E C O N O M I C S   B U L L E T I N

On a Nonlinear Feedback Strategy Equilibrium of a Dynamic Game

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Abstract

This paper reports an intriguing property of a nonlinear feedback Nash strategy equilibrium in a dynamic game with no state variable in the payoff of each player. While the open-loop Nash and linear feedback Nash equilibria coincide with the static Cournot-Nash equilibrium in such a framework, the nonlinear feedback strategy can be properly defined and, furthermore, a particular type of the equilibrium outcomes approximates the bilateral collusion, as is originally proved by Tsutsui and Mino (1990) for a standard differential game with one state variable.

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1 Introduction

The purpose of this note is to report an interesting property of a nonlinear feedback Nash strategy equilibrium of a dynamic game. In our dynamic game model of pollution, no state variable enters the objective functionals of the players, which are two polluting firms. At this stage, it would be natural to presume that both the open-loop and feedback equilibria coincide with the static solution due to the lack of a state variable in their respective payoffs. However, we demonstrate that this conjecture may not hold true if a *nonlinear* feedback strategy is employed while it applies to the equilibria supported by open-loop and *linear* feedback strategies.[1]

The paper is planned as follows. Section 2 lays out our basic model and derives its open-loop Nash equilibrium. After obtaining the linear and nonlinear feedback Nash equilibria, we formally prove this somewhat counter-intuitive result and discuss its cause and economic implications in Section 3. Section 4 concludes the paper.

2 The Model: Open-Loop Nash Equilibrium

2.1 The Model

We consider a homogeneous good duopoly consisting of firms 1 and 2, both of which are completely identical in all characteristics. As a by-product, each firm emits a pollutant, whose amount is proportional to its output level. The inverse demand function of the good is specified linearly as:

$$p = a - x_1 - x_2, \quad a > 0$$

[1] The derivation of a nonlinear feedback strategy equilibrium in this paper follows that of Tsutsui and Mino (1990). For technical details, see Dockner *et al.* (2000).

where p is the price of the good and $x_i, i = 1, 2$, denotes each firm's output. The pollutant is assumed to accumulate in an environmental body according to

$$\dot{Z} = x_1 + x_2 - \delta Z, \quad \delta \in [0, 1], \tag{1}$$

where Z is the pollution stock and δ is its decay rate. Then, letting $c > 0$ be the marginal cost of production of each firm, firm i's problem can be formulated, in a dynamic form, as:

$$\max_{x_i} \quad \int_0^\infty e^{-\rho t}(a - c - x_i - x_j)x_i dt, \quad \text{subject to} \quad (1), \quad \rho > 0,$$

where ρ is the discount rate. Note that these firms do not suffer from the pollution problem.

2.2 Open-Loop Nash Equilibrium

Let us begin by solving the Nash equilibrium of the above model under open-loop formulation. To this end, let us set up firm i's current value Hamiltonian:

$$H_i = (a - c - x_i - x_j)x_i + \lambda_i(x_i + x_j - \delta Z),$$

where λ_i denotes firm i's co-state variable associated with (1). Then, the optimality conditions consist of (1) and

$$0 = a - c - 2x_i - x_j + \lambda_i \tag{2}$$

$$\dot{\lambda}_i = \lambda_i(\rho + \delta) \tag{3}$$

$$0 = \lim_{t \to \infty} e^{-\rho t} \lambda_i Z.$$

Solving the system of equations made of (2) for firm i as well as firm j, the equilibrium output is given by

$$x_i = \frac{a - c + 2\lambda_i - \lambda_j}{3}. \tag{4}$$

Hence, upon substituting (4) into (1), the present system becomes three-dimensional:

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} \rho + \delta & 0 & 0 \\ 0 & \rho + \delta & 0 \\ \frac{1}{3} & \frac{1}{3} & -\delta \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ Z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{2(a-c)}{3} \end{bmatrix}. \tag{5}$$

Based on (5), the following property of the open-loop Nash equilibrium can be obtained:

Proposition 1. *The steady state in the symmetric open-loop Nash equilibrium uniquely exists and is saddle point stable. Moreover, the optimal output is invariant to time and the pollution stock.*

