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On the existence of an equilibrium in the Split-the-Difference Mechanism over an uncountable set with a singular part

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Abstract

Broman (89) analyzes the mixed Nash equilibria of the split-the-difference mechanism over countable sets. She leaves as an open question whether there may be mixed equilibria over uncountable sets with singular parts. In this note, I propose such an equilibrium; the support of the strategies is the union of a countable set and an interval.

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The double auction is an important model that stylizes one-shot simultaneous bargaining between two agents; it was introduced by Chatterjee and Samuelson (1983) who show that double auctions with asymmetric information will typically display inefficiencies[1]. Broman (1989) derives a striking result. She shows that there are many inefficient Nash equilibria in the complete information double auction, just like in the case of asymmetric information. She also argues that there may be no equilibrium such that the mixed strategies of both agents will have an absolutely continuous part[2] (unlike in most models of asymmetric information). In this note, I will show that one can obtain an equilibrium where both agents follow a distribution that includes mass points and a simple absolutely continuous part.

Consider the following game. A buyer may purchase an indivisible item from a seller. If the item is purchased, the buyer achieves a utility of one minus the monetary transfer and the seller obtains the monetary transfer. If the item is not purchased, both traders achieve a utility normalized to zero. The double auction (or split-the-difference) mechanism proceeds as follows. The buyer and the seller simultaneously submit an offer, denoted 'bid', v, for the buyer and 'ask', c, for the seller. If the ask is strictly above the bid, trade fails and both traders obtain zero. If the ask is below the bid, trade occurs. The monetary transfer, τ_{vc}, is the average of both prices: $(v + c)/2$.

Proposition 1 (Broman 89, Proposition 6) *Let p be a probability density function on $[0, 1]$ which can be decomposed into p_1, an absolutely continuous function, and p_2, a probability mass function defined on a set $\{a_i\}$ (i.e. a step function). Then, there is no equilibrium in which the buyer plays p as a mixed strategy.*

Bertomeu and Cheynel (2007) show that this statement is incorrect: there are equilibria such that the support of p is a closed interval. When intervals are excluded, this note will show that if the assumption that p_2 is a step function is relaxed in Proposition 6, so that p is probability density function whose support is the union of an uncountable set (an interval) and a countable set, one can find a mixed equilibrium.

Define F_v (resp. F_c), the distribution of the buyer (resp. seller). Let Γ denote the Gamma function and csc denote the cosecant function.

[1]The authors claim that asymmetric information, in contrast to complete information, accounts for inefficiencies in trade: (p.836) "the complete information approach fails to mirror key features of actual negotiations: (...) the occurrence of 'unreasonable' bargaining outcomes-breakdowns in negotiations, strikes, and work stoppages-when mutually beneficial agreements are possible."

[2]"It is an open question whether or not there are equilibria which have mixed strategies (over uncountable sets) with singular parts. It seems likely that none exist and even more likely that, if they did, no player would ever think of playing such a strategy" (p.141-142).

For all $i \in \mathbb{N} - \{0\}$ and for all $x \in [1/3 - 1/(i+3), 1/3 - 1/(i+4))$,

$$1 - F_v(x) = F_c(1-x) = -\frac{3\sqrt{\frac{5}{2}}\pi\csc(\sqrt{\frac{5}{2}}\pi)\Gamma(1+i)\Gamma(5+i)}{32\Gamma(3-\sqrt{\frac{5}{2}}+i)\Gamma(3+\sqrt{\frac{5}{2}}+i)} \tag{1}$$

For all $x \in [1/3, 2/3)$,

$$1 - F_v(x) = F_c(1-x) = -\frac{\pi\sqrt{\frac{30}{x}}\csc(\sqrt{\frac{5}{2}}\pi)}{64} \tag{2}$$

For all $i \in \mathbb{N} - \{0, 1\}$ and for all $x \in [2/3 + 1/(i+3), 2/3 + 1/(i+2))$,

$$1 - F_v(x) = F_c(1-x) = -\frac{3\sqrt{5\pi}\csc(\sqrt{\frac{5}{2}}\pi)\Gamma(\frac{13-\sqrt{13}}{4}+i)\Gamma(\frac{13+\sqrt{13}}{4}+i)}{2^{1-2i}(35+24i+4i^2)\Gamma(4+2i)} \tag{3}$$

And finally:

$$F_v(11/12) = 1 \text{ and } F_c(1/12) = 1 - F_v(13/15) \tag{4}$$

Equations (1)-(4) correspond to a Nash equilibrium with possibly uncountable support (see Appendix) where both traders will achieve a profit, Π, equal to:

$$\Pi = -\frac{5083\sqrt{\frac{\pi}{5}}\csc(\sqrt{\frac{5}{2}}\pi)\Gamma(\frac{9-\sqrt{13}}{4})\Gamma(\frac{-3+\sqrt{13}}{4})}{86016} \approx 0.27$$

