E C O N O M I C S B U L L E T I N

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We will show that some results in Goyal and Moraga (2001), RAND Journal of Economics 32(4), are incomplete. The results are the social welfare and the total profit of the firms in the complete network is lower than those in some networks. They focus on the symmetric network gk where k is the number of links of each firm and show that the social welfare (the total profit of the firms) in the complete network gn-1 is lower than that in gn-2 where n is the number of the firms. However, their proofs are incomplete because there is no gn-2 if n is odd. Therefore, this paper gives the complete proof of their result. That is, since there is gn-3 if n is odd, we show the social welfare (total profit) in the gn-1 is lower than that in the network gn-3.

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A note on Propositions 7 and 8 of Goyal and Moraga (2001)*

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Abstract

We will show that some results in Goyal and Moraga (2001), RAND Journal of Economics 32(4), are incomplete. The results are the social welfare and the total profit of the firms in the complete network is lower than those in some networks. They focus on the symmetric network g^k where k is the number of links of each firm and show that the social welfare (the total profit of the firms) in the complete network g^{n-1} is lower than that in g^{n-2} where n is the number of the firms. However, their proofs are incomplete because there is no g^{n-2} if n is odd. Therefore, this paper gives the complete proof of their result. That is, since there is g^{n-3} if n is odd, we show the social welfare (total profit) in the g^{n-1} is lower than that in the network g^{n-3}.

D85; D43

Keywords: Symmetric networks; Network formation; R&D; Oligopoly; Graph theory

1 Introduction

Goyal and Moraga (2001) discuss the formation of R&D networks among competing firms. Their main result is that competing firms may have excessive incentives to form links. That is, the complete network in which all firms

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form link with each other is stable, but (i) the social welfare in the complete network is lower than that in some other networks, and (ii) the total profit of the firms in the complete network is also lower than that in some other networks. They show (i) and (ii) in their Proposition 7 (GM, p.696) and Proposition 8 (GM, p.697), respectively, by focusing on symmetric networks in which all firms have the same number of links. A symmetric network is denoted by g^k where k represents the number of links of a firm. That is, they show that the social welfare (the total profit of the firms) in the complete network g^{n-1} is less than that in g^{n-2} where n represents the number of the firms. However, since there exists no g^{n-2} if n is odd,[1] the proofs are not complete in the case that n is odd.

Therefore, we will provide the complete proofs of their results. That is, since there exist g^{n-3} if n is odd, we will prove the social welfare (the total profit of the firms) in g^{n-1} is less than that in g^{n-3} if n is odd.

2 Model and Results

At first, we will provide a review of GM's model. They consider a three-stage model. In the first stage, firms form pairwise links that represent collaborative relationships. In the second stage, firms choose their effort level in R&D. Finally, the firms engage in a Cournot competition, in the third stage.

Let the set of nodes be $N = \{1, 2, \cdots, n\}$ where $n \geq 4$. A node represents a firm. A relationship between two firms i and j is represented by an edge: ij. If there is an (no) edge between i and j, that is, if $ij \in g(\notin g)$, then i and j are (not) linked. A network g is defined as a set of the nodes and the existing links (or edges). Let $N_i(g)$ be the set of the firms with which i forms the link, and $|N_i(g)| = \eta_i(g)$. We call $\eta_i(g)$ the degree of i in g.

The R&D effort level of firm i is given by $e_i \geq 0$. The cost function of each firm depends on a collection of effort levels $\{e_j\}_{j \in N}$ and a network g. In addition, we assume that the production cost function is linear. Specifically, the marginal production cost of i is

$$c_i(\{e_j\}_{j\in N}) = \bar{c} - e_i - \textstyle\sum_{k\in N_i(g)} e_k - \beta \textstyle\sum_{l\notin N_i(g)} e_l,$$

[1]In this paper, the fact that there exists some (no) network means that we can (not) physically construct the network.

where $\beta \in [0,1)$. The cost of i for R&D effort is γe_i^2, $\gamma > 0$. We assume that

$$\gamma > \max\left\{\frac{n^2}{(n+1)^2}, \frac{a}{4\bar{c}}\right\},$$

which ensures interiority of the solutions. The quantity of i is given by q_i. Let the inverse demand function of this market be $p = a - Q$ where $a > \bar{c} > 0$ and $Q = \sum_{i \in N} q_i$.

