E C O N O M I C S   B U L L E T I N

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We investigate a mixed duopoly market where a welfare−maximizing public firm competes against a profit−maximizing private firm, using a linear−city location−then−price model with linear transportation costs. We find that, compared with the results in the purely private duopoly case discussed by Hotelling (1929) and d' Aspremont, Gabszewicz, and Thisse (1979), the condition under which price equilibrium exists for every location of private firm and public firm is changed while the main result of no subgame perfect Nash equilibrium (SPNE) for the game still holds true.

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Hotelling's Location Model in Mixed Duopoly

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We investigate a mixed duopoly market where a welfare-maximizing public firm competes against a profit-maximizing private firm, using a linear-city location-then-price model with linear transportation costs. We find that, compared with the results in the purely private duopoly case discussed by Hotelling (1929) and d' Aspremont, Gabszewicz, and Thisse (1979), the condition under which price equilibrium exists for every location of private firm and public firm is changed while the main result of no subgame perfect Nash equilibrium (SPNE) for the game still holds true.

Keywords

public firm, private firm, mixed duopoly, Hotelling, location

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1. Introduction

Studies of mixed markets, in which welfare-maximizing public firms compete against profit-maximizing private firms, have become increasingly popular in recent years.[1] Most existing work on mixed oligopoly assumes an industry formed by firms selling a homogeneous good. There are also some exceptions. For example, Cremer, Marchand and Thisse (1991) examined a mixed market using a Hotelling-type location-then-price model with quadratic transportation costs. Matsushima and Matsumura (2003a) investigated the sequential choice of location in a mixed oligopoly in which transportation costs are also assumed quadratic. Matsushima and Matsumura (2003b) investigated a mixed oligopoly market using a circular city model with quantity-setting competition. So far, no one has considered mixed duopoly in a linear-city, linear-transportation-cost world.

As for the existence of equilibrium in Hotelling's location-then-price model in the purely private market case, d'Aspremont, et al. (1979) derived the condition under which the price equilibrium exists and demonstrated that there is no pure strategy subgame perfect Nash equilibrium (SPNE) for the game when transportation costs are assumed to be linear. It is interesting to investigate whether the result of no pure strategy SPNE still holds true in the mixed duopoly case. It is also interesting to study under what condition price equilibrium exists in the second stage of the game.

[1] See De Fraja and Delbono (1990) and Nett (1993) for general reviews of the mixed oligopoly model. For recent literature on mixed oligopoly (duopoly), see Pal (1998), Fjell and Heywood (2004), and the references in this introduction, etc.

However, there is no paper discussing these two issues. The purpose of this paper is to investigate these issues: the issue of existence of price equilibrium and the issue of existence of pure strategy SPNE in Hotelling's location-then-price model in mixed duopoly. Transportation costs are assumed to be linear. We find that, compared with the results in the purely private duopoly case discussed by Hotelling (1929) and d' Aspremont, Gabszewicz, and Thisse (1979), the condition under which price equilibrium exists for every location of private firm and public firm is changed while the main result of no pure strategy SPNE for the game still holds true.

The remainder of the paper is organized as follows. The model is presented in section 2. In section 3, we solve the game using backward induction, derive the conditions under which price equilibrium exists in the second stage of the game, and demonstrate that no SPNE exists in this game. Section 4 concludes the paper.

2. The model

We consider the following spatial competition model in a mixed duopoly market. There are two firms producing a homogeneous product at zero marginal cost. Firm a is a private firm and firm b is a public firm. In the first stage, firms choose simultaneously their location in the unit interval [0, 1]; in the second stage, they choose mill prices simultaneously.

Consumers are uniformly distributed over the interval [0, 1] with a unit density. They consume a single unit of the product irrespective of its price. Each consumer chooses to buy from the firm with lower full price (i.e., mill price plus transportation costs). Transportation costs are linear in distance. Hence, the full price paid by a consumer located at x is equal to $p_a + t|x-a|$ if buying from firm *a*, or $p_b + t|x-(1-b)|$ if buying from firm *b*, where *t* is the transportation rate. Here *a* is the distance between the location of firm *a* and the left end of the line, i.e., 0; and *b* is the distance between the location of firm *b* and the right end of the line, i.e., 1 ($a \geq 0$, $b \geq 0$, and $a+b \leq 1$).

It is easy to get the demand of firm *a* and firm *b*:

$$q_a = \begin{cases} 1, & if \quad p_b - p_a \geq t(1-a-b), \\ 0, & if \quad p_b - p_a \leq -t(1-a-b), \\ (1+a-b)/2 + (p_b - p_a)/(2t), & others, \end{cases} \tag{1}$$

and

$$q_b = 1 - q_a . \tag{2}$$

The objective function of firm *a* is given by

$$\pi_a = p_a q_a \tag{3}$$

and firm *b*'s objective is to maximize social surplus. Individual demands being perfectly inelastic, this amounts to minimizing the total transportation costs. The total transportation costs are:

$$TC = \begin{cases} \frac{t}{2}\left[a^2 + (1-a)^2\right], & if \quad p_b - p_a \geq t(1-a-b), \\ \frac{t}{2}\left[b^2 + (1-b)^2\right], & if \quad p_b - p_a \leq -t(1-a-b), \\ \frac{t}{2}\left[a^2 + b^2 + \frac{(1-a-b)^2}{2}\right] + \frac{1}{4t}(p_a - p_b)^2, & others. \end{cases} \tag{4}$$

Our solution-concept is a SPNE in which firms choose locations, looking ahead to the resulting equilibrium prices. We restrict attention to pure strategy equilibrium so that the results are comparable to those in d'Aspremont, et al. (1979).

