E C O N O M I C S   B U L L E T I N

A Stochastic Approach to the Cobb-Douglas Production Function

Constantin Chilarescu
Department of Economics, West University of Timisoara

Nicolas Vaneecloo
Department of Economics, University of Lille 1

Abstract

This paper presents a stochastic approach to the theory of aggregateproduction function, based on the theory of stochastic differentialequations. The main result is that under certain restrictions the productionfunction converges from below to the Cobb-Douglas functionproviding further support for the conclusion drawn by Jones (2005).

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1. Introduction

The best known production function is the famous Cobb-Douglas function

$$F(L, K) = AL^{\alpha}K^{\beta}$$

originally constructed to approximate the output of American manufacturing from 1899 to 1922 as a function of the average number of employed wage earners and the value of fixed capital goods, reduced to dollars of constant purchasing power (Cobb and Douglas 1928).

Despite several stringent criticisms, the continued popularity of the Cobb-Douglas production function may largely be attributed to the remarkable statistical results obtained in fitting this function to time series data (Fraser 2002). Two concepts play a crucial role in the derivation of production functions: returns to scale and the elasticity of substitution. Interpreted in terms of marginalist theory, the Cobb-Douglas function complies with a certain number of requisites. One such requisite is to suppose that the returns are constant. Only with constant returns the sum of the two exponents, α and β is equal to one. It is also assumed that technical progress is neutral. In other words, the efficiencies of capital and of labor increase but their reciprocal substitutability does not change with respect to variations in the relative prices of the same factors. Finally it is assumed that the elasticity of substitution is equal to one. Apparently, we may say that the Cobb-Douglas function is the only linearly homogenous production function with a constant elasticity of substitution in which each factor's share of income is constant over time. Kaldor (1961), for example, insists that an important task of the theory of economic growth is to account for long run constancy.

The first attempt to relax one of these requisites was done by Robert Solow (1957). He found an elementary way of segregating variations in output per head due to technical change from those due to changes in the availability of capital per head. The fundamental hypothesis of Solow says that shifts in the production function are defined as neutral if they leave marginal rates of substitution untouched but simply increase or decrease the output attainable from given inputs. In that case the production function takes the special form

$$F(L, K, t) = A(t)F(L, K).$$

Another production function, known as the CES was introduced by Arrow, Chenery, Minhas and Solow (1961).

$$F(L, K) = \gamma \left[\delta L^{\varphi} + (1 - \delta)K^{\varphi}\right]^{\frac{1}{\varphi}}.$$

Possibly the weakest point of the CES formulation is the assumption of the existence of a relationship between F/L (value added per unit of labor) and W (the wage rates), independent of the stock of capital. If this assumption does not hold, the value of elasticity of substitution derived from the estimated CES function may be biased. The CES function is also subject to the limitation that the value of the elasticity of substitution is constant, although not necessarily equal to one. Soon after the explicit derivation of this production functions, several articles appeared trying to obtain generalized solutions such as the papers of McFadden (1963), and Lu and Fletcher (1968).

There is also some work done related to the dynamic properties of the Cobb-Douglas function. The first one is attributed to Sylos Labini (1995) who claims that the traditional interpretation of the Cobb-Douglas function, which refers to the marginalist theory of income distribution should be abandoned, and another interpretation should be adopted. Traditional theory is founded on the concept of substitution, defined with reference to a given level of production that can be obtained with different combinations of labour and capital, given the technology, already known and available. The passage from one type of technology to another is stimulated by the trend of output to increase. Technology can not be assumed as given and technological change comes to depend on a variation in the ratio of an increasing output trend. This process requires time and the problem of factor substitution is a dynamic, not a static, phenomenon. This is not a new idea. See for example the papers of Solow (1958), Newman and Read (1691) and, Ferguson and Pfouts (1962). The second one is attributed to Barelli and Pessoa (2003). In their paper they proved that the Inada conditions force the production function to be asymptotically Cobb-Douglas that is, its elasticity of substitution is asymptotically equal to one. This is a very important result and it is in accordance with the previous result of Labini and also with the third result, due to Charles Jones (2005). Jones concludes that Pareto distributions lead the production function to take a Cobb-Douglas form and defends the view that the long term production function is Cobb-Douglas.

