E C O N O M I C S   B U L L E T I N

# The Ramsey model with logistic population growth

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## *Abstract*

In standard economic growth theory it is assumed that labor force follows exponential growth, a not realistic assumption. As described in Maynard Smith (1974), the growth of natural populations is more accurately depicted by a logistic growth law. In this paper we analyze how the Ramsey growth model is affected by logistic growth of population, comparing it with the classic Ramsey model. We show that there is a unique nontrivial steady state of the model and that the parameters of the logistic equation play no role in determining long run equilibrium per worker level of consumption and capital. In addition, we study the stability of the model showing that its nontrivial equilibrium is a saddle point.

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# 1 Introduction

The main purpose of this short paper is to improve the classical Ramsey model of optimal economic growth ([4], [6], [9]) by modifying its population growth law. In ([1]), ([3]) and ([5]), the dynamics of the Solow-Swan growth model when the labor growth rate is non-constant but variable and bounded over time is analyzed. In this paper we reproduce this exercise for the Ramsey model.

One of the usual characteristics of any standard economic growth model is the assumption that labor force $L$ grows at a constant rate $n > 0$. In continuous time it is natural to define this growth rate as $r = \frac{\dot{L}}{L} = \frac{\frac{\partial L}{\partial t}}{L}$ which implies that the labor force grows exponentially and for any initial level $L_0$, at time $t$ the level of the labor force is $L(t) = L_0 e^{rt}$. The simple exponential growth model can provide an adequate approximation to such growth only for the initial period because, growing exponentially, as $t \to \infty$, labor force will approach infinity, which is clearly unrealistic. As labor force becomes large enough, crowding, food shortage and environmental effects come into play, so that population growth is naturally bounded. This limit for the population size is usually called the *carrying capacity* of the environment. From the other hand, it is a very well known fact that since the 1950s, population growth rate is decreasing and it is projected to go down to 0 during the next decades. This decrease, prominent in developed countries, can also be seen on a global scale. The decrease in the rate of growth is mainly due to the population aging and, consequently, a dramatic increase in number of deaths.

Then, a more realistic law of growth of the labor force $L(t)$ must verify the following properties:

1. when population is small enough in proportion to environmental carrying capacity $L_\infty$, then $L$ grows at a constant rate $r > 0$;

2. when population is large enough in proportion to environmental carrying capacity $L_\infty$, the economic resources become more scarce and this affect negatively growth of the population;

3. population growth rate is decreasing to 0.

In this paper we assume that labor force $L(t)$ verify all these properties. In particular, in section 2 we introduce the logistic law that is a very general equation verifying the previous conditions frequently used to describe and analyze population processes. Under this population growth law we obtain a generalization of the Ramsey model. In section 3 we find the equilibria and analyze the stability of the model and in section 4 we study the speed of convergence and transitional dynamics of the model. Finally, in section 5 we present some concluding remarks.

# 2 The model

Consider an economy developing over time, where $K = K(t)$ denotes capital stocks, $C = C(t)$ consumption and $Y = Y(t)$ net national product at time $t$. We suppose that labor force $L = L(t)$ (identified with population) follows a logistic law (see [10]). Then $L(t)$ is the solution of the initial value problem:

$$\begin{aligned} &\dot{L} = aL - bL^2, \ a, b > 0, \\ &L(0) = L_0, \end{aligned} \tag{1}$$

and therefore

$$L(t) = \frac{aL_0 e^{at}}{a + bL_0\left(e^{at} - 1\right)}. \tag{2}$$

A detailed description of how logistic equation describes a real population, can be found in [7]. Note that if $L_0 < \frac{a}{b} = L_\infty$ then $\dot{L}(t) > 0, \forall t \geq 0$ and $\lim\limits_{t \to +\infty} L(t) = L_\infty$ and that logistic growth rate is:

$$n(t) = \frac{\dot{L}}{L} = a - bL = \frac{a\left(a - bL_0\right)}{a + bL_0\left(e^{at} - 1\right)}, \tag{3}$$

which goes to zero in the long run.