3. Equilibrium and the Existence of Equilibrium

3.1 The Second Stage

In the second stage, firm *a* chooses p_a to maximize its profit and firm *b* chooses p_b to minimize the total transportation costs. We examine the issue of existence of price equilibrium for every location *a* and *b*.

Proposition 1 (the existence of price equilibrium): For $a+b=1$, there are infinite equilibria ($p_b-\varepsilon, p_b$). For $a+b<1$, there is an equilibrium point if and only if

$$\sqrt{a}+\sqrt{b}\le 1, \tag{5}$$

and, whenever it exists, the equilibrium is $p_a^*=p_b^*=t(1+a-b)$.

Proof: The case $a+b=1$ is immediate. Then both firms are located in the same place. Since the total transportation costs are constant, the public firm *b* will not change price for any $p_a=p_b-\varepsilon$. The private firm *a* will not change price either.

For case $a+b<1$, it is clear that, when $|p_a-p_b|<t(1-b-a)$, TC is minimized when $p_a=p_b$ and the minimum TC is $\frac{t}{2}\left[a^2+b^2+\frac{(1-a-b)^2}{2}\right]$, which is less than $\frac{t}{2}\left[a^2+(1-a)^2\right]$ and $\frac{t}{2}\left[b^2+(1-b)^2\right]$. So TC is minimized when $p_a=p_b$ and the public firm will choose its price equal to the price charged by the private firm. In other words, its reaction function is

$$p_b=p_a. \tag{6}$$

When $|p_a-p_b|<t(1-b-a)$, the private firm's reaction function is

$$p_a = \frac{p_b}{2} + \frac{t(1+a-b)}{2}. \qquad (7)$$

Solving (6) and (7) and then substituting the solution into firms' objective function gives us equilibrium prices, firm *a*'s profit, and total transportation costs:

$$p_a^* = p_b^* = t(1+a-b), \qquad (8)$$

$$\pi_a = \frac{t}{2}(1+a-b)^2, \qquad (9)$$

$$TC = \frac{t}{2}\left[a^2 + b^2 + \frac{(1-a-b)^2}{2}\right]. \qquad (10)$$

(6) ((8) also) means that two firms choose the same price. Intuitively, the reason for this result is as follows. By equating p_a and p_b, firm *b* shares equally the demand between the interval [*a*, 1-*b*] so that the total transportation costs are minimized. Next, we shall verify that the pair of prices given by (8) is indeed an equilibrium.

Since the total transportation costs are minimized when $p_a = p_b$, the public firm *b* will not change its price if the private firm *a* does not. It means p_b^* is an equilibrium strategy against p_a^*. For p_a^* to be an equilibrium strategy against p_b^*, we must have in particular that, for any $\varepsilon > 0$,

$$\pi_a(p_a^*, p_b^*) = \frac{t}{2}(1+a-b)^2 \geq p_b^* - t(1-a-b) - \varepsilon = 2ta - \varepsilon, \qquad (11)$$

where, $p_b^* - t(1-a-b) - \varepsilon$ is the profit firm *a* would obtain if it changed its price to $p_b^* - t(1-a-b) - \varepsilon$ and captured the entire demand. Since $p_a^* = p_b^* = t(1+a-b)$ is the unique pure strategy price equilibrium when $|p_a - p_b| < t(1-b-a)$, the aforementioned deviation is the only relevant one to consider.

Let $\varepsilon \to 0$, (11) becomes $(1+a-b)^2 \geq 4a$, which can be written as $1+a-b \geq 2\sqrt{a}$ and be further simplified to (5).

This completes the proof of the proposition.

3.2 The First Stage

We will neglect the case $a+b=1$ since the public firm b can lower the total transportation costs by moving away from the private firm *a*. Hence, *a* and *b* such that $a+b=1$ cannot be sustained as a subgame perfect location equilibrium.

Next, we will consider the case $a+b<1$. In the first stage, firm *a* chooses location *a* to maximize (9) and firm *b* chooses location *b* to minimize (10). Differentiating (9) with respect to a gives us $\partial \pi_a / \partial a = t(1+a-b) > 0$, which means that *a* should be as large as possible. Differentiating (10) with respect to *b* yields the first-order condition: $a+3b=1$. So the location equilibrium is $a = 1-3\varepsilon$ and $b=\varepsilon$, where ε is an infinitesimally positive number. However, this location equilibrium cannot be sustained as a subgame perfect location equilibrium because the condition (5) is violated.

Thus, we get the following proposition:

Proposition 2: Like in the purely private duopoly case, there does not exist a SPNE in Hotelling's linear-city location-then-price model when the transportation costs are linear in the mixed duopoly case.

4. Concluding Remarks

There is a large literature on mixed oligopoly. However, until now, there is no paper investigating the issue of the existence of SPNE in Hotelling's linear-city location-then-price model in mixed duopoly and the issue of the existence of price equilibrium in the second stage of the game. The purpose of this paper is to investigate these two issues.

We find that, compared with the results in the purely private duopoly case discussed by Hotelling (1929) and d' Aspremont, Gabszewicz, and Thisse (1979), the condition under which price equilibrium exists for every location of private firm and public firm is changed while the main result of no subgame perfect Nash equilibrium (SPNE) for the game still holds true.

Finally, we point out that in Hotelling's linear-city location-then-price game with quadratic transportation costs in a mixed duopoly, there exists pure strategy SPNE, and that for every location, there exists price equilibrium in the second stage of the game. [2]

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[2] Cremer, et al. (1991) also solves the SPNE in Hotelling's linear-city location-then-price game with quadratic transportation costs in a mixed duopoly. However, the authors did not prove the existence of SPNE in the game and the existence of price equilibrium in the second stage of the game. Following the same procedures as we use in the proof of proposition 1 and 2, we can easily demonstrate the results listed in this paragraph.

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