An equilibrium existence theorem for atomless economies without the monotonicity assumption

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Abstract

We consider an exchange economy with an atomless space of consumers whose preference relations need not be monotone. We prove that if for every commodity, there is a group of consumers who regard it as potentially desirable, in a sense to be made precise, then there exists a competitive equilibrium, however small the proportion of such a group may be in the entire economy.

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1 Introduction

This note is concerned with the existence of a competitive equilibrium in an exchange economy with an atomless space of consumers whose preference relations need not be monotone.

The problem of the existence of a competitive equilibrium is quite different between finite (or atomic) economies and atomless economies when the monotonicity assumption is dropped. McKenzie (1959, 1981), Debreu (1962), and their followers established the existence of a competitive equilibrium in finite economies in which the preference relations need not be monotone and, at the same time, free disposal is not allowed. On the other hand, Aumann (1966) and Schmeidler (1969) established the existence of a competitive equilibrium in atomless economies in which free disposal is not allowed but the preference relations are monotone. Hildenbrand (1970) dispensed with the monotonicity assumption in his equilibrium existence theorem for atomless economies, but allowed for free disposal. Hara (2005) gave an example to show that a competitive equilibrium need not exist in atomless economies when the preference relations are not monotone and, at the same time, the free disposal is not allowed.

In the example of Hara (2005), there are two types of commodities. The first one is a good, which increases utility, and the second one is a bad, which causes disutility. The crux of the example does not lie in this unanimous perception of goods and bads, but lies in the way in which the intensities (levels) of marginal disutilies are distributed across consumers. Subsequently, Martins-da-Rocha and Monteiro (2006) established equilibrium existence theorems in atomless economies without monotonicity or free disposal, by identifying sufficient condition to eliminate the types of distributions leading to non-existence.

In this note, we also offer an equilibrium existence theorem for atomless economies without monotonicity or free disposal. But it is of a somewhat different nature from those of Martins-da-Rocha and Monteiro (2006). We show that if for every commodity (of which there can be more than two types), there is a group of consumers of positive measure for whom the commodity is a good, then there always exists a competitive equilibrium. Two aspects of our existence theorem should be emphasized. First, the measure of the group of consumers for whom the commodity is a good can be made arbitrarily small, as long as it is strictly positive. Second, the notion of being a good, which we refer to as *potential desirability*, can be made quite weak. Specifically, we do not require the addition of an arbitrary positive quantity of the commodity to increase the utility levels of the consumers in the group. Rather, we only require the addition of some positive quantity (which could be very large) of the commodity to increase the utility levels.

2 Setup

There are L types of commodities, which are all perfectly divisible. Denote by $\mathscr{P}$ the set of all complete, transitive, continuous, and locally non-satiated preference relations on the non-negative orthant $\boldsymbol{R}^L_+$. Write X for $\boldsymbol{R}^L_+$. The space $\mathscr{P}$ is metrizable (Theorem 1 of Section 2 of Chapter 1 of Hildenbrand (1974)). The space of (names of) consumers is given as an atomless probability measure space $(A, \mathscr{A}, \nu)$. An *economy* $\mathscr{E}$ is a mapping $\mathscr{E} : A \to \mathscr{P} \times \boldsymbol{R}^L$ measurable with respect to the product σ-field of the Borel measurable subsets of $\mathscr{P}$ and $\boldsymbol{R}^L$. We write $\mathscr{E}(a) = (\succsim_a, e(a))$ with $e(a) \in \boldsymbol{R}^L$ and $\succsim_a \in \mathscr{P}$. Throughout this note, we assume that e is integrable and $e(a) \in \boldsymbol{R}^L_{++}$ for almost every $a \in A$, to avoid complications arising from the violation of the so-called minimum income condition. Denote the asymmetric part of $\succsim_a$ by $\succ_a$. For any integrable mapping or correspondence defined on a $B \in \mathscr{A}$, we write $\int_B f$ for the integral $\int_B f(a)\, \nu(da)$. When $B = A$, we suppress B and write simply $\int f$.