Proof. In the steady state where $\dot{\lambda_1} = \dot{\lambda_2} = \dot{Z} = 0$, $\lambda_1 = \lambda_2 = 0$ can be easily confirmed from (3). Substituting this into (4), each firm's equilibrium output level becomes

$$x^O = \frac{a-c}{3}, \tag{6}$$

where the superscript O indicates the open-loop Nash equilibrium. (6) tells us that the optimal output depends neither on time nor on the state variable. Further substitution of (6) into the equation of $\dot{Z} = 0$, the steady state stock of the pollutant is

$$Z^O = \frac{2(a-c)}{3\delta} \tag{7}$$

The saddle point stability can be checked easily as well. Letting the eigenvalue associated with the coefficient matrix in (5) denoted by y, the characteristic equation is defined as

$$\begin{vmatrix} \rho+\delta-y & 0 & 0 \\ 0 & \rho+\delta-y & 0 \\ \frac{1}{3} & \frac{1}{3} & -\delta-y \end{vmatrix} = (-\delta-y)(\rho+\delta-y)^2 = 0,$$

from which the two eigenvalues are positive and given by $y = \rho+\delta$, and the other is negative and given by $y = -\rho$. Since (5) contains one state variable and two jump variables, this establishes the saddle point stability. **Q. E. D.**

3 Feedback Nash Equilibria

3.1 Feedback Formulation

This section turns to another solution concept: a feedback Nash equilibrium. In order to pay attention to not only a linear feedback strategy but also a nonlinear feedback strategy, we adopt the derivation method of a nonlinear feedback strategy equilibrium introduced by Tsutsui and Mino (1990). It begins by defining each player's Hamilton-Jacobi-Bellman equation:

$$\rho V(Z) = \max_{x_i} \left\{ [a - c - x_i - x_j(Z)]x_i + V'(Z)[x_i + x_j(Z) - \delta Z] \right\}, \tag{8}$$

where the function $V(\cdot)$ is firm i's value function:[2]

$$V(Z) \equiv \max_{x_i} \left\{ \int_t^\infty e^{-\rho(s-t)}[a - c - x_i - x_j(Z)]x_i ds \;\middle|\; \dot{Z} = x_i + x_j(Z) - \delta Z \right\}.$$

The first-order condition for maximizing the right-hand side of (8), combined with the symmetry condition, $x_i = x_j$, yields

$$V'(Z) = 3x(Z) - (a - c), \tag{9}$$

Then, substituting (9) into (8), we have an identity in Z:

$$\rho V(Z) = [a - c - 2x(Z)]x(Z) + [3x(Z) - (a - c)][2x(Z) - \delta Z].$$

Differentiating both sides with respect to Z and rearranging the terms, we have the following auxiliary equation:

$$x'(Z) = \frac{(\rho + \delta)[3x(Z) - (a - c)]}{8x(Z) - 3\delta Z - (a - c)}, \tag{10}$$

which gives the slope of an uncountable number of feedback strategies. Resorting to the diagrammatic method of Tsutsui and Mino (1990) and Dockner and Long (1993), such nonlinear strategies can be depicted by the integral curves in Figures 1 and 2.

3.2 Linear Feedback Nash Equilibrium

Before moving on to a nonlinear feedback strategy equilibrium, let us briefly consider some properties of the linear feedback Nash equilibrium. The linear strategy can be obtained as follows. Let us assume that the strategy is linearly dependent on the state variable: $x(Z) = \alpha Z + \beta$. In such a case, $x'(Z) = \alpha$ and (10) takes the form of

$$\alpha = \frac{(\rho + \delta)[3(\alpha Z + \beta) - (a - c)]}{8(\alpha Z + \beta) - 3\delta Z - (a - c)},$$

[2]The subscript i to the value function is dropped for notational simplicity.

which is equivalent to

$$\alpha(8\alpha - 3\rho - 6\delta)Z + (8\alpha - 3\rho - 3\delta)\beta - (\alpha - \rho - \delta)(a - c) = 0.$$

The two unknown coefficients α and β must satisfy this equation, which lead to the following pairs of α and β:

$$(\alpha, \beta) = \left(0, \frac{a-c}{3}\right), \quad \left(\frac{3\rho + 6\delta}{8}, -\frac{(5\rho + 2\delta)(a - c)}{24\delta}\right).$$

While the former pair corresponds to x_b in Figure 2 and the latter pair to x_a, only x_b survives the condition of asymptotic stability. As a result, we can state:

