E C O N O M I C S   B U L L E T I N

Nested Potential Games

Hiroshi Uno
Graduate School of Economics, Osaka University

Abstract

This paper proposes a new class of potential games, the nested potential games, which generalize the potential games defined in Monderer and Shapley (1996), as well as the pseudo-potential games defined in Dubey et al. (2006). We show that each maximizer of a nested potential is a Nash equilibrium.

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1 Introduction

This paper proposes a class of nested potential games that generalize potential games introduced in Monderer and Shapley (1996).

In the literature, various classes of potential games have been proposed and analyzed.[1] These potential games have in common an attractive feature that every maximizer of a potential, a real valued function over the set of action profiles, is a (pure) Nash equilibrium of the game.[2] That is, in potential games, the problem of finding a Nash equilibrium is a simple maximization problem rather than a fixed point problem.[3]

This paper shows that the above feature is inherited by some games that do not have a potential in the conventional sense. Specifically, we propose a new concept of 'nested' potentials as follows. In a potential game, it is as if the potential is a payoff function of one representative agent who chooses strategies for all players. In considering a nested potential, we think of a representative agent for a subset T of players instead of all of them: for each player i in T, given any strategy profile of other players, maximizing this representative agent's payoff f_T yields a best response for player i.

Suppose that there is a partition $\mathcal{T}$ of players such that, for each member T of $\mathcal{T}$, there is such a representative agent whose payoff function is f_T. Then the collection of f_T's can be seen as a new strategic form game, where each member T in $\mathcal{T}$ is regarded as a single player. That is, the original game is reduced to a game with a smaller number of players.

Notice that such reduction can be nested: the new game among step 1 representative agents may be reduced to a game with an even smaller number of players, by considering a step 2 representative agent for step 1 representative agents, and then a representative agent of these, and so on. We say that a game has a nested potential if a game is reduced to a game with one representative agent through this process.

This paper shall show that the nested potentials are natural generalizations of the existing notions of potentials:

- The nested potential games strictly expand the class of pseudo-potential games defined in Dubey *et al.* (2006) (see Example 2.5).

[1] For example, exact potentials, weighted potentials, ordinal potentials, generalized ordinal potentials are introduced in Monderer and Shapley (1996); (ordinal) best response potentials in Voorneveld (2000); pseudo-potentials in Dubey *et al.* (2006); best response potentials and better response potentials in Morris and Ui (2004); generalized potentials, monotone potentials, and local potentials in Morris and Ui (2005); iterated potentials in Oyama and Tercieux (2004), and so on.

[2] An exception is the generalized potential defined in Morris and Ui (2005), which is a function on a covering of the set of action profiles.

[3] Rosenthal (1973) first shows that there exists a function which is later called a potential in congestion games and demonstrates that every congestion game possesses a pure Nash equilibrium which corresponds to the maximizer of the function. It is also known that potential games admit refinements of equilibria. See for example Blume (1993), Hofbauer and Sorger (1999,2002), Morris and Ui (2005), Oyama *et al.* (2003), Oyama and Tercieux (2004), and Ui (2000).

- In a nested potential game, each maximizer of the nested potential is a Nash equilibrium (see Proposition 2.8).

The organization of this paper is as follows. Section 2 introduces the nested potentials. It is shown that the nested potential is a strict generalization of the pseudo-potential and that its maximizer is a Nash equilibrium. Section 3 shows that a similar nesting procedure does not expand the class of weighted potential games.

