ECONOMICS BULLETIN

Risk Aversion, Intertemporal Elasticity of Substitution and Correlation Aversion

Antoine Bommier
CNRS and University of Toulouse

Abstract

Intertemporal correlation aversion is an intuitive concept indicating whether an individual prefers lotteries concerning consumption at different moments in time to be positively or negatively correlated. I show that the difference between the coefficient of relative risk aversion and the inverse of the intertemporal elasticity of substitution is related, in a simple way, to the index of intertemporal correlation aversion.

I am grateful to Bertrand Villeneuve for very helpful discussions about this paper.
Citation: Bommier, Antoine, (2007) "Risk Aversion, Intertemporal Elasticity of Substitution and Correlation Aversion." *Economics Bulletin,* Vol. 4, No. 29 pp. 1-8
Submitted: June 16, 2006. **Accepted:** August 13, 2007.
URL: http://economicsbulletin.vanderbilt.edu/2007/volume4/EB-06D90002A.pdf

1 Introduction

An unappealing feature of the standard additively separable intertemporal choice model is that it assumes that the inverse of the intertemporal elasticity of substitution (IES) is equal to the coefficient of relative risk aversion (RRA). Separation between IES and RRA can be achieved in different ways. For example, it is obtained in Epstein and Zin (1989) or Weil (1990) who consider preferences that do not comply with the axioms of the expected utility theory. But it can also be achieved by considering von Neumann-Morgenstern utility functions that are not additively separable, as is explained in Kihlstrom and Mirman (1974) and Epstein (1992, p. 15-17).

This paper deals with this latter possibility. I show that, within the expected utility framework, the difference between the inverse of the IES and the coefficient of RRA is simply related to the intuitive concept of "correlation aversion". This concept was originally introduced by Richard (1975) under the name of "multivariate risk aversion" and renamed "correlation aversion" by Epstein an Tanny (1980)[1]. Basically, when applied to intertemporal choice theory, it indicates whether the individual prefers that lotteries concerning consumption at different moments in time show a positive or a negative correlation.

In Section 2, I explain correlation aversion and define a measure thereof. In Section 3, I discuss the relationship between RRA, IES and intertemporal correlation aversion in the discrete time model. The mathematics of that model are simple but lead to a rather unaesthetic relation. However, this relation simplifies when considering infinitely small periods of time. This is formalized in Section 4 that treats of the continuous time model. Two examples of preferences with intertemporal correlation aversion are provided in Section 5. The main results are summarized in the concluding section.

2 Correlation aversion

Consider the case of preferences over two attributes measured by the variables x and y. Correlation aversion is defined as follows:

Definition 1 *The individual is correlation averse if and only if, for all x_1, x_2, y_1, y_2, such that $x_1 < x_2$ and $y_1 < y_2$, the lottery:*

$$\left\{ \begin{array}{lll} (x_1, y_2) & w.p. & \frac{1}{2} \\ (x_2, y_1) & w.p. & \frac{1}{2} \end{array} \right. \text{ is preferred to the lottery } \left\{ \begin{array}{lll} (x_1, y_1) & w.p. & \frac{1}{2} \\ (x_2, y_2) & w.p. & \frac{1}{2} \end{array} \right.$$

Remark that in both lotteries of the above definition, the first attribute takes the value x_1 with probability $\frac{1}{2}$ and x_2 with probability $\frac{1}{2}$ and the second attribute takes the value y_1 with probability $\frac{1}{2}$ and y_2 with probability $\frac{1}{2}$. The only

[1] Throughout this paper, we will stick to Epstein and Tanny's terminology, because it is more intuitive and avoids confusion with the "multivariate risk aversion" of Kihlstrom and Mirman (1974). Finkelshtain, Kella and Scarsini (1999) indicate that this notion of "correlation aversion" had already been presented by de Finetti (1952).