By backward induction, we can have the subgame perfect equilibrium of the second stage and third stage of this game for given g. That is, at first, we solve the Cournot-Nash equilibrium for given $\{e_j\}_{j \in N}$ and g, and second we solve the Nash equilibrium levels of $\{e_j\}_{j \in N}$ for given g. When we will write the solutions as $\{q_j(g)\}_{j \in N}$ and $\{e_j(g)\}_{j \in N}$, the profit function is

$$\pi_i(g) = [a - Q(g) - c_i(g)]\, q_i(g) - \gamma e_i^2(g),$$

where $c_i(g) = c_i(\{e_j(g)\}_{j \in N})$ and $Q(g) = \sum_{j \in N} q_j(g)$. Also, for given g, social welfare $W(g)$, which is the sum of consumers' surplus and producers' surplus, is

$$W(g) = \frac{Q(g)^2}{2} + \sum_{j \in N} \pi_j(g).$$

GM especially focus on symmetric networks. Following GM, the profit of $i \in N$ and the social welfare in g^k are,

$$\pi_i(g^k) = \frac{(a-\bar{c})^2 \gamma[\gamma(n+1)^2 - (n-k)^2]}{[\gamma(n+1)^2 - (n-k)(k+1)]^2} := \pi(g^k) \text{ for all } i \in N, \quad (1)$$

$$W(g^k) = \frac{(a-\bar{c})^2 n\gamma[\gamma(n+2)(n+1)^2 - 2(n-k)^2]}{2[\gamma(n+1)^2 - (n-k)(k+1)]^2}, \quad (2)$$

respectively. GM provide the following two propositions:

Proposition 1 *(GM, Proposition 7, p.696) There exists an intermediate level of collaborative activity $\bar{k}$ with $0 < \bar{k} < n-1$ for which firms' profits are maximized.*

Proposition 2 *(GM, Proposition 8, p.697) There exists an intermediate level of collaborative activity $\tilde{k}$ with $0 < \tilde{k} < n-1$ for which social welfare are maximized.*

GM prove Proposition 1 and 2 by showing $\pi(g^{n-2}) > \pi(g^{n-1}) > \pi(g^0)$ and $W(g^{n-2}) > W(g^{n-1}) > W(g^0)$, respectively. However, these proofs are not complete. This is because there is no g^{n-2} if and only if n is odd. This fact is direct from *Euler's handshaking lemma*: the sum of the degrees of all nodes in any network is even. Thus, the proofs of Proposition 1 and 2 by GM are complete if n is even, but they are incomplete if n is odd.

Therefore, we need to derive sufficient conditions for Proposition 1 and 2 in the case of n is odd. By the theorem on existence of a simple graph with given degrees the by Erdos-Gallai (1960)[2], there exists g^{n-3} if n is odd. Thus, sufficient conditions for Proposition 1 and 2 will be $\pi(g^{n-3}) > \pi(g^{n-1}) > \pi(g^0)$ and $W(g^{n-3}) > W(g^{n-1}) > W(g^0)$, respectively, if n is odd. By an elementary calculation, we can see $\pi(g^{n-3}) > \pi(g^{n-1})$ and $W(g^{n-3}) > W(g^{n-1})$ for all $n \geq 5$. The proofs are in the Appendix. Thus, we have complete proofs of Proposition 1 and 2.