In our paper we try to prove that under certain restrictions the production function converges from below to the Cobb-Douglas function providing further support for the conclusion drawn by Jones. To arrive to our main result we claim that our variables (output, labor and capital) are stochastic processes. To our knowledge, the stochastic extension of the deterministic theory of production functions has so far not been subject to a widely investigation. It is not possible to suppose a purely deterministic process for the evolution of output, labor and capital stock. The purpose of this paper is to provide a stochastic alternative of production function and to prove that under certain restrictions this function converges from below to the Cobb-Douglas function. This result will be given in section 2. In the last section we present some concluding remarks.

2. Stochastic Production Functions

We define an aggregate production function as one for which the output has to be a value measure, rather than a physical measure. The value measure has to be used due to the heterogeneity of output, capital stock and labor. In order to derive our production function we consider that the inputs L and K and the output $F(L, K)$ are aggregate stochastic value functions, we follow the concepts of Karatzas (1991) and consider a time interval $[0, T]$ with $T > 0$, possibly infinite. We define capital $K = K(t)$, and labor $L = L(t)$, as the solutions to the following stochastic differential equations:

$$\begin{cases} dL = L[\mu_L dt + \sigma_L dW_L] \\ \\ dK = K[\mu_K dt + \sigma_K dW_K]. \end{cases} \quad (1)$$

We assume that W_L and W_K are standard Brownian motions satisfying

$$E_t(dW_L, dW_K) = \rho dt.$$

$\rho \in (0, 1)$ is given and constant and the coefficients μ_L, μ_K, σ_L, and σ_K are given by:

$$\mu_l = \lim_{h \to 0} \frac{E_t\left[logL(t+h) - logL(t)\right]}{h}, \ \sigma_L^2 = \lim_{h \to 0} \frac{E_t\left[logL(t+h) - logL(t)\right]^2}{h}$$

$$\mu_k = \lim_{h \to 0} \frac{E_t\left[logK(t+h) - logK(t)\right]}{h}, \ \sigma_K^2 = \lim_{h \to 0} \frac{E_t\left[K(t+h) - logK(t)\right]^2}{h}$$

with $\mu_L = \mu_l + \frac{\sigma_L^2}{2}$ and $\mu_K = \mu_k + \frac{\sigma_K^2}{2}$. Using Itô's lemma, at any instant t, $0 \le t \le T$ we can write

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial L}dL + \frac{\partial F}{\partial K}dK + \frac{1}{2}\frac{\partial^2 F}{\partial L^2}dL^2 + \frac{1}{2}\frac{\partial^2 F}{\partial K^2}dK^2 + \frac{\partial^2 F}{\partial L \partial K}dLdK. \quad (2)$$

Putting (1) and (2) together, we find that

$$dF = \left[\frac{\partial F}{\partial t} + \mu_L L\frac{\partial F}{\partial L} + \mu_K K\frac{\partial F}{\partial K} + \frac{\sigma_L^2 L^2}{2}\frac{\partial^2 F}{\partial L^2} + \frac{\sigma_K^2 K^2}{2}\frac{\partial^2 F}{\partial K^2} + \rho\sigma_L\sigma_K LK\frac{\partial^2 F}{\partial L\partial K}\right]dt + \sigma_L L\frac{\partial F}{\partial L}dW_L + \sigma_K K\frac{\partial F}{\partial K}dW_K. \quad (3)$$

Here we suppose that the production function can be written as follows

$$F(L,K,t) = F_d(L,K,t) + \Delta_L L(t) + \Delta_K K(t) \quad (4)$$

where Δ_L and Δ_K are two unknowns to be determined such that

$$dF_d(L,K,t) = rF_d(L,K,t)dt$$

where r is the risk-free interest rate. If we choose $\Delta_L = \frac{\partial F}{\partial L}$ and $\Delta_K = \frac{\partial F}{\partial K}$, then the stochastic terms vanish, and we arrive at

$$\frac{\partial F}{\partial t} + rL\frac{\partial F}{\partial L} + rK\frac{\partial F}{\partial K} + \frac{\sigma_L^2 L^2}{2}\frac{\partial^2 F}{\partial L^2} + \frac{\sigma_K^2 K^2}{2}\frac{\partial^2 F}{\partial K^2} + \rho\sigma_L\sigma_K LK\frac{\partial^2 F}{\partial L\partial K} - rF = 0$$