Proposition 2. *Suppose that each firm plays a linear feedback strategy. Then, the resulting Nash equilibrium output is invariant to time and the pollution stock, and it is identical with the static Cournot-Nash outcome.*

3.3 Nonlinear Feedback Nash Equilibrium

Finally, we explore the implication of a nonlinear feedback strategy equilibrium. While there are numerous nonlinear feedback strategies, we focus on one particular strategy, which is denoted by x^N in Figure 2. If the domain of the initial pollution stock is properly defined, x^N can be supported by an equilibrium strategy. Then, it converges to N over time. At N, the steady state condition implies that $x = \delta Z/2$ and the slope of x^N is equal to that of the $\dot{Z} = 0$ line. Hence, substituting these results into (10), we have

$$\frac{(\rho + \delta)\left[\frac{3\delta Z}{2} - (a - c)\right]}{\delta Z - (a - c)} = \frac{\delta}{2},$$

where the left-hand side gives the slope of x^N at $x = \delta Z/2$, whereas the right-hand side is the slope of the $\dot{Z} = 0$ line. Solving this equation for Z, the steady state stock of pollution is immediately obtained as

$$Z^N = \frac{(2\rho + \delta)(a - c)}{\delta(3\rho + 2\delta)}, \tag{11}$$

and the corresponding output in the steady state becomes

$$x^N = \frac{(2\rho + \delta)(a - c)}{2(3\rho + 2\delta)}, \tag{12}$$

where the superscript N refers to this particular nonlinear feedback Nash equilibrium.

Then, from (12), we can state a seemingly surprising result:

Proposition 3. *As $\rho \to 0$, this equilibrium output approaches the static monopoly output in the steady state.*

Proof. Through a simple calculation, the monopoly output is given by $(a-c)/2$. On the other hand, under $\rho \to 0$, (12) simplifies to

$$x^N\Big|_{\rho \to 0} = \frac{a-c}{4},$$

and hence the industry output is $(a-c)/2$, which is nothing but the monopoly output. **Q. E. D.**

It might appear at first that all of the open-loop, linear feedback, and nonlinear feedback strategy equilibrium outcomes coincide with one another and are the same as the static solution due to the lack of the state variable in the payoff function. However, Proposition 3 states that such a view is incorrect. Then, a question arises: why does a nonlinear strategy equilibrium outcome deviate from the static solution?

In the open-loop formulation, no state variable enters each player's objective function literally. In contrast, in the feedback formulation, firm i's problem becomes

$$\begin{array}{rl} \max\limits_{x_i} & \displaystyle\int_0^\infty e^{-\rho t} \left[a - c - x_i - x_j(Z)\right] x_i dt \\ \text{subject to} & \dot{Z} = x_i + x_j(Z) - \delta Z. \end{array}$$

That is, the state variable indirectly enters each player's payoff since player i seeks to maximize its payoff by anticipating that its rival's strategy is a function of the state variable. As a result, this indirect influence of the state variable on player i's decision can make a nonlinear strategy equilibrium different from the open-loop and linear feedback strategy equilibrium. The same can be observed in any game situation where each player i's payoff depends not only on its own choice variable but on its rival's choice variable. This implies that the all of the equilibrium outcomes above coincide with the static solution when each player's payoff depends only on its own control variable and no state variable.

Concerning the implications of this result, it suffices to cite Tsutsui and Mino's (1990) statement in a context of dynamic duopoly: $\cdots$ *as the discount rate r approaches to zero, the upper bound p_H asymptotically approaches the collusive stationary price p^*_{joint}* (Tsutsui and Mino, 1990, p. 154).[3] What should be stressed here is that the same observation can hold even when no state variable enters each player's payoff function. Hence, the scope of their finding is actually larger than is generally known. Indeed, the monopoly outcome can be achieved in the limiting case with $\rho \to 0$ according to Proposition 3. When a stock pollution issue is a significant concern of a government, one way to ameliorate the pollution problem would be simply to encourage firms to employ non-linear feedback strategies and let them act non-cooperatively.

4 Concluding Remarks

In this article, we have demonstrated that, even in a dynamic game model where no state variable enters the players' objective function, a *nonlinear* feedback strategy equilibrium can deviate from the static solution while the open-loop and *linear* feedback strategy equilibria coincide with the static solution. This seemingly counter-intuitive result originates from the fact that the state variable enters player i's objective indirectly through its anticipation of the rival's strategy. Our result indicates broader applicability of the novel finding of Tsutsui and Mino (1990), i.e., a certain feedback strategy equilibrium approximates the monopoly outcome, provided that the discount rate is sufficiently small.