2 Nested pseudo-potential games

A strategic form game consists of a finite player set $N = \{1, \ldots, n\}$, a finite or infinite action set A_i for $i \in N$, and the payoff function $g_i : A \to \mathbb{R}$ for $i \in N$, where $A := \prod_{i \in N} A_i$. Since we fix the set A of action profiles, we denote a strategic form game $(N, (A_i)_{i \in N}, (g_i)_{i \in N})$ simply by $\mathbf{g}^N := (g_i)_{i \in N}$. For notational convenience, we write $a = (a_i)_{i \in N} \in A$; for $i \in N$, $A_{-i} = \prod_{j \neq i} A_j$ and $a_{-i} = (a_j)_{j \neq i} \in A_{-i}$; and for $T \subseteq N$, $A_T = \prod_{i \in T} A_i$, $a_T = (a_i)_{i \in T} \in A_T$, $A_{-T} = \prod_{i \in N \setminus T} A_i$, and $a_{-T} = (a_i)_{i \in N \setminus T} \in A_{-T}$. We write $(a_T, a_{-T}) \in A_T \times A_{-T}$. We write (a_i, a_{-i}) instead of $(a_{\{i\}}, a_{-\{i\}})$ for simplicity.

Let $\mathbf{g}^N$ be a strategic form game. Beginning with Monderer and Shapley (1996), various notions of potential games have been proposed. Among them, one of the weakest notions is the pseudo-potential games introduced in Dubey *et al.* (2006). A pseudo-potential of game $\mathbf{g}^N$ is a real valued function f on the set A of action profiles such that, for each player i, i's best-response against the other players' actions a_{-i} in the alternative game where i's payoff function is given by f is that in the original game $\mathbf{g}^N$:

Definition 2.1 A function $f : A \to \mathbb{R}$ is a *pseudo-potential* of $\mathbf{g}^N$ if, for each $i \in N$,

$$\arg\max_{a_i \in A_i} f(a_i, a_{-i}) \subseteq \arg\max_{a_i \in A_i} g_i(a_i, a_{-i}) \tag{1}$$

for all $a_{-i} \in A_{-i}$. If $\mathbf{g}^N$ has a pseudo-potential, $\mathbf{g}^N$ is called a *pseudo-potential game.*[4]

We shall extend this idea by introducing a weaker notion where a 'pseudo-potential' is considered for each subset of players instead of the entire set. For a partition $\mathcal{T}$ of N, we define the $\mathcal{T}$-pseudo-potentials as follows:

Definition 2.2 Let $\mathcal{T}$ be a partition of N. A *$\mathcal{T}$-pseudo-potential* of $\mathbf{g}^N$ is a tuple $(\mathcal{T}, (A_T)_{T \in \mathcal{T}}, (f_T)_{T \in \mathcal{T}})$, where, for each $T \in \mathcal{T}$, $f_T : A \to \mathbb{R}$ satisfies

[4] If the inclusion of (1) can be replaced by the equality, f is called an (ordinal) *best-response potential*, which is introduced in Voorneveld (2000). The pseudo-potentials generalize the (ordinal) best-response potentials. Morris and Ui (2004, 5) also introduced alternative best-response potentials, which are special classes of (ordinal) best-response potentials of Voorneveld (2000) and the pseudo-potentials in Dubey *et al.* (2006). See Morris and Ui (2004) for more discussion of this notion. We can apply the analogous arguments in this section to these best response potential of Morris and Ui (2004).

that, for each $i \in T$,

$$\arg\max_{a_i \in A_i} f_T(a_i, a_{-i}) \subseteq \arg\max_{a_i \in A_i} g_i(a_i, a_{-i}) \tag{2}$$

for all $a_{-i} \in A_{-i}$.[5]

We denote such a $\mathcal{T}$-pseudo-potential $(\mathcal{T}, (A_T)_{T\in\mathcal{T}}, (f_T)_{T\in\mathcal{T}})$ by $\mathbf{f}^{\mathcal{T}} := (f_T)_{T\in\mathcal{T}}$ since action sets $(A_T)_{T\in\mathcal{T}}$ can be derived from the partition $\mathcal{T}$ of N and the set A of action profiles in the original game $\mathbf{g}^N$.

Remark 2.3 The $\mathcal{T}$-pseudo-potential generalizes Monderer (2007)'s q-potential: a strategic form game $\mathbf{g}^N$ has a q-potential if and only if $\mathbf{g}^N$ has a $\mathcal{T}$-potential, where q refers to the number of elements in $\mathcal{T}$ and the potential is meant to be the exact potential in Monderer and Shapley (1996). If $\mathbf{g}^N$ is a q-potential game, then it has a $\mathcal{T}$-pseudo-potential such that the number of elements of $\mathcal{T}$ is q. The converse is not true, since there is a pseudo-potential game without an exact potential.

