficiently close) regular equilibria intersects the codimension one stratum of critical equilibria in a finite number of points.

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1 Introduction

In this paper we construct a Riemannian metric on the equilibrium manifold with fixed total resources, $E(r)$, when preferences are smooth. This paper represents the smooth version of Matta (2005), where preferences were assumed to be real analytic. The complications arising from the smoothness assumption make this paper different both for the result obtained and the techniques used.

The motivation for our analysis is that of choosing a continuous path connecting two points (regular equilibria) of the equilibrium manifold, which enjoys the properties of minimizing distance (Balasko 1988, p. 70) and *catastrophes* (Balasko 1978). We recall that catastrophes can originate when a path crosses the set of critical equilibria (see Dierker 1982, and Balasko 1988).

The following three features characterize our analysis. First, we will only be concerned with the codimension one stratum of critical equilibria (S_1 henceforth). This means that we want that our path intersects S_1 in a finite number of points. Observe that ignoring the strata of codimension greater than one is not relevant since S_1 is the only stratum which can disconnect $E(r)$. Second, the endpoints of the path are assumed to be "sufficiently close" regular equilibria, i.e., equilibria which belong to a *geodesic convex neighborhood* (see Proposition 2.5). Third, we content ourselves to consider the issue of the existence of a metric with desirable economic properties without investigating the issue of the economic meaning of such a metric.

We tackle the distance-catastrophes minimization problem by showing the existence of a Riemannian metric on $E(r)$ such that the minimal geodesic (path) connecting two (sufficiently close) regular equilibria intersects S_1 in a finite number of points (see Theorem 3.2).

The three features mentioned above are reflected in the nature of the result we have obtained, that can be regarded as a preliminary result: natural directions of research would be to consider all the strata of critical equilibria, to allow the endpoints of the path to be arbitrary points of $E(r)$ and to explore the connections between the Riemannian metric and the economic meaning of a metric.

This paper is organized as follows. Section 2 recalls the economic setting and mathematical background. Section 3 contains our main results.

2 Preliminaries

Balasko (1988) has shown that the equilibrium manifold E is a smooth submanifold of $S \times \Omega$ globally diffeomorphic to $(\mathbb{R}^l)^m$, where l and m represent the number of goods and consumers, respectively, $S = \{p = (p_1, \dots p_l) | p_j > 0, j = 1, \dots l, p_l = 1\}$ is the set of normalized prices and $\Omega = (\mathbb{R}^l)^m$ is the space of endowments $\omega = (\omega_1, \dots, \omega_m)$, $\omega_i \in \mathbb{R}^l$.

An explicit global parametrization of E is given through the map defined by:

$$\Phi : E \to S \times \mathbb{R}^m \times \mathbb{R}^{(l-1)(m-1)} \cong \mathbb{R}^{lm}$$

$$(p, \omega_1, \dots \omega_m) \mapsto (p, p \cdot \omega_1, \dots, p \cdot \omega_m, \bar{\omega}_1, \dots, \bar{\omega}_{m-1}), \tag{1}$$

where $\bar{\omega}_i$ denotes the vector of $\mathbb{R}^{l-1}$ defined by the first $(l-1)$ -coordinates of ω_i.

Let $r \in \mathbb{R}^l$ denote the vector representing the total resources of the economy and $\Omega(r)$ denote the space of economies associated with the fixed total resources, i.e., $\Omega(r) = \{\omega \in (\mathbb{R}^l)^m | \sum_{i=1}^m \omega_i = r\}$. Define the equilibrium manifold when total resources are fixed as

$$E(r) = \{(p, \omega) \in S \times \Omega(r) | \sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = r\},$$

where $\sum_{i=1}^m f_i(p, p \cdot \omega_i)$ is the aggregate demand function.

The restriction of the map (1) given above to $E(r)$ defines a diffeomorphism, still denoted by Φ,

$$\Phi : E(r) \to B(r) \times \mathbb{R}^{(l-1)(m-1)}, \tag{2}$$

where $B(r) = \{(p, w) \in S \times \mathbb{R}^m | \sum_{i=1}^m f_i(p, w_i) = r)\}$. For $(p, w) \in B(r)$, let $F_{(p,w)}(r)$ be the fiber over (p, w), i.e. the pairs $(p, \omega) \in E(r)$ such that $p \cdot \omega_i = w_i$.