which is a second order linear, two-dimensional partial differential equation. For the sake of simplicity we denote by A the operator defined as follows:

$$A =: \frac{\partial}{\partial t} + rL\frac{\partial}{\partial L} + rK\frac{\partial}{\partial K} + \frac{\sigma_L^2 L^2}{2}\frac{\partial^2}{\partial L^2} + \frac{\sigma_K^2 K^2}{2}\frac{\partial^2}{\partial K^2} + \rho\sigma_L\sigma_K LK\frac{\partial^2}{\partial L\partial K} - r$$

As we pointed out above, in a recent paper Jones defends the point of view that the long term production function is Cobb-Douglas and concludes that Pareto distributions lead the production function to take a Cobb-Douglas form. In order to find a solution to the above equation we adopt here Jones′ hypothesis.

Theorem 0.1. *If the dynamics of the function $F(L,K,t)$ is described by*

$$AF = 0 \quad (5)$$

and

$$\begin{cases} F(0,K,t) = F(L,0,t) = F(0,0,t) = 0, \text{ for all } t \in [0,T] \\ F(L,K,0) \geq 0, \text{ for all } L \in I\!R_+ \text{ and } K \in I\!R_+ \\ F(L,K,t) = AL^\alpha K^\beta, \ t \geq T \end{cases} \quad (6)$$

then the solution is given by

$$F(L,K,t) = F(L,K,T)e^{\theta(T-t)} = AL^{\alpha}K^{\beta}e^{\theta(T-t)} \tag{7}$$

where

$$\theta = \frac{\alpha(\alpha-1)(\sigma_L^2 + \sigma_K^2 - 2\rho\sigma_L\sigma_K)}{2}. \tag{8}$$

Proof. We consider the following changes of variables:

$$\begin{cases} F(L,K,t) = e^{-r\tau}V(x_1,x_2,\tau), \\ \tau = T-t, \\ x_1 = \lambda_1 ln(L) + \delta_1\tau, \\ x_2 = \lambda_2 ln(K) + \delta_2\tau \end{cases} \tag{9}$$

where λ_1, λ_2, δ_1 and δ_2 are unknowns to be determined so that the equation (5) contains no terms in V, $\frac{\partial V}{\partial x_1}$ and $\frac{\partial V}{\partial x_2}$. Choosing

$$\lambda_1 = \frac{\sqrt{2}}{\sigma_L},\ \lambda_2 = \frac{\sqrt{2}}{\sigma_K},\ \delta_1 = \frac{\sqrt{2}}{\sigma_L}\left(r - \frac{\sigma_L^2}{2}\right),\ \delta_2 = \frac{\sqrt{2}}{\sigma_K}\left(r - \frac{\sigma_K^2}{2}\right)$$

we obtain

$$\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2} + 2\rho\frac{\partial^2 V}{\partial x_1 \partial x_2} \tag{10}$$

with

$$V(x_1,x_2,0) = A\exp\left[\frac{\alpha\sigma_L}{\sqrt{2}}x_1 + \frac{\beta\sigma_K}{\sqrt{2}}x_2\right] \tag{11}$$

We will proceed here to another change of variables (see Logan 1998).

$$\begin{cases} V(x_1,x_2,\tau) = H(y_1,y_2,\tau), \\ y_1 = -\frac{\rho}{\sqrt{1-\rho^2}}\,x_1 + \frac{1}{\sqrt{1-\rho^2}}\,x_2, \\ y_2 = x_1 \end{cases} \tag{12}$$

to arrive to the following prototype two-dimensional diffusion equation

$$\frac{\partial H}{\partial \tau} = \frac{\partial^2 H}{\partial y_1^2} + \frac{\partial^2 H}{\partial y_2^2} \tag{13}$$

subject to the initial condition

$$h(y_1, y_2) = H(y_1, y_2, 0) = A \exp\left[\frac{\beta\sigma_K\sqrt{1-\rho^2}}{\sqrt{2}} y_1 + \frac{\alpha\sigma_L + \rho\beta\sigma_K}{\sqrt{2}} y_2\right]. \quad (14)$$