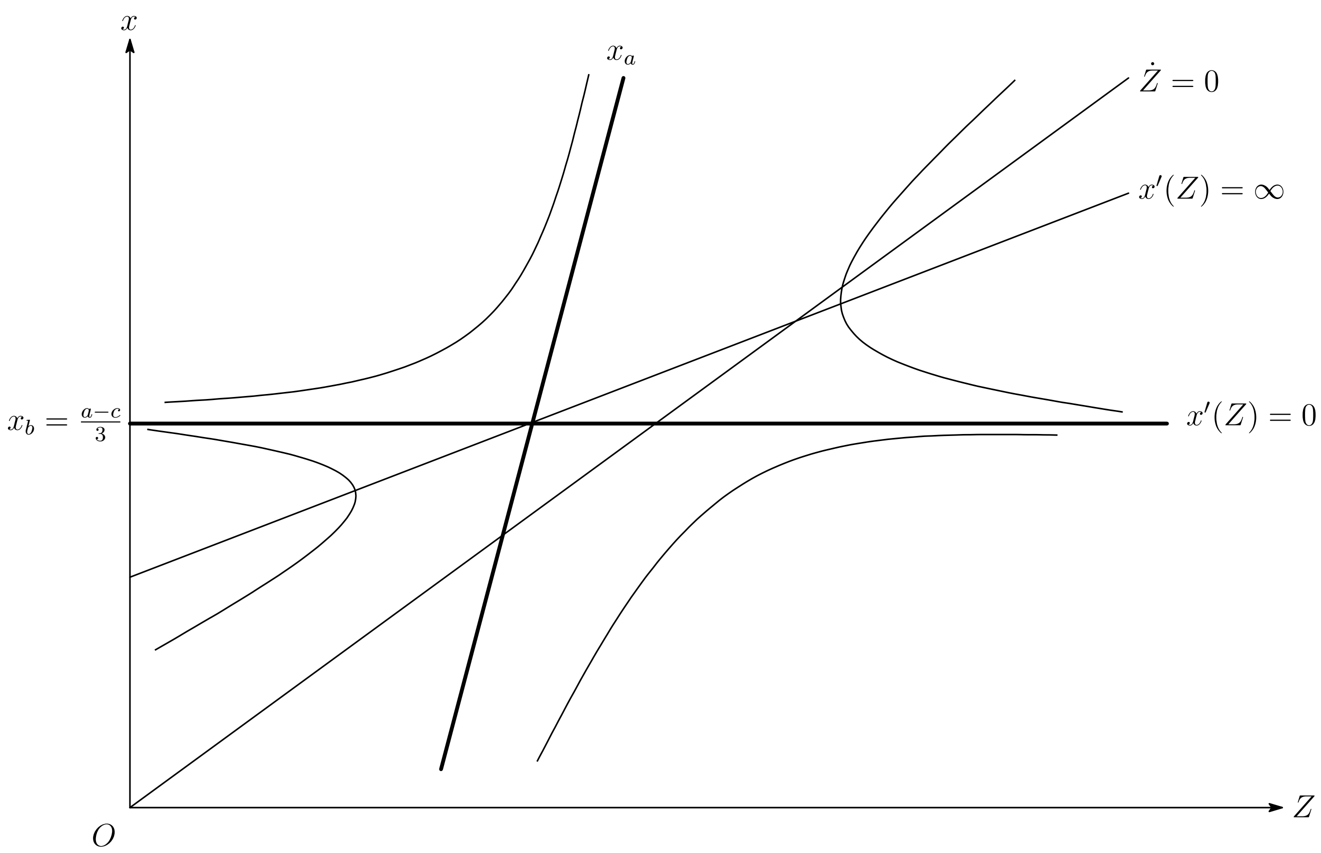
References

[1] Dockner, E. J. and N. V. Long (1993), 'International Pollution Control: Cooperative versus Noncooperative Strategies', *Journal of Environmental Economics and Management*, 24, 13-29.