If we have a production function $f$ and $y = \frac{Y}{L}$ and $k = \frac{K}{L}$ are income and capital per worker respectively, we can express $y$ as a function of $k$, $y = f(k)$, where as usual we suppose that:

- $f(0) = 0$;
- $f'(k) > 0, \forall k \in R^+$
- $\lim\limits_{k \to +\infty} f'(k) = 0$
- $\lim\limits_{k \to 0^+} f'(k) = +\infty$
- $f''(k) < 0, \forall k \in R^+$

We suppose that output is consumed or invested

$$Y(t) = C(t) + I(t) \tag{4}$$

and that capital stock changes $\dot{K}\left(t\right)$ equal the gross investment $I(t)$ minus the capital depreciation $\delta K\left(t\right)$:

$$\dot{K}(t) = I(t) - \delta K(t). \tag{5}$$

Hence

$$\frac{Y(t)}{L(t)} = \frac{C(t)}{L(t)} + \frac{\dot{K}(t)}{L(t)} + \frac{\delta K(t)}{L(t)} \tag{6}$$

i.e.

$$y(t) = c(t) + \frac{\dot{K}(t)}{L(t)} + \delta k(t) \tag{7}$$

where $c = \frac{C}{L}$ denotes the per capita consumption. But

$$\dot{k} = \frac{L\dot{K} - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - k\frac{\dot{L}}{L} = \frac{\dot{K}}{L} - k(a - bL) \tag{8}$$

and then

$$y = c + \dot{k} + \left(a - bL + \delta\right)k \tag{9}$$

We therefore have the condition

$$\dot{k} = f(k) - c - (a - bL + \delta)k \tag{10}$$

Suppose that $u(c)$ denotes the utility of consumption per head. As usual, we require that $u'(c) > 0$ and $u''(c) < 0$. Then the aim is to maximize the discounted value of utility subject to equations (10) and (1); i.e.

$$\begin{cases} \max_c \int_0^\infty u(c)e^{-\rho t}dt \\ \dot{k} = f(k) - c - (a - bL + \delta)k \\ \dot{L} = aL - bL^2 \\ k(0) = k_0, \ L(0) = L_0 \\ 0 \leq c \leq f(k). \end{cases} \tag{11}$$

The current value Hamiltonian of the problem is

$$\mathcal{H}_c = u(c) + \lambda \left[ f(k) - c - (a - bL + \delta)k \right] + \mu \left[ aL - bL^2 \right] \tag{12}$$

with first order conditions

$$\begin{cases} \frac{\partial \mathcal{H}_c}{\partial k} = u'(c) - \lambda = 0 \\ \dot{\lambda} = -\frac{\partial \mathcal{H}_c}{\partial k} + \rho\lambda = -\lambda \left[ f'(k) - (a - bL + \delta + \rho) \right] \\ \dot{k} = f(k) - (a - bL + \delta)k - c \\ \dot{L} = aL - bL^2 \end{cases} \tag{13}$$

If we differentiate the condition $u'(c) = \lambda$ with respect to $t$ we have that

$$u''(c)\dot{c} = \dot{\lambda} \tag{14}$$

and then it is

$$u''(c)\dot{c} = -\lambda \left[ f'(k) - (a - bL + \delta + \rho) \right] \tag{15}$$

and using that $u'(c) = \lambda$, this condition can be expressed as:

$$-\frac{u''(c)}{u'(c)}\dot{c} = f'(k) - (a - bL + \delta + \rho) \tag{16}$$

But $\sigma(c) = -\frac{cu''(c)}{u'(c)}$ is the measure of relative risk aversion of Arrow-Pratt (see [8]) and then it is

$$\dot{c} = \frac{\left[ f'(k) - (a - bL + \delta + \rho) \right] c}{\sigma(c)} \tag{17}$$

