E C O N O M I C S   B U L L E T I N

A pitfall in joint stationarity, weak exogeneity and autoregressive distributed lag models

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Abstract

We prove weak exogeneity is an not an impossibility with an ADL structure in the marginal and in the conditional. We show that joint stationarity requirements is driving such common belief

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1 Introduction

It is sometimes argued that an autoregressive distributed lag conditional model is incompatible with weak exogeneity. Below, we proof this does not need to be the case, unless joint stationarity is imposed in the marginal and in the conditional model. It shall be obvious that imposing joint stationarity will create a cross link between parameters, whilst imposing stationarity in the marginal and stationarity in the conditional allows the parameter space to remain the cartesian product of the respective parameter spaces. Hence, several literature claims (see, *inter alia*, Psaradakis and Sola 1996) are incorrect in this domain.

2 Weak Exogeneity and Joint Stationarity

Consider the bivariate DGP:

$$\begin{pmatrix} y_t \\ z_t \end{pmatrix} |X_{t-1} \sim N_2 \left[\begin{pmatrix} \pi_{10} \\ \pi_{20} \end{pmatrix} + \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \begin{pmatrix} y_{t-1} \\ z_{t-1} \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \right] \tag{1}$$

,where $x_t = (y_t : z_t)'$, $t = 1, ..., T$, and hence $X_{t-1}$is the information set containing the history of y_t and z_t, such that:

$$X_{t-1} = (Y_{t-1} : Z_{t-1})' \tag{2}$$

It is well known that the joint density in (1) may be factorized into the product of a conditional and a marginal density:

$$D_X (y_t, z_t|X_{t-1}; \theta) = D_{y|z} (y_t|z_t, X_{t-1}; \phi_1) D_z (z_t|X_{t-1}; \phi_2) \tag{3}$$

where $\theta \in \Theta$, the parameter space for the joint density, $\phi_1 \in \Phi_1 \wedge \phi_2 \in \Phi_2$, where $\Phi_1$and Φ_2 are the spaces for the parameters in the conditional density and in the marginal density, respectively. Weak exogeneity (Engle, Hendry and Richard, 1983) entails that, given a parameter of interest ψ, $\psi = h(\phi_1)$ and ϕ_1 and ϕ_2 are variation-free, that is the joint space A is the cartesian product of the individual spaces $A = \{(\phi_1; \phi_2) : \phi_1 \in \Phi_1 \wedge \phi_2 \in \Phi_2\}$.Given the normality assumption imposed in (1), it follows that both the marginal and the conditional densities in (3) are gaussian. Therefore,

$$y_t|z_t, X_{t-1} \sim N \left[\begin{array}{c} \pi_{10} + \pi_{11}y_{t-1} + \pi_{12}z_{t-1}+ \\ +\sigma_{12}\sigma_{22}^{-1} \left(z_t - \pi_{20} - \pi_{21}y_{t-1} - \pi_{22}z_{t-1}\right); \sigma_{11} - \sigma_{12}\sigma_{22}^{-1}\sigma_{12} \end{array} \right] \tag{4}$$

and also,

$$z_t|X_{t-1} \sim N \left[\pi_{20} + \pi_{21}\pi_{21}y_{t-1} + \pi_{22}z_{t-1}; \sigma_{22}\right] \tag{5}$$

where (4) is the conditional density $D_{y|z}$ in (3), and (5) is the marginal density D_z in (3). It follows that (6) is the conditional expectation of y_t:

$$E\left[y_t|z_t, X_{t-1}\right] = \pi_{10}+\pi_{11}y_{t-1}+\pi_{12}z_{t-1}+\sigma_{12}\sigma_{22}^{-1}\left(z_t - \pi_{20} - \pi_{21}y_{t-1} - \pi_{22}z_{t-1}\right) \tag{6}$$

and that (7) is the marginal model for $z_t|X_{t-1}$:

$$z_t|X_{t-1} = \pi_{20} + \pi_{21}y_{t-1} + \pi_{22}z_{t-1} + \varepsilon_{z,t} \tag{7}$$

where $\varepsilon_{z,t} \sim IN\left(0; \sigma_{22}\right)$.