[3] In a model of international pollution control, Dockner and Long (1993) also state, '$\cdots$ *in the limiting case when r tends to zero, the collusive long-run pollution stock can be supported as a steady state of nonlinear differential Markov strategies* $\cdots$'.

[2] Dockner, E. J., S. Jorgensen, N. V. Long and G. Sorger (2000), *Differential Games in Economics and Management Science*, Cambridge: Cambridge University Press.

[3] Tsutsui, S. and K. Mino (1990), 'Nonlinear Strategies in Dynamic Duopolistic Competition with Sticky Prices', *Journal of Economic Theory*, 52, 136-161.



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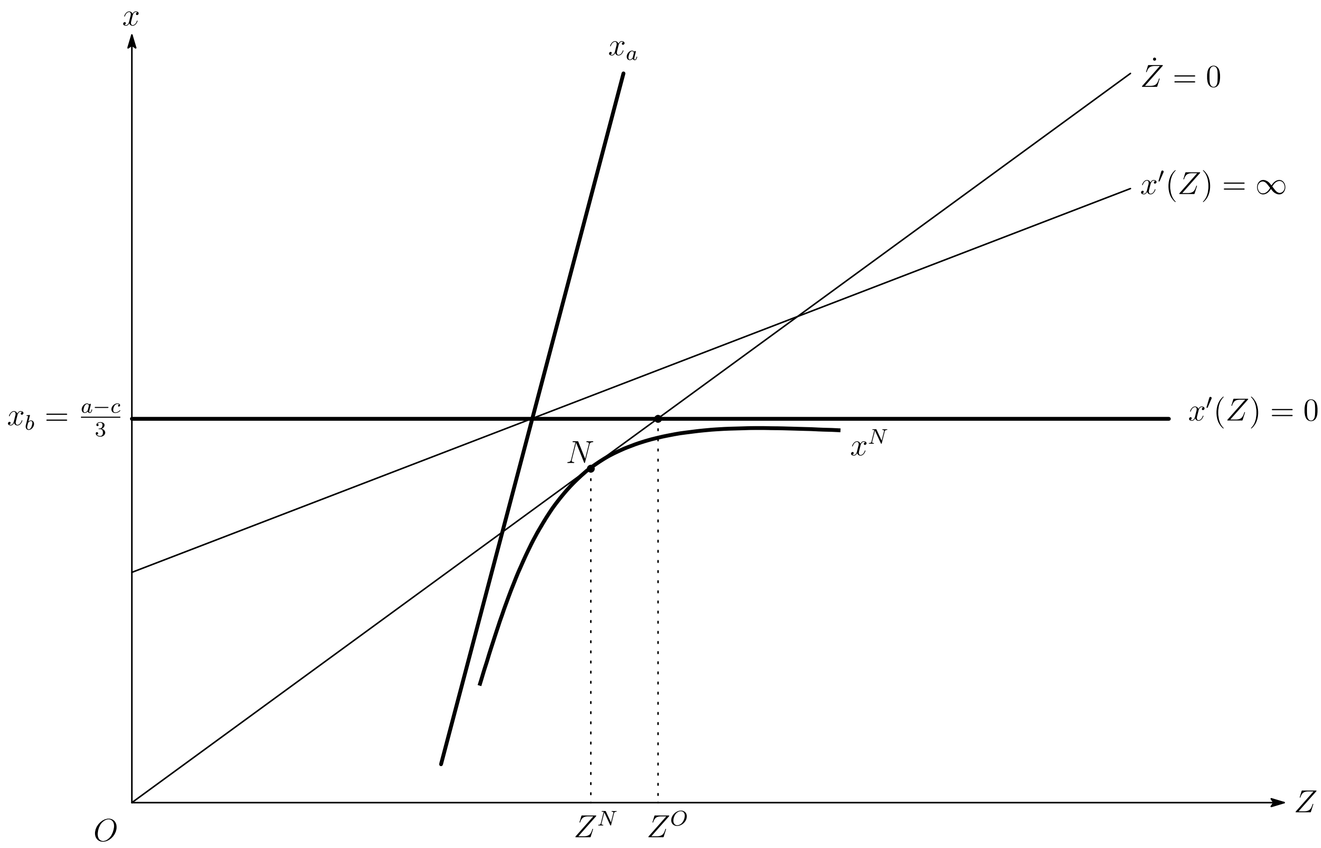
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