Then the model can be expressed by the following system of three differential equations:

$$\begin{cases} \dot{c} = \frac{c}{\sigma(c)} \left[ f'(k) - (a - bL + \delta + \rho) \right] \\ \dot{k} = f(k) - (a - bL + \delta)k - c \\ \dot{L} = aL - bL^2 \end{cases} \tag{18}$$

## 3 Equilibria and stability

The equilibria of the model are the solutions of the system

$$\begin{cases} \dot{c} = 0 \\ \dot{k} = 0 \\ \dot{L} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{c}{\sigma(c)}\left[f'(k) - (a - bL + \delta + \rho)\right] = 0 \\ f(k) - (a - bL + \delta)k - c = 0 \\ aL - bL^2 = 0 \end{cases} \tag{19}$$

If we exclude the trivial solutions obtained by considering $c = 0, k = 0$ or $L = 0$, then the model has a unique positive equilibrium $\left(\hat{c}, \hat{k}, \hat{L}\right)$ verifying:

$$\begin{cases} \hat{L} = \frac{a}{b} \\ f'(\hat{k}) = \delta + \rho \\ \hat{c} = f(\hat{k}) - \delta\hat{k} \end{cases} \tag{20}$$

To study the stability of the equilibrium $\left(\hat{c}, \hat{k}, \hat{L}\right)$ we consider the linear approximation of the system around this point. The Jacobian matrix of the linear approximation is given by

$$J_G\left(\hat{c}, \hat{k}, \hat{L}\right) = \begin{pmatrix} 0 & \dfrac{f''(\hat{k})\hat{c}}{\sigma(\hat{c})} & \dfrac{b\hat{c}}{\sigma(\hat{c})} \\ -1 & \rho & b\hat{k} \\ 0 & 0 & -a \end{pmatrix} \tag{21}$$

Then the characteristic polynomial of this matrix is

$$-\left(X^2 - \rho X + \frac{f''(\hat{k})\hat{c}}{\sigma(\hat{c})}\right)(X + a) \tag{22}$$

and, being $\rho > 0$ and $\dfrac{f''(\hat{k})\hat{c}}{\sigma(\hat{c})} < 0$, this polynomial has two negative roots and one positive root. Then the steady state $\left(\hat{c}, \hat{k}, \hat{L}\right)$ is a saddle point and the stable transitional path is a two dimensional locus.

## 4 Stability and transitional dynamics

The classic Ramsey model is a good approximation to the real world, nevertheless it describes the economic reality incompletely. Our model has richer dynamics. As we will show, in the modified model we can have positive growth at the equilibrium levels of capital and consumption.

### 4.1 Analysis of the steady state

When population attains the stationary level $\hat{L} = \frac{a}{b}$, dynamics at the equilibrium of capital and consumption of both models are equivalent. But the level of capital $k_g^m$ defined by the gold rule[1]

[1] $k$ follows the gold rule if it maximize the consumption at the trajectory $\dot{k} = 0$.

for the modified model is bigger than the corresponding level $k_g^c$ in the classical model. This is a consequence of (20) and the concavity of the production function.

In the modified model we can study the behavior of orbits verifying $\dot{c} = \dot{k} = 0$ and where the population is not in its stationary state. This can be seen as a pseudo-equilibrium. In this case consumption per capita follows the trajectory given by the equation $f'(k) = -bL + a + \delta + \rho$, implying that capital per capita increases with population. Similarly, $\dot{k} = 0$ implies: $c = f(k) - (a - bL + \delta)k$ and then consumption per-capita is an increasing function of the population.

### 4.2 Transitional dynamics and speed of convergence

If we assume that the population is at the steady state $\hat{L} = \frac{a}{b}$, then the unique difference of the dynamics of transition between the two models is that $k_g^m > k_g^c$. However in other case the transition of the optimal trajectories of $k$ and $c$ to the pseudo-equilibrium $\dot{k} = \dot{c} = 0$ depends on population growth.