The set of critical equilibria when total resources are fixed is denoted $E_c(r)$. The structure of $E_c(r)$ is described in the following theorem where we summarize some results due to Balasko (1979, 1988).

Theorem 2.1 (Balasko). *$E_c(r)$ is a disjoint union of closed smooth submanifolds $\mathcal{S}_i$, $i = 1, \dots, inf(l-1, m-1)$ of $E(r)$. The manifold $\mathcal{S}_i$ has dimension $l(m-1) - i^2$ and $\mathcal{S}_i = \emptyset$ for $i > inf(l-1, m-1)$.*

Remark 2.2. Observe that for each $i = 1, \dots, \inf(l-1, m-1)$, the dimension of each $\mathcal{S}_i$ is strictly less than the dimension of $E(r)$ and therefore $E_c(r) = \cup_i \mathcal{S}_i$ has measure zero in $E(r)$ (see Balasko 1992).

Remark 2.3. Each manifold $\mathcal{S}_i$ could not be connected. For later use, we write $\mathcal{S}_1$, the stratum we are concerned with, as the countable union of its connected components $\mathcal{S}_1^j$, i.e., $\mathcal{S}_1 = \cup_{j=1}^{+\infty} \mathcal{S}_1^j$. Observe also that each $\mathcal{S}_1^j$ is a closed, connected, codimension 1 smooth submanifold of $E(r)$.

For proofs and details of the following mathematical preliminaries we refer the reader to Chapters $1, 2, 3, 6$ of Do Carmo (1992), unless otherwise specified.
A *Riemannian metric* g on a smooth k-dimensional manifold $X \subset \mathbb{R}^n$ is a family of inner products g_x on T_xX, $x \in X$, varying smoothly with x. This means that for every local parametrization $\psi : U \subset \mathbb{R}^k \to X$ and $v_1, v_2 \in \mathbb{R}^k$, $\psi(u) = x$, the following map

$$u \mapsto g_x(d\psi_u(v_1), d\psi_u(v_2))$$

is smooth on U. A smooth manifold with a Riemannian metric is called a *Riemannian* manifold. We denote by g_{can} the Euclidean metric on $\mathbb{R}^n$, i.e., the metric defined by $g_{can}(v_1, v_2) = v_1 \cdot v_2$, where $v_1 \cdot v_2$ denotes the standard inner product in $\mathbb{R}^n$. If X is a smooth submanifold of a Riemannian manifold (Y, h), we can define a Riemannian metric g on X by restricting h to TX, the tangent bundle of X. Therefore every manifold $X \subset \mathbb{R}^n$ admits a Riemannian metric, obtained by restricting g_{can} to TX. Let $\Phi : X \to Y$ be a diffeomorphism. A Riemannian metric h on Y induces a Riemannian metric on X, called the *pull-back metric*, which we denote by $\Phi^*(h)$. This is defined by:

$$(\Phi^*(h))_x(v, w) = h_{\Phi(x)}(d\Phi_x(v), d\Phi_x(w)), \forall x \in X, \forall v, w \in T_xX. \qquad (3)$$

Two Riemannian manifolds (X, g) and (Y, h) are *isometric* if there exists a diffeomorphism $\Phi : X \to Y$ such that $g = \Phi^*(h)$. The map Φ is then called an *isometry* between (X, g) and (Y, h). The Cartesian product $X \times Y$ of two Riemannian manifolds (X, g) and (Y, h) can be endowed with a natural metric $g \oplus h$ called the *product metric* (see Do Carmo 1992, Example 2.7 p. 42).

A smooth curve $\gamma : I \to X$ on a Riemannian manifold (X, g) is a *geodesic* if the component of the acceleration $\gamma''(t)$ on the tangent space $T_{\gamma(t)}X$ of X is zero for all t.