Notice that we can regard each $\mathcal{T}$-pseudo-potential $\mathbf{f}^{\mathcal{T}}$ as a strategic form game, where $\mathcal{T}$ is the player set; for each $T \in \mathcal{T}$, A_T is the action set of T; and for each $T \in \mathcal{T}$, f_T is the payoff function of T. The idea of the nested pseudo-potential games is to construct such games iteratively:

Definition 2.4 A function $f : A \to \mathbb{R}$ is a *nested pseudo-potential* of $\mathbf{g}^N$ if there exist a positive integer K and a sequence $(\mathbf{f}^{\mathcal{T}^k})_{k=0}^K = ((f_T^k)_{T\in\mathcal{T}^k})_{k=0}^K$ such that

- $\{\mathcal{T}^k\}_{k=0}^K$ is a nest sequence of N: $\{\mathcal{T}^k\}_{k=0}^K$ is an increasingly coarser sequence of partitions of N with $\mathcal{T}^0 = \{\{i\}|i \in N\}$ and $\mathcal{T}^K = \{N\}$;
- $\mathbf{f}^{\mathcal{T}^0} = (f_T^0)_{T\in\mathcal{T}^0}$ is the original game $\mathbf{g}^N$: for each $i \in N$, $f_{\{i\}}^0(a) = g_i(a)$ for all $a \in A$;
- for each $k = 1, 2, \ldots, K$, $\mathbf{f}^{\mathcal{T}^k} = (f_T^k)_{T\in\mathcal{T}^k}$ is a $\mathcal{T}^k$-pseudo-potential of $\mathbf{f}^{\mathcal{T}^{k-1}} = (f_T^{k-1})_{T\in\mathcal{T}^{k-1}}$, where $\mathbf{f}^{\mathcal{T}^{k-1}}$ is regarded as a strategic form game as above: for each $T^k \in \mathcal{T}^k$ and for each $T^{k-1} \in \mathcal{T}^{k-1}$ with $T^{k-1} \subseteq T^k$,

  $$\arg\max_{a_{T^{k-1}} \in A_{T^{k-1}}} f_{T^k}^k(a_{T^{k-1}}, a_{-T^{k-1}}) \subseteq \arg\max_{a_{T^{k-1}} \in A_{T^{k-1}}} f_{T^{k-1}}^{k-1}(a_{T^{k-1}}, a_{-T^{k-1}}) \tag{3}$$

  for all $a_{-T^{k-1}} \in A_{-T^{k-1}}$; and
- $\mathbf{f}^{\mathcal{T}^K} = (f_N^K)$ is such that $f_N^K(a) = f(a)$ for all $a \in A$.

A game that admits a nested pseudo-potential is called *a nested pseudo-potential game.*

[5]Note that $\mathbf{g}^N$ has a $\mathcal{T}$-pseudo-potential if and only if, for each member T of $\mathcal{T}$, for any strategies a_{-T} of all players outside T, the subgame restricted by a_{-T} is a pseudo-potential game. Note also that $\mathbf{g}^N$ is a pseudo-potential game if and only if it has a $\{N\}$-pseudo-potential.

For two-person games, it is clear that the set of nested pseudo-potential games is equivalent to that of pseudo-potential games. For games with more than two players, the set of nested pseudo-potential games strictly includes that of pseudo-potential games. Indeed, since each pseudo-potential is a nested pseudo-potential, if $\mathbf{g}^N$ is a pseudo-potential game, then it is a nested pseudo-potential game. The next example demonstrates the strict inclusion.