A *feasible allocation* is an integrable mapping $f : A \to X$ that satisfies $\int f = \int e$. Note that this requires supply be exactly equal to demand, not allowing for free disposal. A *price vector* is simply a vector of $\boldsymbol{R}^L$. A *competitive equilibrium* is a pair of a price vector p and a feasible allocation f such that for almost every $a \in A$, $p \cdot f(a) \leq p \cdot e(a)$ and $p \cdot x > p \cdot e(a)$ whenever $x \in X$ and $x \succ_a f(a)$.

3 Equilibrium Existence Theorem

The following definition formulates the key assumption of our existence theorem. For each $\ell = 1, \dots, L$, denote by s_ℓ the ℓ-th unit vector of $\boldsymbol{R}^L$, that is, the vector of which the ℓ-th coordinate is equal to 1 and all the other coordinates are equal to 0.

Definition 1 Let $\ell \in \{1, \dots, L\}$ and $\succsim \in \mathscr{P}$. Commodity ℓ is *potentially desirable* for $\succsim$ if for every $x \in X$, there exists a $t > 0$ such that $x + ts_\ell \succ x$.

Every commodity is potentially desirable for every strongly monotone preference relation. However, even if every commodity is potentially desirable for a $\succsim \in \mathscr{P}$, $\succsim$ need not be strongly monotone. This is because in the definition of potential desirability, we only require $x + ts_\ell \succ x$ for some $t > 0$, not for every $t > 0$. As an example, let $L = 2$ and a preference relation $\succsim \in \mathscr{P}$ be represented by $u(x) = x_1 + x_2 \sin x_2$. Then every commodity is potentially desirable for $\succsim$, but $\succsim$ is not strongly monotone. Yet, if every commodity is potentially desirable for a convex $\succsim \in \mathscr{P}$, then $\succsim$ is strongly monotone. The crucial implication of potential desirability that we will utilize when proving the existence theorem is that if commodity ℓ is potentially desirable for $\succsim$ and if $p_\ell \leq 0$, then there is no preference-maximizing consumption vector under the price vector p.

We are now ready to state our equilibrium existence theorem.

Theorem 1 *If for every $\ell = 1, \dots, L$, there exists an $A_\ell \in \mathscr{A}$ such that $\nu(A_\ell) > 0$ and commodity ℓ is potentially desirable for $\succsim_a$ for every $a \in A_\ell$, then there exists a competitive equilibrium and every equilibrium price vector belongs to $\boldsymbol{R}^L_{++}$.*

This theorem states that if every commodity is potentially desirable for a group of consumers with positive measure, then there exists at least one competitive equilibrium and, moreover, the price of every commodity is strictly positive at every competitive equilibrium. Thus, it implies that if a commodity is always a good for a group of consumers, say $C \in \mathscr{A}$, and it is always a bad for the complementary group of consumers, $A \setminus C$, then the consumption of the commodity is concentrated on C, as long as its proportion $\nu(C)$ is strictly positive, however small it may be.

Proof of Theorem 1 It follows immediately from the definition of potential desirability that every competitive equilibrium price vector, if any, is strictly positive. To prove its existence, we shall rely on Section 2 of Chapter 2 of Hildenbrand (1974). Define a correspondence $\varphi : A \times \boldsymbol{R}^L_{++} \to X$ by letting $\varphi(a, p)$ be the set of all $x \in X$ such that $p \cdot x \leq p \cdot e(x)$ and $x \succsim_a y$ for every $y \in X$ with $p \cdot y \leq p \cdot e(a)$. This is nonempty-valued because the budget set $\{x \in X \mid p \cdot x \leq p \cdot e(a)\}$ is compact and $\succsim_a$ is complete, transitive, and continuous. By local nonsatiation, $p \cdot x = p \cdot e(a)$ for every $x \in \varphi(p, a)$. For a fixed $p \in \boldsymbol{R}^L_{++}$, the correspondence $a \mapsto \varphi(a, p)$ is measurable. Moreover, it is bounded from above by an integrable mapping

$$a \mapsto \left(\frac{p \cdot e(a)}{p_1}, \ldots, \frac{p \cdot e(a)}{p_L} \right).$$