Collecting terms in (6), we obtain:

$$\begin{aligned} E\left[y_t|z_t, X_{t-1}\right] &= \pi_{10} - \sigma_{12}\sigma_{22}^{-1}\pi_{20} + \left(\pi_{11} - \sigma_{12}\sigma_{22}^{-1}\pi_{21}\right)y_{t-1} + \\ &\quad + \left(\pi_{12} - \sigma_{12}\sigma_{22}^{-1}\pi_{22}\right)z_{t-1} + \sigma_{12}\sigma_{22}^{-1}z_t \end{aligned} \tag{8}$$

which we can rewrite as:

$$E\left[y_t|z_t, X_{t-1}\right] = \beta_0 + \beta_1 y_{t-1} + \beta_2 z_{t-1} + \beta_3 z_t \tag{9}$$

Equation (9) leads to the conditional econometric model:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 z_{t-1} + \beta_3 z_t + \varepsilon_t \tag{10}$$

where $\beta_0 = \pi_{10} - \sigma_{12}\sigma_{22}^{-1}\pi_{20}$, $\beta_1 = \pi_{11} - \sigma_{12}\sigma_{22}^{-1}\pi_{21}$,$\beta_2 = \pi_{12} - \sigma_{12}\sigma_{22}^{-1}\pi_{22}$ and $\beta_3 = \sigma_{12}\sigma_{22}^{-1}$. Furthermore, $\varepsilon_t \sim IN\left(0; \sigma_\varepsilon^2\right)$, $\sigma_\varepsilon^2 = \sigma_{11} - \sigma_{12}\sigma_{22}^{-1}\sigma_{12}$. The parameter set for the conditional is: $\phi_1 = \left(\beta_0 : \beta_1 : \beta_2 : \beta_3 : \sigma_\varepsilon^2\right)'$, whereas for the marginal model we have $\phi_2 = \left(\pi_{20} : \pi_{21} : \pi_{22} : \sigma_{22}\right)'$. So ϕ_1 is the vector of parameters of interest.

Theorem 1 *Joint Stationarity of the conditional and the marginal model contradicts weak exogeneity.*

Proof. Assume for simplicity that $\sigma_{12} = 0$. From (7) and (10), it follows that,

$$y_t = \pi_{10} + \pi_{11} y_{t-1} + \pi_{12} \left(\pi_{20} + \pi_{21} y_{t-2} + \pi_{22} z_{t-1} \right) + v_t \tag{11}$$

Joint stationarity (as used in, e.g. Psaradakis and Sola (1996) would then entail that the roots of the lag polynomical

$$(1 - \pi_{11} L - \pi_{12} \pi_{21} L^2) \tag{12}$$

whould all be outside the unit circle. Since $\pi_{21} \in \phi_2$, it follows that joint stationarity imposes a cross equation restriction that violates weak exogeneity. Indeed, under the condition that the roots of (12) would all lie outside the unit circle, ϕ_1 and ϕ_2 would no longer be variation free. The cross link between parameter spaces is even more obvious for $\sigma_{12} = 0$. ■

Claim 1 *The ADL (1,1) structure in the marginal and in the conditional does not impose per se a violation of weak exogeneity.*

Proof. Consider in models (7) and (10) the conditions $|\pi_{11}| < 1 \wedge |\pi_{21}| < 1$, maintaining the assumption $\sigma_{12} = 0$. In this case, it follows that both the marginal and the conditional are stationary, irrespective of their ADL(1,1) structure. Such a restriction does now fail to impose a violation of weak exogeneity, since the resulting joint parameter space is rectangular. In this case, $\sigma_{12} = 0$ allows the ADL structure and weak exogeneity to coexist with stationarity of marginal and conditional. If $\sigma_{12} \neq 0$ but the parameters of interest is β_3, weak exogeneity also holds. ■

3 Conclusion

The claim made in some literature that the ADL(1,1) conditional model would necessarily entail failure of weak exogeneity is shown to be false. Rather, we show that failure of weak exogeneity in these settings is due to joint stationarity (substituting the lagged marginal in the conditional and finding the roots of the lag polynomial). We show that sationarity of both models can coexist with weak exogeneity.

References

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