Example 2.5 Consider the three-person game $\mathbf{g}^{\{1,2,3\}}$ represented as Table 1, where player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix; players 1 and 2 have identical interests, player 3's payoff is the same as others when player 1 chooses a_1, but is reversed otherwise as in the matching pennies game.

a_3	a_2	a_2'	a_3'	a_2	a_2'
a_1	$3,3,3$	$0,0,0$	a_1	$0,0,0$	$2,2,2$
a_1'	$-1,-1,1$	$1,1,-1$	a_1'	$1,1,-1$	$-1,-1,1$

Table 1: (g_1, g_2, g_3)

Note that $\mathbf{g}^{\{1,2,3\}}$ has a strict best-response cycle $(a_1', a_2, a_3) \to (a_1', a_2', a_3) \to (a_1', a_2', a_3') \to (a_1', a_2, a_3') \to (a_1', a_2, a_3)$. Since pseudo-potential games cannot have strict best response cycles as shown by Schipper (2004), this game is not a pseudo-potential game.

However, $\mathbf{g}^{\{1,2,3\}}$ is a nested pseudo-potential game. Indeed, $(f^1_{\{1,2\}}, f^1_{\{3\}})$ given in Table 2 is a $\{\{1,2\},\{3\}\}$-pseudo-potential of $\mathbf{g}^{\{1,2,3\}}$, where $f^1_{\{1,2\}}(\cdot) = g_1(\cdot) = g_2(\cdot)$ and $f^1_{\{3\}}(\cdot) = g_3(\cdot)$.

	a_3	a_3'
(a_1, a_2)	$3,3$	$0,0$
(a_1, a_2')	$0,0$	$2,2$
(a_1', a_2)	$-1,1$	$1,-1$
(a_1', a_2')	$1,-1$	$-1,1$

Table 2: $(f^1_{\{1,2\}}, f^1_{\{3\}})$

	a_3	a_3'
(a_1, a_2)	3	0
(a_1, a_2')	0	2
(a_1', a_2)	1	-1
(a_1', a_2')	-1	1

Table 3: $f^2_{\{1,2,3\}}$ or f

Regarding the $\{\{1,2\},\{3\}\}$-pseudo-potential $(f^1_{\{1,2\}}, f^1_{\{3\}})$ as a strategic form game, we can show that $(f^2_{\{1,2,3\}})$ defined in Table 3 is a $\{\{1,2,3\}\}$-pseudo-potential of $(f^1_{\{1,2\}}, f^1_{\{3\}})$. Thus $\mathbf{g}^{\{1,2,3\}}$ is a nested pseudo-potential game.

Remark 2.6 For a nested pseudo-potential game, the sequence $(\mathbf{f}^{\mathcal{T}^k})_{k=0}^K$ cannot be freely chosen. That is, even if a game has a nested pseudo-potential, one may not arrive at it with a wrong choice of partitions or payoff numbers. This point is illustrated below.

Example 2.7 Consider the game $\mathbf{g}^{\{1,2,3\}}$ in Example 2.5 again. Note that $(\hat{f}^1_{\{1,2\}}, f^1_{\{3\}})$ given in Table 4 is also a $\{\{1,2\},\{3\}\}$-pseudo-potential of $\mathbf{g}^{\{1,2,3\}}$.

However, regarding a $\{\{1,2\},\{3\}\}$-pseudo-potential $(\hat{f}^1_{\{1,2\}}, f^1_{\{3\}})$ as a strategic form game, we find a strict best-response cycle $((a'_1, a_2), a_3) \to ((a'_1, a'_2), a_3) \to ((a'_1, a'_2), a'_3) \to ((a'_1, a_2), a'_3) \to ((a'_1, a_2), a_3)$ in $(\hat{f}^1_{\{1,2\}}, f^1_{\{3\}})$, and thus $(\hat{f}^1_{\{1,2\}}, f^1_{\{3\}})$ has no $\{\{1,2,3\}\}$-pseudo-potential by Schipper (2004).

	a_3	a'_3
(a_1, a_2)	$3,3$	$0,0$
(a_1, a'_2)	$0,0$	$2,2$
(a'_1, a_2)	$-1,1$	$4,-1$
(a'_1, a'_2)	$4,-1$	$-1,1$