A geodesic is uniquely determined by its initial conditions, namely its starting point and its tangent vector at this point as expressed by the following proposition (see Do Carmo 1992, Proposition 2.5 p. 64).

Proposition 2.4. *Given $x \in X$ and $v \in T_x X$, there exists a unique geodesic γ such that $x = \gamma(t_0)$ and $v = \gamma^{'}(t_0)$, for some $t_0 \in I$.*

A submanifold C of a Riemannian manifold (X, g) is *totally geodesic* if every geodesic of C is a geodesic of X.

An important property of the geodesic is that it minimizes the distance between two nearby points of X (see Do Carmo 1992, Section 3 p. 67). In other words, given a point x in X there exists a sufficiently small neighborhood U of x such that for every point y in this neighborhood there exists a geodesic γ on X joining x to y and such that its length equals $d(x, y)$. We call such a geodesic a *minimal geodesic* between the points x and y. Observe that this is not always true if the two points are not sufficiently close. Observe also that the minimal geodesic joining two points on U is in general not unique and, even worse, it does not belong to U. Nevertheless one can prove the following fundamental result (Do Carmo 1992, Section 4 p. 74).

Proposition 2.5. *Let (X, g) be a Riemannian manifold. Every point of X has an open neighborhood U such that for any pair of points x and y on U there exists a unique minimal geodesic joining them and lying entirely on U. The set U is called a geodesic convex neighborhood.*

We often use the following characterization of totally geodesic submanifolds whose proof follows by Proposition 2.4.

Proposition 2.6. *A submanifold C of a Riemannian manifold (X, g) is* totally geodesic *if and only if for every $x \in C$ and $v \in T_x C$ the geodesic $\gamma : [0, 1] \to X$, such that $\gamma(0) = x$ and $\gamma'(0) = v$, is a geodesic of C.*

Example 2.7. Let (X, g) and (Y, h) be two Riemannian manifolds. Then (X, g) and (Y, h) are totally geodesic submanifolds of $(X \times Y, g \oplus h)$ (see Do Carmo 1992, Exercise 1 p. 139).

Our interest for totally geodesic submanifolds is due to the following lemma.

Lemma 2.8. *Let (C, h) be a closed and totally geodesic submanifold of a Riemannian manifold (X, g). Then any geodesic of X joining two points x and y in the complement of C in X intersects C in a finite number of points.*

Proof: Let $\gamma : [0,1] \to X$ be any geodesic joining x and y in $X \setminus C$, i.e. $\gamma(0) = x$ and $\gamma(1) = y$. We need to prove that $\mathrm{Im}\,\gamma = \{\gamma([0,1])\}$ intersects C transversally and hence in a finite number of points. Assume, by contradiction, that there exists a point $x_0 = \gamma(t_0) \in \mathrm{Im}\,\gamma \cap C$, $t_0 \in (0,1)$ such that $\dim(T_{x_0}(\mathrm{Im}\,\gamma) \cap T_{x_0}C) = 1$. This implies that the vector $v = \gamma'(t_0)$ belongs to $T_{x_0}C$. Since, by assumption, (C,h) is totally geodesic in (X,g), it follows that $\mathrm{Im}\,\gamma$ is forced to be entirely contained in C which gives the desired contradiction since the points x and y do not belong to C by hypothesis. Therefore the set $C \cap \gamma([0,1])$ consists of isolated points which are forced to be finite for the compactness of $C \cap \gamma([0,1])$. $\square$

We conclude this section with the following result of general topology (see e.g. Guillemin and Pollack 1974, Exercises 16 and 18 on p. 76 and 106, respectively).

Lemma 2.9. *Let $X \subset \mathbb{R}^n$ be a closed, connected codimension 1 smooth submanifold of $\mathbb{R}^n$. Then there exists a closed neighborhood $\mathcal{T}$ of X in $\mathbb{R}^n$ diffeomorphic to $X \times [-\delta, \delta]$, for some $\delta > 0$.*

3 Main results

Let $\Phi : E(r) \to B(r) \times \mathbb{R}^{(l-1)(m-1)} \cong \mathbb{R}^{l(m-1)}$ be the diffeomorphism given by formula (2) above. Define

$$g_\Phi = \Phi^*(g_{can}), \tag{4}$$

where g_{can} denotes the Euclidean metric on $\mathbb{R}^{l(m-1)}$ (see equation 3 for the definition of $\Phi^*(g_{can})$).

