E C O N O M I C S   B U L L E T I N

A robust definition of possibility for biseparable preferences

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Abstract

This note presents several preference-based definitions of a likely event, and shows that they induce (in the sense of Lo 2005b) the same set of possible states for biseparable preferences.

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1. Introduction

According to the expected utility model of Savage (1954), and more generally the probabilistic sophistication model of Machina and Schmeidler (1992, 1995), the beliefs of a decision maker are represented by a probability measure. For any probabilistically sophisticated decision maker confronted with a finite set of states of the world, it is standard to say that a state of the world is possible if the decision maker assigns positive probability to that state. In contrast, for models of preference without the probabilistic sophistication property, especially Choquet expected utility (Schmeidler 1989) and maxmin expected utility (Gilboa and Schmeidler 1989), the issue of defining possibility has been controversial. Dow and Werlang (1994) is an early paper raising the issue. Subsequently, Morris (1997), Ryan (2002) and Lo (2005a) propose several preference-based definitions of a possible state; Lo (2005b) even provides a recipe, which has the potential for generating different sets of possible states from different definitions of a likely event.

As a concrete illustration of his recipe, Lo (2005b, Section 3) proposes a specific preference-based definition of a likely event, and uses it to generate what he calls $\mathcal{I}$-possible states. Interestingly, for the class of biseparable preferences (Ghirardato and Marinacci 2001), which includes Choquet expected utility and maxmin expected utility as special cases, *$\mathcal{I}$-possibility* is equivalent to the notions of *subjective possibility* of Ryan (2002) and *strict possibility* of Lo (2005a). Those equivalence results (established in Lo 2005a,b) suggest that there may be at least a "robust" definition of possibility for biseparable preferences. The purpose of this note is to provide further evidence. More precisely, we show that a couple of other intuitive definitions of a likely event also induce the set of $\mathcal{I}$-possible states. So, confining to biseparable preferences, the issue of defining possibility may not be as controversial as it first appeared.

2. Likely events and possible states

This section is a brief review of Lo (2005b). Let Ω be a finite set of states of the world, and 2^Ω the set of events. Objects of choice are acts, where an act is a function from Ω to $\mathbb{R}$. Let $\succeq$ be a preference ordering over acts. Consider the following (indirect) approach of defining a possible state. First, derive from $\succeq$ a subset $\mathcal{L}$ of 2^Ω, which is interpreted as the collection of likely events. Then possible states are induced from $\mathcal{L}$ as follows. (We use $\subseteq$ for subset, and $\subset$ for strict subset. For any state $\omega \in \Omega$, ω also denotes the event $\{\omega\}$.)

Definition 1. A state $\omega \in \Omega$ is *$\mathcal{L}$-possible* if there exists an event $E \subset \Omega$ such that $E \notin \mathcal{L}$ and $\omega \cup E \in \mathcal{L}$.

According to Definition 1, a state ω is $\mathcal{L}$-possible if there exists an event E such that ω has the following impact on E: E is not a likely event, but E with ω attached becomes a likely event. As long as Ω is a likely event and $\emptyset$ is not, there exists at least one $\mathcal{L}$-possible state. If any superset of a likely event is also a likely event, then the set of all $\mathcal{L}$-possible states is equal to

$$\cup\{D | D \in \min \mathcal{L}\}, \tag{1}$$

where

$$\min \mathcal{L} = \{D | D \in \mathcal{L}; \text{and for all } E \subset D, E \notin \mathcal{L}\}. \tag{2}$$

In words, the set of $\mathcal{L}$-possible states is the union of all minimal likely events. It follows that if every element of $\min \mathcal{L}$ is a singleton, then every $\mathcal{L}$-possible state can be identified with a likely event; on the other hand, if $\min \mathcal{L}$ is a singleton, then $\mathcal{L}$ is closed under intersection, and every element of $\mathcal{L}$ can be regarded as a believed event (in the sense of Morris 1997, p. 220). In general, an event containing a $\mathcal{L}$-possible state may not be a likely event, and a likely event may not be a believed event.

Back to the derivation of $\mathcal{L}$. Let $\succ$ be the strict preference ordering corresponding to $\succeq$. For all $D \subseteq \Omega$ and for all $x, y \in \mathbb{R}$, $x_D y$ denotes the binary act that yields the outcome x if the event D happens, and the outcome y otherwise. The following preference-based notion is defined and justified in Lo (1999).