Table 4: $(\hat{f}^1_{\{1,2\}}, f^1_{\{3\}})$

The essential property shared by all existing versions of potential games is that maximizers of a potential function are Nash equilibria. The nested pseudo-potential proposed here inherits this property. To see this, let $\mathbf{g}^N$ be a game with a nested pseudo-potential f. We say that an action profile a^* is a *nested pseudo-potential maximizer* (NPP-maximizer) of $\mathbf{g}^N$ if $f(a^*) \geq f(a)$ for all $a \in A$. A NPP-maximizer, if it exists, is a Nash equilibrium of the underlying game:

Proposition 2.8 *Let $\mathbf{g}^N$ be a nested pseudo-potential games with a NPP-maximizer a^*. Then, a^* is a Nash equilibrium of $\mathbf{g}^N$.*

In order to establish the above proposition, the following lemma concerning $\mathcal{T}$-pseudo-potentials is useful:

Lemma 2.9 *Suppose that $\mathbf{g}^N$ has a $\mathcal{T}$-pseudo-potential $\mathbf{f}^{\mathcal{T}}$, where $\mathcal{T}$ is a partition of N. If an action profile a^* is a pure Nash equilibrium of $\mathbf{f}^{\mathcal{T}}$, where $\mathbf{f}^{\mathcal{T}}$ is regarded as a strategic form game, it is a pure Nash equilibrium of $\mathbf{g}^N$.*

This lemma is a generalization of a claim concerning q-potential games in Monderer (2007).

Proof. Suppose an action profile a^* is a pure Nash equilibrium of $\mathbf{f}^{\mathcal{T}}$. Fix any $i \in N$. Take $T \in \mathcal{T}$ such that $i \in T$. Since a^* is a Nash equilibrium of $\mathbf{f}^{\mathcal{T}}$, $f_T(a^*) - f_T(a_T, a^*_{-T}) \geq 0$ for all $a_T \in A_T$. Since $(a_i, a^*_{T\setminus\{i\}}) \in A_T$ for any $a_i \in A_i$, we have $f_T(a^*) - f_T(a_{\{i\}}, a^*_{T\setminus\{i\}}, a^*_{-T}) \geq 0$. Since $\mathbf{f}^{\mathcal{T}}$ is a $\mathcal{T}$-pseudo-potential of $\mathbf{g}$, it implies that $g_i(a^*) - g_i(a_i, a^*_{-i}) \geq 0$ for all $a_i \in A_i$, where $a^*_{-i} = (a^*_{T\setminus\{i\}}, a^*_{-T})$, which completes the proof. ■

We prove Proposition 2.8 by applying Lemma 2.9 iteratively.

Proof of Proposition 2.8. Let a^* be a NPP-maximizer of $\mathbf{g}^N$. Let K be a positive integer and $(\mathbf{f}^{\mathcal{T}^k})_{k=0}^K$ be a sequence such that $\mathbf{f}^{\mathcal{T}^0}$ is the original game $\mathbf{g}^N$; for each $k = 1, 2, \ldots, K$, $\mathbf{f}^{\mathcal{T}^k}$ is a $\mathcal{T}^k$-pseudo-potential of $\mathbf{f}^{\mathcal{T}^{k-1}}$, where $\mathbf{f}^{\mathcal{T}^{k-1}}$ is regarded as a strategic form game as above; and $\mathbf{f}^{\mathcal{T}^K}$ is such that $\mathcal{T}^K = \{N\}$ and $f^K_N(a) = f(a)$ for all $a \in A$.