The economic and mathematical properties of g_Φ are summarized in the following theorem, which also gives a partial solution to the problem raised in the Introduction (see property 4) below).

Theorem 3.1. *The metric g_Φ satisfies the following properties:*

1) given two points in $E(r)$, there exists a unique minimal geodesic joining them;

2) the fiber $F_{(p,w)}(r), (p,w) \in B(r)$, is totally geodesic in $(E(r), g_\Phi)$ and therefore the geodesic joining two points on the same fiber lies on this fiber;

3) the geodesic joining two points not belonging to the same fiber intersects each fiber in a finite number of points;

4) let x and y be two regular equilibria and let γ be the (minimal) geodesic joining them. Then, there exists a smooth curve σ joining x and y, arbitrarily close to γ, which intersects S_1 in a finite number of points.

Proof: Property 1) is a straightforward consequence of the fact that $(E(r), g_\Phi)$ is isometric to $(\mathbb{R}^{l(m-1)}, g_{can})$. For $(p, w) \in B(r)$, the image of the fiber $F_{(p,w)}(r)$ via Φ is equal to $\Phi(F_{(p,w)}(r)) = \{(p, w)\} \times \mathbb{R}^{(l-1)(m-1)}$ which is totally geodesic in $(\mathbb{R}^{l(m-1)}, g_{can})$ by Example 2.7 and this proves property 2). Property 3) follows by Lemma 2.8. Finally, property 4) follows from Theorem 2.1 and transversality theory (see e.g. Guillemin and Pollack 1974). Indeed, any curve, and in particular our geodesic γ, can be deformed by an arbitrary small perturbation to a smooth curve σ intersecting $E_c(r)$ in a finite number of points. □

As we can observe, g_Φ does not represent a definitive solution to the minimization problem raised in the introduction: its geodesic needs to be perturbed in order to intersect S_1 in a finite number of points. The following theorem represents our main result.

Theorem 3.2. *There exists a Riemannian metric g on $E(r)$ which coincides with g_Φ outside an arbitrary small neighborhood of $\mathcal{S}_1$ in $E(r)$ and satisfies the following properties:*

1. *A geodesic joining any two regular equilibria (if it exists!) intersects $\mathcal{S}_1$ in a finite number of points;*

2. *Every regular equilibrium admits an open neighborhood U, disjoint from $E_c(r)$, such that every pair of regular equilibria in U can be joined by a unique and minimal geodesic which lies entirely on U.*

3. *Every critical equilibrium belonging to $\mathcal{S}_1$ admits an open neighborhood V such that every pair of regular equilibria in V can be joined by a unique and minimal geodesic lying on V and intersecting $\mathcal{S}_1$ in at most one point.*

Proof: Write $\mathcal{S}_1 = \cup_{j=1}^{+\infty} \mathcal{S}_1^j$. For each $j = 1, \ldots,$ let h_j be the metric on $\mathcal{S}_1^j$ given by the restriction of g_Φ to $\mathcal{S}_1^j$ and let $\mathcal{T}_j$ be a closed neighborhood of $\mathcal{S}_1^j$ in $E(r)$ diffeomorphic to $\mathcal{S}_1^j \times [-\delta_j, \delta_j]$, via a diffeomorphism