Define a correspondence $\psi : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ by $\psi(p) = \int \varphi(p, \cdot) - \int e$. Then p is a competitive equilibrium price vector if and only if $0 \in \psi(p)$. Moreover, ψ is homogeneous of degree zero, and satisfies Walras Law because $p \cdot x = p \cdot e(a)$ for every $x \in \varphi(p, a)$. It is nonempty- and compact-valued. It is also convex-valued by Lyapunov Theorem. It is bounded from below by $-\int e$. It is also upper semi-continuous. According to Proposition 3 of Section 2 of Chapter 2 of Hildenbrand (1974), therefore, it suffices to prove that if $(p^n)_n$ is a sequence in $\boldsymbol{R}^L_{++}$, $p \in \boldsymbol{R}^L_{+} \setminus \left(\boldsymbol{R}^L_{++} \cup \{0\}\right)$, $p^n \to p$, and $z^n \in \psi(p^n)$ for every n, then $s \cdot z^n \to \infty$ as $n \to \infty$, where s is a vector of $\boldsymbol{R}^L$ whose coordinates are all equal to 1. In the rest of the proof of this theorem, we fix such $(p^n)_n$ and $(z^n)_n$ and prove that $s \cdot z^n \to \infty$ as $n \to \infty$ for these $(p^n)_n$ and $(z^n)_n$. We let ℓ satisfy $p_\ell = 0$.

Claim 1 For each n, let $x^n \in \int_{A_\ell} \varphi(\cdot, p^n)$. Then $s \cdot x^n \to \infty$ as $n \to \infty$.

Proof of Claim 1 For every n, there exists a measurable mapping $\xi^n : A_\ell \to X$ such that $x^n = \int_{A_\ell} \xi^n$ and $\xi^n(a) \in \varphi(a, p^n)$ for every $a \in A_\ell$. Then $s \cdot \xi^n(a) \to \infty$ as $n \to \infty$ for every $a \in A_\ell$. Indeed, if not, then, there would be a cluster point of the sequence $(\xi^n(a))_n$. Then, since $X = \boldsymbol{R}^L_+$ and $e(a) \in \boldsymbol{R}^L_{++}$, every such cluster point would be a preference-maximizing consumption vector under the price vector p. But this is a contradiction because $a \in A_\ell$ and $p_\ell = 0$. Thus $s \cdot \xi^n(a) \to \infty$ as $n \to \infty$ for every $a \in A_\ell$.

For positive integers n and r, define $B^n_r = \{a \in A_\ell \mid s \cdot \xi^n(a) \geq r\}$. Since $s \cdot \xi^n(a) \to \infty$ as $n \to \infty$ for every $a \in A_\ell$, we have $\bigcup_{m=1}^\infty \bigcap_{n=m}^\infty B^n_r = A_\ell$ for every r. Thus, for every r, there exists an m such that $\nu\left(\bigcap_{n=m}^\infty B^n_r\right) \geq (1/2)\nu(A_\ell)$. Hence

$$\int_{A_\ell} s \cdot \xi^h \geq \int_{\bigcap_{n=h}^\infty B^n_r} s \cdot \xi^h \geq \frac{r}{2}\nu(A_\ell)$$

for every $h \geq m$. Since r is arbitrary, this implies that

$$s \cdot x^n = s \cdot \int_{A_\ell} \xi^n = \int_{A_\ell} s \cdot \xi^n \to \infty$$

as $n \to \infty$. The proof of this claim is thus completed. ///

To complete the proof of our existence theorem, for each n, let $\xi^n : A \to X$ be an integrable mapping such that $\xi^n(a) \in \varphi(a, p^n)$ for every $a \in A$ and $z^n = \int \xi^n - \int e$. Then $s \cdot \int_{A \setminus A_\ell} \xi^n = \int_{A \setminus A_\ell} s \cdot \xi^n \geq 0$. By Claim 1, $s \cdot \int_{A_\ell} \xi^n \to \infty$. Hence

$$s \cdot z^n = s \cdot \int_{A_\ell} \xi^n + s \cdot \int_{A \setminus A_\ell} \xi^n - s \cdot \int e \to \infty$$

as $n \to \infty$. The proof of the theorem is thus completed. ///

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