By regarding $\mathbf{f}^{\mathcal{T}^K} = (f_N^K) = (f)$ as a one-person game, since f is a pseudo-potential of $\mathbf{f}^{\mathcal{T}^{k-1}}$, a^* is a pure Nash equilibrium of $\mathbf{f}^{\mathcal{T}^{K-1}}$ by Lemma 2.9. Similarly, for any $k = 1, 2, \ldots, K-1$, if a^* is a pure Nash equilibrium $\mathbf{f}^{\mathcal{T}^k}$, then it is a pure Nash equilibrium of $\mathbf{f}^{\mathcal{T}^{k-1}}$. Therefore, a^* is a Nash equilibrium of $\mathbf{g}^N = \mathbf{f}^{\mathcal{T}^0}$, which completes the proof. ■

The idea of nested pseudo-potential games is well defined for games with arbitrary action sets. For games with finite strategy sets, Proposition 2.8 implies that if the game has a nested pseudo-potential, then there exists a pure Nash equilibrium:

Corollary 2.10 *Every nested potential game with finite action sets possesses a pure Nash equilibrium.*

3 Nested weighted potential games

The idea of iterative construction of potential functions is general and so one might be interested in applying it to other potentials. But this is not necessarily fruitful: we show that the class of weighted potential games defined in Monderer and Shapley (1996) does not expand through nesting.

Definition 3.1 A function $f : A \to \mathbb{R}$ is a *weighted potential* of $\mathbf{g}^N$ if, for each $i \in N$, there exists a positive weight $w_i > 0$ such that

$$f(a_i, a_{-i}) - f(a'_i, a_{-i}) = w_i[g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i})]$$

for all $a_i, a'_i \in A_i$ and for all $a_{-i} \in A_{-i}$. If $\mathbf{g}^N$ has a weighted potential, then it is called a *weighted potential game*.[6]

We define nested weighted potential games following the same steps as before:

Definition 3.2 A *weighted $\mathcal{T}$-potential* of $\mathbf{g}^N$ is a tuple $(\mathcal{T}, (A_T)_{T\in\mathcal{T}}, (f_T)_{T\in\mathcal{T}})$, where, for each $T \in \mathcal{T}$, $f_T : A \to \mathbb{R}$ satisfies that, for each $i \in T$, there exists a weight $w_i > 0$ such that

$$f_T(a_i, a_{-i}) - f_T(a'_i, a_{-i}) = w_i[g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i})]$$

for all $a_i, a'_i \in A_i$ and $a_{-i} \in A_{-i}$.

We denote such a weighted $\mathcal{T}$-potential $(\mathcal{T}, (A_T)_{T\in\mathcal{T}}, (f_T)_{T\in\mathcal{T}})$ by $\mathbf{f}^{\mathcal{T}} := (f_T)_{T\in\mathcal{T}}$.

Definition 3.3 A function $f : A \to \mathbb{R}$ is a *nested weighted potential* of $\mathbf{g}^N$ if there exist a positive integer K and a sequence $(\mathbf{f}^{\mathcal{T}^k})_{k=0}^K$ such that

- $\{\mathcal{T}^k\}_{k=0}^K$ is a nest sequence of N;

[6] If $w_i = 1$ for all $i \in N$, $\mathbf{g}^N$ is called an *exact potential game* and f is called an *exact potential.*

- $\mathbf{f}^{\mathcal{T}^0}$ is the original game $\mathbf{g}^N$;
- for each $k = 1, 2, \ldots, K$, $\mathbf{f}^{\mathcal{T}^k}$ is a weighted $\mathcal{T}^k$-potential of $\mathbf{f}^{\mathcal{T}^{k-1}}$; and
- $\mathbf{f}^{\mathcal{T}^K} = (f_N^K)$ is such that $f_N^K(a) = f(a)$ for all $a \in A$.