$$T_j : \mathcal{T}_j \to \mathcal{S}_1^j \times [-\delta_j, \delta_j],$$

whose existence is guaranteed by Lemma 2.9 (here δ_j are suitable strictly positive real numbers). Without loss of generality, by shrinking the $\mathcal{T}_j$ if necessary, we can assume $\mathcal{T}_j \cap \mathcal{T}_k = \emptyset$, $\forall j, k = 1, \ldots$. Let $k_j = h_j \oplus g_{\delta_j}$ be the product metric on $\mathcal{S}_1^j \times [-\delta_j, \delta_j]$, where g_{δ_j} denotes the canonical metric on $[-\delta_j, \delta_j]$ and let $\tilde{g}_j = T_j^*(k_j)$ be the pull-back metric on $\mathcal{T}_j$. Let $0 < \epsilon_j < \delta_j$ and let $\mathcal{T}_j(\epsilon_j) = T_j^{-1}(\mathcal{S}_1^j \times [-\epsilon_j, \epsilon_j])$ and consider a *partition of unity* $\lambda_\alpha^j : E(r) \to [0, 1], \alpha = 1, 2$, subordinate to the open cover of $E(r)$ given by two open sets $U_1^j = E(r) \setminus \mathcal{T}_j(\epsilon_j)$ and $U_2^j = \operatorname{Int} \mathcal{T}_j$ (the interior of $\mathcal{T}_j$). This means to take two smooth real valued functions $\lambda_\alpha^j : E(r) \to [0, 1], \alpha = 1, 2$, such that:

- $\lambda_1^j(x) + \lambda_2^j(x) = 1, \ \forall x \in E(r)$;
- $\operatorname{supp} \lambda_\alpha^j \subset U_\alpha^j, \ \alpha = 1, 2$, namely each λ_α^j vanishes outside a closed subset contained in U_α^j for all $\alpha = 1, 2$.

Consider now the Riemannian metric g on $E(r)$ given by

$$g_j = \lambda_1^j g_\Phi \oplus \lambda_2^j \tilde{g}_j$$

on $\mathcal{T}_j$ and equal to g_Φ outside $\cup_j \mathcal{T}_j$. Since we can choose the $\delta_j' s$ arbitrary small, the metric g coincides with g_Φ outside an arbitrary small neighborhood of $\mathcal{S}_1$. In order to prove property 1, observe first that, by Lemma 2.7, $(\mathcal{S}_1^j, h_j)$ is totally geodesic in $(\operatorname{Int} \mathcal{T}_j(\epsilon_j), \tilde{g}_j)$, where $\operatorname{Int} \mathcal{T}_j(\epsilon_j) = T_j^{-1}(\mathcal{S}_1^j \times (-\epsilon_j, \epsilon_j))$. On the other hand, the metric g coincides with $\tilde{g}_j$ on the open set $\operatorname{Int} \mathcal{T}_j(\epsilon_j)$. Hence $(\mathcal{S}_1^j, h_j)$ is totally geodesic also in $(E(r), g)$. Therefore property 1 follows by applying Lemma 2.8. Property 2 follows by taking a geodesic convex neighborhood U (see Proposition 2.5) around the regular equilibrium x such that U does not intersect $E_c(r)$ (U exists since $E(r)$ is closed). Finally, let V be a geodesic convex neighborhood around a critical equilibrium $y \in S_1^j$ such that $V \cap S_1^k = \emptyset$, $\forall k \neq j$. Let γ be the (unique) minimal geodesic joining two regular equilibria $\gamma(t_1) = x_1 \in V$, $\gamma(t_2) = x_2 \in V$, $t_1, t_2 \in [0, 1]$. Suppose, by contradiction, that γ intersects S_1^j in more than one point. By property 1, γ must intersect S_1^j in a finite number of points, say $\{y_1, \ldots y_k\}$, $k \geq 2$. Let $\gamma(t) = x$ be a point in $\gamma([0, 1])$, $t_1 < t < t_2$, such that $\gamma([t_1, t]) \cap S_1^j = \{y_1, y_2\}$. Let $\sigma : [0, 1] \to S_j$ be the minimal geodesic of S_1^j joining $y_1 = \gamma(s_1)$ and $y_2 = \gamma(s_2)$, $t_1 < s_1 < s_2 < t_2$. Since S_1^j is totally geodesic in $E(r)$, σ is also a geodesic of $E(r)$. But then we have $\sigma([0, 1]) = \gamma([s_1, s_2])$ because V is a geodesic convex neighborhood. This gives the desired contradiction since S_1^j is totally geodesic in $E(r)$ (see Lemma 2.8). $\square$

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