Definition 2. For any $D, E \subseteq \Omega$, D is *infinitely more likely than* E if for all $x, y, z \in \mathbb{R}$ with $x > y$ and $z > y$, $x_D y \succ z_E y$.

To elaborate, an event D is infinitely more likely than another event E if the decision maker strictly prefers to bet on D rather than on E, and the strict preference persists no matter how much bigger is the outcome for winning the E bet than that for winning the D bet. With Definition 2, Lo (2005b) proposes

$$\mathcal{L} = \mathcal{I} \equiv \{D | D \text{ is infinitely more likely than } \Omega \setminus D\}, \tag{3}$$

where $\Omega \setminus D$ denotes the complement of D. Say that a state is *$\mathcal{I}$-possible* if it satisfies Definition 1, with $\mathcal{L} = \mathcal{I}$.

3. Equivalence results

Equation (3) leads us to consider the following obvious alternatives:

- $\mathcal{I}_1 \equiv \{D | D \text{ is infinitely more likely than } \emptyset\}$.
- $\mathcal{I}_2 \equiv \{D | \Omega \text{ is infinitely more likely than } \Omega \setminus D\}$.
- $\mathcal{I}_3 \equiv \{D | \Omega \text{ is not infinitely more likely than } D\}$.
- $\mathcal{I}_4 \equiv \{D | \Omega \setminus D \text{ is not infinitely more likely than } D\}$.
- $\mathcal{I}_5 \equiv \{D | \Omega \setminus D \text{ is not infinitely more likely than } \emptyset\}$.

For every $j = 1, \ldots, 5$, the notion of $\mathcal{I}_j$-possibility is parallel to that of $\mathcal{I}$-possibility; to be exact, ω is *$\mathcal{I}_j$-possible* if it satisfies Definition 1, with $\mathcal{L} = \mathcal{I}_j$.

A biseparable preference ordering gets its name from the following property: For any $x, x', y, y' \in \mathbb{R}$ with $x \geq y$ and $x' \geq y'$, and any $D, E \subseteq \Omega$, $x_D y \succeq x'_E y'$ if and only if

$$\rho(D)u(x) + (1 - \rho(D))u(y) \geq \rho(E)u(x') + (1 - \rho(E))u(y'), \tag{4}$$

where $u\colon \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing, and $\rho : 2^\Omega \to [0, 1]$ is normalized and monotone. In the literature, ρ is called a capacity (or nonadditive probability measure). According to Equation (4), $\succeq$ is separable in binary acts; more precisely, a binary act is evaluated according to its "expected utility," where expectation is taken with respect to a capacity in a rank dependent fashion.

We end this note with the following proposition, which establishes the equivalence of $\mathcal{I}_j$-possibility and $\mathcal{I}$-possibility for biseparable preferences. Almost all the equivalence results do not carry over to cumulative prospect theory (Tversky and Kahneman 1992) and lexicographically biseparable preferences (Ryan 2002), which are two noticeable generalizations of biseparable preferences.

Proposition. *Suppose $\succeq$ is biseparable. Then for every $j = 1, \ldots, 5$, a state is $\mathcal{I}_j$-possible if and only if it is $\mathcal{I}$-possible.*

Proof of Proposition. As $\succeq$ satisfies Equation (4), D is infinitely more likely than E if and only if $\rho(D) > 0$ and $\rho(E) = 0$. Based on this observation, further observations can be made. (In the sequel, $\min \mathcal{I}_j$ is as defined in Equation (2), with $\mathcal{L} = \mathcal{I}_j$; similarly for $\min \mathcal{I}$.)

OBSERVATION 1. $D \in \min \mathcal{I}_1$ if and only if both Conditions A and B below are satisfied.

A. $\rho(D) > 0$.

B. For every $E \subset D$, $\rho(E) = 0$.

OBSERVATION 2. $D \in \min \mathcal{I}_2$ if and only if both Conditions A and B below are satisfied.

A. $\rho(\Omega \setminus D) = 0$.

B. For every $E \subset D$, $\rho(\Omega \setminus E) > 0$.

OBSERVATION 3. $D \in \min \mathcal{I}_3$ if and only if both Conditions A and B below are satisfied.

A. $\rho(D) > 0$.

B. For every $E \subset D$, $\rho(E) = 0$.

OBSERVATION 4. $D \in \min \mathcal{I}_4$ if and only if both Conditions A and B below are satisfied.