It turns out that the class of nested weighted potential games is no larger than that of weighted potential games:

Proposition 3.4 $\mathbf{g}^N$ *is a weighted potential game if and only if it is a nested weighted potential game.*

Proof. It is clear that if $\mathbf{g}^N$ is a weighted potential game then it is a nested weighted potential game. Conversely, suppose that $\mathbf{g}^N$ is a nested weighted potential game. Fix any $i \in N$. Let a positive integer K, sequences $(\mathbf{f}^{\mathcal{T}^k})_{k=0}^K$, $(T^k)_{k=0}^K$ and $(w_{T^k}^k)_{k=0}^K$ be such that $\{i\} = T^0 \subseteq T^1 \subseteq \cdots \subseteq T^{K-1} \subseteq T^K = N$, $T^k \in \mathcal{T}^k$ and $w_{T^k}^k > 0$ for each $k = 0, \ldots, K$,

$$\begin{aligned} & f_{T^k}^k(a_{T^{k-1}}, a_{-T^{k-1}}) - f_{T^k}^k(a'_{T^{k-1}}, a_{-T^{k-1}}) \\ & = w_{T^{k-1}}^{k-1}[f_{T^{k-1}}^{k-1}(a_{T^{k-1}}, a_{-T^{k-1}}) - f_{T^{k-1}}^{k-1}(a'_{T^{k-1}}, a_{-T^{k-1}})] \end{aligned}$$

for any $a_{T^{k-1}}, a'_{T^{k-1}} \in A_{T^{k-1}}$, $a_{-T^{k-1}} \in A_{-T^{k-1}}$, and $k = 0, \ldots, K$; and $f_{T^0}^0(a) = g_i(a)$ for all $a \in A$, $f_{T^K}^K(a) = f(a)$ for all $a \in A$. Fix any $a_i, a'_i \in A_i$ and $a_{-i} \in A_{-i}$. Let $a_{-i} = (a_{T^{K-1}\setminus\{i\}}, a_{-T^{K-1}}) = (a_{T^{K-2}\setminus\{i\}}, a_{-T^{K-2}}) = \cdots = (a_{-T^0})$. Then, we have

$$\begin{aligned} & f(a_i, a_{-i}) - f(a'_i, a_{-i}) \\ = \ & f_{T^K}^K((a_i, a_{T^{K-1}\setminus\{i\}}), a_{-T^{K-1}}) - f_{T^K}^K((a'_i, a_{T^{K-1}\setminus\{i\}}), a_{-T^{K-1}}) \\ = \ & w_{T^{K-1}}^{K-1}[f_{T^{K-1}}^{K-1}((a_i, a_{T^{K-1}\setminus\{i\}}), a_{-T^{K-1}}) - f_{T^{K-1}}^{K-1}((a'_i, a_{T^{K-1}\setminus\{i\}}), a_{-T^{K-1}})] \\ = \ & w_{T^{K-1}}^{K-1}[f_{T^{K-1}}^{K-1}((a_i, a_{T^{K-2}\setminus\{i\}}), a_{-T^{K-2}}) - f_{T^{K-1}}^{K-1}((a'_i, a_{T^{K-2}\setminus\{i\}}), a_{-T^{K-2}})] \\ = \ & w_{T^{K-1}}^{K-1} w_{T^{K-2}}^{K-2}[f_{T^{K-2}}^{K-2}((a_i, a_{T^{K-2}\setminus\{i\}}), a_{-T^{K-2}}) - f_{T^{K-2}}^{K-2}((a'_i, a_{T^{K-2}\setminus\{i\}}), a_{-T^{K-2}})] \\ = \ & \cdots \\ = \ & \left(\prod_{k=1}^K w_{T^{k-1}}^{k-1}\right)[f_{T^0}^0(a_{T^0}, a_{-T^0}) - f_{T^0}^0(a'_{T^0}, a_{-T^0})] \\ = \ & \left(\prod_{k=1}^K w_{T^{k-1}}^{k-1}\right)[g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i})]. \end{aligned}$$

Thus, $\mathbf{g}^N$ is a weighted potential game. ■

We can also construct a nested version of exact potential games, ordinal potential games and generalized ordinal games as defined by Monderer and Shapley (1996). However, a similar argument shows that $\mathbf{g}^N$ is an exact (resp. ordinal and generalized ordinal) potential game if and only if it is a nested exact (resp. nested ordinal and nested generalized ordinal) potential game.[7]

[7]This means that the nested construction of q-potentials is also not fruitful.

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