As it is well known the solution of (13) with the initial condition (14) is given by

$$H(y_1, y_2, \tau) = \frac{1}{4\pi\tau}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(z_1, z_2)\exp\left[-\frac{(y_1 - z_1)^2 + (y_2 - z_2)^2}{4\tau}\right] dz_1 dz_2.$$

To evaluate the above integral we make the following change of variables:

$$u_1 = \frac{z_1 - y_1 - \sqrt{2(1-\rho^2)}\beta\sigma_K\tau}{\sqrt{2\tau}},\ u_2 = \frac{z_2 - y_2 - \sqrt{2}(\alpha\sigma_L + \rho\beta\sigma_K)\tau}{\sqrt{2\tau}}$$

to obtain

$$H(y_1, y_2, \tau) = A\exp\left[\xi_1(\tau)\right]\exp\left[\xi_2(y_1, y_2)\right] \quad (15)$$

where

$$\xi_1 = \frac{\alpha^2\sigma_L^2 + \beta^2\sigma_K^2 + 2\rho\alpha\beta\sigma_L\sigma_K}{2}\tau,\ \xi_2 = \frac{\beta\sqrt{1-\rho^2}\sigma_K y_1 + (\alpha\sigma_L + \rho\beta\sigma_K)y_2}{\sqrt{2}}.$$

Considering the change of variables (9) and (12) and the relation (15) we finally obtain

$$F(L, K, t) = F(L, K, T)e^{\theta(T-t)} = AL^\alpha K^\beta e^{\theta(T-t)} \quad (16)$$

where

$$\theta = \frac{(\alpha^2 - \alpha)\sigma_L^2 + (\beta^2 - \beta)\sigma_K^2 + 2\rho\alpha\beta\sigma_L\sigma_K + 2(\alpha + \beta - 1)r}{2}. \quad (17)$$

Under the above hypotheses we conclude that $\alpha + \beta = 1$ for all $\alpha, \beta \in (0, 1)$ and consequently

$$\theta = \frac{\alpha(\alpha - 1)(\sigma_L^2 + \sigma_K^2 - 2\rho\sigma_L\sigma_K)}{2} \quad (18)$$

and thus the proof is completed. □

Remark 0.1. *Because* $\alpha \in [0,1]$ *and* $\sigma_L^2 + \sigma_K^2 - 2\rho\sigma_L\sigma_K \geq 0$*, it follows that* $\theta \leq 0$

$$F(L, K, t) \leq AL^{\alpha}K^{\beta} \quad \textit{for all} \quad t \leq T, \tag{19}$$

and consequently the production function converges from below to the Cobb-Douglas function. Equivalently, that production function is asymptotically efficient.

Remark 0.2. *If* $t = T$*, meaning that we are at the steady-state, we obtain*

$$F(L, K, T) = AL^{\alpha}K^{1-\alpha}, \tag{20}$$

which is the Cobb-Douglas production function.

Remark 0.3. *If* $\theta = 0$ *and* $t \leq T$*, we claim that*

$$F(L, K, t) = AL^{\alpha}K^{1-\alpha} \quad \textit{and} \quad \rho = \frac{\sigma_L^2 + \sigma_K^2}{2\sigma_L\sigma_K} \tag{21}$$

3. Some Properties and Concluding Remarks

3.1 As one can see from the hypotheses of the above theorem, the function $F(L, K, t)$ is not known on the interval $(0, T)$. Only the boundary condition stipulates that the analytic form of the function is Cobb-Douglas. This is one of the main hypotheses of our approach, and was inspired by the paper of Jones (2005).

3.2 We may interpret θ as a penalty coefficient. This coefficient measures "the distance" between the actual level of production and the steady state level of production. This means that under certain restrictions the production function converges from below to the Cobb-Douglas function providing further support for the conclusion drawn by Jones (2005). Equivalently, that production function is asymptotically efficient. This is the main result of our paper.

3.3 If $\alpha + \beta \neq 1$ then the production function is not Cobb-Douglas. As it is shown by Jones (see Jones, page 537) the CES setup still delivers a Cobb-Douglas global production function, at least on average, and the theory suggests that we should expect a Cobb-Douglas production function with a capital exponent of $\beta/(\alpha + \beta)$.

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