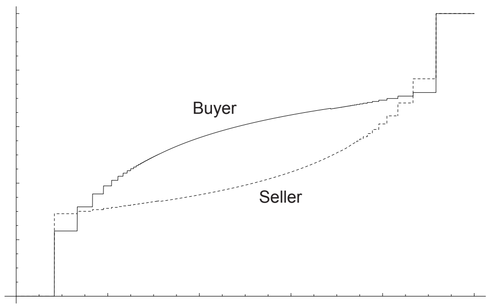

Figure 1: CDF of Proposal: Buyer (Solid) and Seller (Dashed)

There are several reasons why equilibria of this form may matter for the analysis of mixed strategies in the complete information double auction. First, as the example shows, the additional complexity of the equilibrium does not originate from the strategy that is chosen on the uncountable subset of the support (i.e. on the interval), but rather from the coordination of the players on the countable part. Therefore, a mixed strategy with an absolutely continuous part is not less reasonable than a mixed strategy with a countably infinite support[3]. Second, a probability density function that satisfies the conditions Proposition 6 may be made to approximate the equilibrium as closely as desired, just by truncating the countable part of the support. Third, the construction proposed in Equations (1)-(4) is obtained by taking the limit of equilibrium strategies with a finite support, for which a closed-form expression exists[4].

Appendix:

I shall prove here that Equations (1), (2), (3) and (4) are such that the buyer is indifferent to any value in $\{1/3-1/(3+i)\}_{i=1}^{\infty}\cup[1/3,2/3]\cup\{2/3+1/(3+i)\}_{i=1}^{\infty}$; the indifference of the seller follows by symmetry. Clearly, any value outside of this range cannot be optimal. Along the proof, $\overline{C}$ and $\overline{C}$ will refer to functions of i whose expression is omitted for notational simplicity.

Consider first values in the set $\{1/3-1/(3+i)\}_{i=1}^{\infty}$. Denote $\underline{\Delta}(i)$, the difference between the expected profit achieved by playing $1/3-1/(3+i)$ minus the expected profit achieved by playing $1/3-1/(4+i)$.

$$\underline{\Delta}(i)=-(F_c(1/3-1/(4+i))-F_c(1/3-1/(3+i)))(2/3+1/(4+i))+F_c(1/3+1/(3+i))(1/(3+i)-1/(4+i))$$

Substituting the strategy from Equation (3) and simplifying yields:

$$\underline{\Delta}(i)=\underline{C}\left(\frac{-(69+34i+4i^2)\Gamma(\frac{17}{4}-\frac{\sqrt{13}}{4}+i)\Gamma(\frac{17+\sqrt{13}}{4}+i)}{4\Gamma(\frac{21}{4}-\frac{\sqrt{13}}{4}+i)\Gamma(\frac{21+\sqrt{13}}{4}+i)}+1\right)$$

The term on the right-hand side is readily verified to be zero. Note also that the buyer must be indifferent to a sequence of values converging to $1/3$ and F_c is continuous at $1/3$, therefore playing $1/3$ yields the same profit as any value in $\{1/3-1/(3+i)\}_{i=1}^{\infty}$.

Denote now $\Pi(x)$ the profit of the buyer for any value in $[1/3,2/3]$.

$$\Pi(x)=\int_0^x(1-u/2-x/2)dF_c(u)$$

Since F_s admits a density on the interval, $\Pi(x)$ can be differentiated:

$$\Pi'(x)=F_c'(x)(1-x)-F_c(x)/2$$

Substituting the distribution F_c' yields that $\Pi'(x)$ is zero.

[3] Such strategies are the focus of Propositions 2, 3 and 4, p.139).

[4] See Proposition 1 in Broman (1989)

Finally, denote $\overline{\Delta}(i)$, the difference between the expected profit achieved by playing $2/3 + 1/(4+i)$ minus the expected profit achieved by playing $2/3 + 1/(3+i)$. Similar arguments as before yield:

$$\underline{\Delta}(i) = \overline{C}\left(\frac{-(2i^2+8i+3)\Gamma(2-\sqrt{\frac{5}{2}}+i)\Gamma(2+\sqrt{\frac{5}{2}}+i)}{2\Gamma(3-\sqrt{\frac{5}{2}}+i)\Gamma(3+\sqrt{\frac{5}{2}}+i)}+1\right)$$

Again, the right-hand side term is verified to be zero.

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