Appendix

At first, we will prove $\pi(g^{n-3}) > \pi(g^{n-1})$ for all $n \geq 5$. By (1), we have

$$\begin{aligned}
\pi(g^{n-1}) &= \frac{(a-\bar{c})^2\gamma[\gamma(n+1)^2-1]}{[\gamma(n+1)^2-n]^2}, \\
\pi(g^{n-3}) &= \frac{(a-\bar{c})^2\gamma[\gamma(n+1)^2-9]}{[\gamma(n+1)^2-3(n-2)]^2}.
\end{aligned}$$

Thus,

$$\pi(g^{n-3}) - \pi(g^{n-1}) = \frac{(a-\bar{c})^2\gamma\left\{[\gamma(n+1)^2-9][\gamma(n+1)^2-n]^2 - [\gamma(n+1)^2-1][\gamma(n+1)^2-3(n-2)]^2\right\}}{[\gamma(n+1)^2-n]^2[\gamma(n+1)^2-3(n-2)]^2}.$$

This implies that $\pi(g^{n-3}) - \pi(g^{n-1}) > 0$ if and only if

$$\gamma^2(n-5)(n+1)^4 > 2\gamma(n^2-6n+3)(n+1)^2 + 9n - 9. \tag{3}$$

First, suppose $n \geq 6$. Then, both the RHS and LHS of (3) are increasing in γ. The LHS of (3) rises at the rate $2\gamma(n-5)(n+1)^4$ and the RHS of (3) rise at the rate $2(n^2-6n+3)(n+1)^2$. It is easy to show that, since

[2] See, for example, Berge (1976, p.115).

$\gamma > n^2/(n+1)^2$, the LHS is increases at a higher rate than the RHS. Thus, if (3) is satisfied for minimum γ and $n \geq 6$, we have $\pi(g^{n-3}) - \pi(g^{n-1}) > 0$ for all $n \geq 6$. If $\gamma = n^2/(n+1)^2$, then (3) is

$$n^4(n-5) > 2n^2(n^2 - 6n + 3) + 9n - 9.$$

This inequality holds for all $n \geq 6$.

Second, if $n = 5$, then (3) is

$$-144\gamma + 36 < 0.$$

Since $\gamma > 25/36$, $\pi(g^{n-3}) - \pi(g^{n-1}) > 0$ for $n = 5$.

Therefore, we have $\pi(g^{n-3}) > \pi(g^{n-1})$ for all $n \geq 5$.

Next, we will prove $W(g^{n-3}) > W(g^{n-1})$ for all $n \geq 5$. By (2), we have

$$\begin{aligned} W(g^{n-1}) &= \frac{(a-\bar{c})^2 n\gamma[\gamma(n+2)(n+1)^2 - 2]}{2[\gamma(n+1)^2 - n]^2}, \\ W(g^{n-3}) &= \frac{(a-\bar{c})^2 n\gamma[\gamma(n+2)(n+1)^2 - 18]}{2[\gamma(n+1)^2 - 3(n-2)]^2}. \end{aligned}$$

Thus, $W(g^{n-3}) - W(g^{n-1}) > 0$ if and only if

$$\begin{aligned} &\gamma^2(n^6 + 3n^5 - 8n^4 - 42n^3 - 63n^2 - 41n - 10) > \\ &\gamma(2n^5 - n^4 - 23n^3 - 23n^2 + 9n + 12) + 18n - 18 \end{aligned} \quad (4)$$

For all $n \geq 5$, both the RHS and the LHS of (4) are increasing in γ, and the LHS is increases at a higher rate than the RHS, because $\gamma > n^2/(n+1)^2$. Thus, if (4) is satisfied for minimum γ and $n \geq 5$, we have $W(g^{n-3}) - W(g^{n-1}) > 0$ for all $n \geq 5$. If $\gamma = n^2/(n+1)^2$, then (4) holds for all $n \geq 5$. Thus, for all $n \geq 5$, $W(g^{n-3}) > W(g^{n-1})$.

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