A. $\rho(D) > 0$ or $\rho(\Omega \setminus D) = 0$.

B. For every $E \subset D$, $\rho(E) = 0$ and $\rho(\Omega \setminus E) > 0$.

OBSERVATION 5. $D \in \min \mathcal{I}_5$ if and only if both Conditions A and B below are satisfied.

A. $\rho(\Omega \setminus D) = 0$.

B. For every $E \subset D$, $\rho(\Omega \setminus E) > 0$.

OBSERVATION 6. $D \in \min \mathcal{I}$ if and only if both Conditions A and B below are satisfied.

A. $\rho(D) > 0$ and $\rho(\Omega \setminus D) = 0$.

B. For every $E \subset D$, $\rho(E) = 0$ or $\rho(\Omega \setminus E) > 0$.

The set of $\mathcal{I}_j$-possible states can be derived using Equation (1), with $\mathcal{L} = \mathcal{I}_j$; likewise for the set of $\mathcal{I}$-possible states. We are now prepared to show that

- *a state is $\mathcal{I}_1$-possible only if it is $\mathcal{I}_4$-possible.* Fix any $D \in \min \mathcal{I}_1$, and consider the following two exhaustive cases.

  CASE 1: Suppose $\rho(\omega \cup [\Omega \setminus D]) > 0$ for every $\omega \in D$. Then we can immediately use Observations 1 and 4 to conclude that $D \in \min \mathcal{I}_4$, and hence every $\omega \in D$ is $\mathcal{I}_4$-possible. (Obviously, we rely on the fact that ρ is monotone; it is also the case in various places below.)

  CASE 2: Suppose there exists $\omega \in D$ such that

$$\rho(\omega \cup [\Omega \setminus D]) = 0. \tag{5}$$

  Equation (5) enables us to fix $D^* \subset D$ such that

$$\rho(\Omega \setminus D^*) = 0 \tag{6}$$

  and

$$\rho(\Omega \setminus E) > 0 \quad \forall E \subset D^*. \tag{7}$$

  By Observation 1, $D \in \min \mathcal{I}_1$ and $D^* \subset D$ imply

$$\rho(E) = 0 \quad \forall E \subseteq D^*. \tag{8}$$

  By Observation 4, Equations (6)–(8) imply $D^* \in \min \mathcal{I}_4$. For any $\omega \in D$, if ω violates Equation (5), then $\omega \in D^*$; otherwise $D^* \subset D$ and $\omega \notin D^*$ would imply $\omega \cup [\Omega \setminus D] \subseteq \Omega \setminus D^*$, which can be combined with $\rho(\omega \cup [\Omega \setminus D]) > 0$ to arrive at $\rho(\Omega \setminus D^*) > 0$, contradicting Equation (6). So any $\omega \in D$ violating Equation (5) is $\mathcal{I}_4$-possible. Next, consider any $\omega \in D$ such that Equation (5) holds. By Observation 1, $D \in \min \mathcal{I}_1$ implies

$$\rho(D) > 0 \tag{9}$$

  and

$$\rho(D \setminus \omega) = 0. \tag{10}$$

  Equation (10) enables us to fix

$$D' \subseteq \omega \cup [\Omega \setminus D] \tag{11}$$

such that

$$\rho(\Omega \setminus D') = 0 \tag{12}$$

and

$$\rho(\Omega \setminus E) > 0 \quad \forall E \subset D'. \tag{13}$$

Equations (5) and (11) imply

$$\rho(D') = 0. \tag{14}$$

By Observation 4, Equations (12)–(14) imply $D' \in \min \mathcal{I}_4$. Finally, Equations (9), (11) and (12) imply $\omega \in D'$. This completes the proof that any ω satisfying Equation (5) is also $\mathcal{I}_4$-possible.

- *a state is $\mathcal{I}_4$-possible only if it is $\mathcal{I}$-possible.* Fix any $D \in \min \mathcal{I}_4$. Condition A of Observation 4 can be broken into three exhaustive cases.

  CASE 1: Suppose $\rho(D) > 0$ and $\rho(\Omega \setminus D) = 0$. Then we can immediately use Observations 4 and 6 to conclude that $D \in \min \mathcal{I}$, and therefore every $\omega \in D$ is $\mathcal{I}$-possible.