The transitional dynamics around the steady state $\left(\hat{c}, \hat{k}, \hat{L}\right)$ can be quantified by using the linearization of system (18):

$$\begin{pmatrix} \dot{c} \\ \dot{k} \\ \dot{L} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{f''(\hat{k})\hat{c}}{\sigma(\hat{c})} & \dfrac{b\hat{c}}{\sigma(\hat{c})} \\ -1 & \rho & b\hat{k} \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} c - \hat{c} \\ k - \hat{k} \\ L - \hat{L} \end{pmatrix}$$

We know that the matrix representing this linear system has one positive eigenvalue and two negative eigenvalues. Then, there is a straight line of solutions that tend away from the equilibrium and a plane of solutions that tend toward the equilibrium as time increases. All other solution will eventually move away from the equilibrium. This give us the local dynamics of system (18) near the equilibrium $\left(\hat{c}, \hat{k}, \hat{L}\right)$: the model exhibits saddle-path stability. Then the dynamic equilibrium follows a stable saddle path expressing the equilibrium $\hat{c}$ as a function of $\hat{k}$ and $\hat{L}$. This relation is called the policy function (see [2]).

We want now to provide a quantitative assessment of the speed of transitional dynamics. The speed of convergence depends on the parameters of technology and preferences and can be computed from the matrix $J_G\left(\hat{c}, \hat{k}, \hat{L}\right)$. The positive eigenvalue of $J_G\left(\hat{c}, \hat{k}, \hat{L}\right)$ is excluded from the analysis to ensure the convergence to the equilibrium. The negative eigenvalues are the analogous to the convergence coefficient in standard growth models. The first negative eigenvalue $-a$ corresponds to the speed of convergence of population to the carrying capacity $\hat{L}$. The second negative eigenvalue $\lambda = \frac{\rho}{2} - \sqrt{\frac{\rho^2}{4} - \dfrac{f''(\hat{k})\hat{c}}{\sigma(\hat{c})}}$ is identical with the negative eigenvalue for the basic Ramsey model (see [2]) and depends on parameters of technology and preferences. Each eigenvalue corresponds to one source of convergence and each stable transition path to the steady state of the system takes the form

$$\begin{cases} c(t) = \hat{c} + xe^{\lambda t} + (c(0) - \hat{c} - x)\, e^{-at} \\ k(t) = \hat{k} + ye^{\lambda t} + \left(k(0) - \hat{k} - y\right) e^{-at} \\ L(t) = \hat{L} + \left(L(0) - \hat{L}\right) e^{-at} \end{cases} \tag{23}$$

where $x$ and $y$ depends on the coefficients of $J_G\left(\hat{c}, \hat{k}, \hat{L}\right)$. Then the speed of convergence of consumption and capital depends on eigenvalues $-a$ and $\lambda$ and that of population only depends on $-a$. In fact, their transition depend on the smaller of the eigenvalue in absolute value. If $|\lambda| < a$, then the speed of convergence of $L$ is faster than that of $c$ and $k$ and if $|\lambda| \geq a$ then all variables converge at speed $a$. In any case, the speed of convergence of the system is the absolute value of the biggest negative eigenvalue.

## 5 Concluding Remarks

In growth theory it is usually assumed that population growth follows an exponential law. This is clearly unrealistic because it implies that population goes to infinity when time goes to infinity. In this paper we have developed an improved version of the Ramsey growth model suggesting a more realistic approach by considering that population growth follows the logistic law. In this case, the model can be represented by an autonomous system of three differential equations and one of the merits of the paper is in showing that the model is tractable. We show that the intrinsic rate of population growth $n(t)$ plays no role in determining the long run equilibrium levels of per capita consumption, capital and output and that these values are greater than the steady states values of the classical Ramsey model. Thus, in the long run, economic growth is improved if labor force growth rate decreases to zero. This can be viewed as a motivation for policy makers to have an efficient population growth rate. We study the stability of the system showing that the equilibrium is a saddle point with two negative eigenvalues. This implies that the model is saddle stable and that the stable transitional path is two dimensional. Finally, we show that the model has finite speed of convergence, which depends on the parameters of technology and preferences.

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