  CASE 2: Suppose $\rho(D) > 0$ and $\rho(\Omega \setminus D) > 0$. Fix any $\omega \in D$. By Observation 4,

$$\rho(D \setminus \omega) = 0 \text{ and } \rho(\omega \cup [\Omega \setminus D]) > 0. \tag{15}$$

  By Observation 6, Equation (15) implies that there exists $D^* \in \min \mathcal{I}$ such that

$$D^* \subseteq \omega \cup [\Omega \setminus D]. \tag{16}$$

  If $\omega \notin D^*$, then Equation (16) would imply $D \subseteq \Omega \setminus D^*$, which can be combined with $\rho(D) > 0$ to arrive at $\rho(\Omega \setminus D^*) > 0$, contradicting $D^* \in \min \mathcal{I}$. So we must have $\omega \in D^*$, and therefore ω is $\mathcal{I}$-possible.

  CASE 3: Suppose $\rho(D) = 0$ and $\rho(\Omega \setminus D) = 0$. As in Case 2 above, fix any $\omega \in D$ and any $D^* \in \min \mathcal{I}$ satisfying Equation (16). If $\omega \notin D^*$, then (16) would imply $D^* \subseteq \Omega \setminus D$, which can be combined with $\rho(\Omega \setminus D) = 0$ to arrive at $\rho(D^*) = 0$, contradicting $D^* \in \min \mathcal{I}$. So once again we must have $\omega \in D^*$, and therefore ω is $\mathcal{I}$-possible.

- *a state is $\mathcal{I}$-possible only if it is $\mathcal{I}_2$-possible.* Fix any $D \in \min \mathcal{I}$. Observations 2 and 6 allow us to fix $D^* \in \min \mathcal{I}_2$ such that $D^* \subseteq D$. If $D^* = D$, we can immediately conclude that every $\omega \in D$ is $\mathcal{I}_2$-possible. Suppose that $D^* \subset D$. Fix any ω such that $\omega \in D$ and $\omega \notin D^*$. Then $\omega \cup [\Omega \setminus D] \subseteq \Omega \setminus D^*$. Since $D^* \in \min \mathcal{I}_2$, we have $\rho(\Omega \setminus D^*) = 0$, which can be combined with $\omega \cup [\Omega \setminus D] \subseteq \Omega \setminus D^*$ to arrive at

$$\rho(\omega \cup [\Omega \setminus D]) = 0. \tag{17}$$

  By Observation 6, $D \in \min \mathcal{I}$ and Equation (17) imply

$$\rho(D \setminus \omega) = 0. \tag{18}$$

By Observation 2, Equation (18) implies that there exists $D' \in \min \mathcal{I}_2$ such that

$$D' \subseteq \omega \cup [\Omega \setminus D]. \tag{19}$$

Since $D \in \min \mathcal{I}$, we have $\rho(D) > 0$. If $\omega \notin D'$, then Equation (19) would imply $D \subseteq \Omega \setminus D'$, which can be combined with $\rho(D) > 0$ to arrive at $\rho(\Omega \setminus D') > 0$, contradicting $D' \in \min \mathcal{I}_2$. So we must have $\omega \in D'$ and thus ω is $\mathcal{I}_2$-possible.

- *a state is $\mathcal{I}_2$-possible only if it is $\mathcal{I}_1$-possible.* Fix any $D \in \min \mathcal{I}_2$ and any $\omega \in D$. According to Observation 2,

$$\rho(\omega \cup [\Omega \setminus D]) > 0. \tag{20}$$

By Observation 1, Equation (20) implies that there exists $D^* \in \min \mathcal{I}_1$ such that

$$D^* \subseteq \omega \cup [\Omega \setminus D]. \tag{21}$$

Since $D \in \min \mathcal{I}_2$, we have $\rho(\Omega \setminus D) = 0$. If $\omega \notin D^*$, then Equation (21) would imply $D^* \subseteq \Omega \setminus D$, which can be combined with $\rho(\Omega \setminus D) = 0$ to arrive at $\rho(D^*) = 0$, contradicting $D^* \in \min \mathcal{I}_1$. So we must have $\omega \in D^*$ and thus ω is $\mathcal{I}_1$-possible.[1]

- *a state is $\mathcal{I}_1$-possible if and only if it is $\mathcal{I}_3$-possible.* This follows immediately from Observations 1 and 3.

- *a state is $\mathcal{I}_2$-possible if and only if it is $\mathcal{I}_5$-possible.* This follows immediately from Observations 2 and 5.

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[1] The equivalence of $\mathcal{I}$-possibility, $\mathcal{I}_1$-possibility and $\mathcal{I}_2$-possibility can also be established by suitably combining Observations 1 and 2 with Lo (2005b, p. 15, Proof of Proposition 8).

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