distinction between the two lotteries is in the manner with which the attributes are associated. An individual is correlation adverse if she prefers being lucky in either one or the other attribute to taking a chance on being lucky or unlucky in both attributes. Richard (1975) explained that if preferences are represented by a twice continuously differentiable von Neumann-Morgenstern utility function, $U(x,y)$, the individual is correlation averse if and only if $\frac{\partial^2 U}{\partial x \partial y} < 0$. However, he did not provide a coefficient for measuring correlation aversion. For that purpose, I suggest the following definition:

Definition 2 *For any twice continuously differentiable von Neumann-Morgenstern utility function, $U(x,y)$, the coefficient of correlation aversion with respect to the attributes x and y is defined by:*

$$\rho_{x,y} = -2\frac{\frac{\partial^2 U}{\partial x \partial y}}{\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y}} \tag{1}$$

It is clear that the individual is correlation averse if and only if $\rho_{x,y} > 0$. This coefficient can be simply interpreted in terms of "correlation premium". Indeed, consider two bivariate lotteries l_A and l_B that have the same univariate margins but that may differ in the manner they associate the first and second attributes. Note, $f_A(x,y)$ and $f_B(x,y)$ their density functions, $\overline{x}$ and $\overline{y}$ the means of their univariate margins and

$$\sigma^i = \begin{pmatrix} \sigma^i_{xx} & \sigma^i_{xy} \\ \sigma^i_{xy} & \sigma^i_{yy} \end{pmatrix} \qquad i = A, B$$

their matrices of variance-covariance. Define ε as the scalar that makes the individual indifferent between the lottery l_A and the lottery $l_B + (\varepsilon, \varepsilon)$ (the lottery obtained by adding $(\varepsilon, \varepsilon)$ to all the outcomes of lottery l_B). Indifference is reached whenever:

$$\int U(x,y) f_A(x,y) dxdy = \int U(x+\varepsilon, y+\varepsilon) f_B(x,y) dxdy \tag{2}$$

By a Taylor expansion, it follows that when $\sigma^A, \sigma^B \to 0$:

$$\varepsilon \approx \frac{1}{2}(\sigma^B_{xy} - \sigma^A_{xy})\rho_{x,y}(\overline{x}, \overline{y}) \tag{3}$$

Thus, in a first order approximation, the premium $(\varepsilon, \varepsilon)$ that compensates for the difference between lotteries l_A and l_B is simply half of the product of the coefficient of correlation aversion by the difference between their covariances.

3 The discrete time inter-temporal choice model

Now consider preferences over N attributes, $c_1, c_2, ..., c_N$, that represent consumptions in N successive periods of time. Assume that these preferences are increasing and can be represented by a twice continuously differentiable von Neumann-Morgenstern utility function $U(c_1, c_2, ..., c_N)$. We define:

1. The marginal rate of substitution between consumption in periods i and j:

$$m_{i,j} = \frac{\frac{\partial U}{\partial c_i}}{\frac{\partial U}{\partial c_j}} \tag{4}$$

2. The coefficient of RRA with respect to consumption in period i:

$$r_i = -\frac{c_i \frac{\partial^2 U}{(\partial c_i)^2}}{\frac{\partial U}{\partial c_i}} \tag{5}$$

3. The direct elasticity of substitution between consumption in periods i and j:

$$\sigma_{i,j} = \frac{\frac{1}{c_i \frac{\partial U}{\partial c_i}} + \frac{1}{c_j \frac{\partial U}{\partial c_j}}}{-\frac{\frac{\partial^2 U}{(\partial c_i)^2}}{\left(\frac{\partial U}{\partial c_i}\right)^2} + 2\frac{\frac{\partial^2 U}{\partial c_i \partial c_j}}{\frac{\partial U}{\partial c_i}\frac{\partial U}{\partial c_j}} - \frac{\frac{\partial^2 U}{(\partial c_j)^2}}{\left(\frac{\partial U}{\partial c_j}\right)^2}} \tag{6}$$

4. The coefficient of correlation aversion with respect to consumption in periods i and j :

$$\rho_{i,j} = -2\frac{\frac{\partial^2 U}{\partial c_i \partial c_j}}{\frac{\partial U}{\partial c_i} + \frac{\partial U}{\partial c_j}} \tag{7}$$

The first three concepts are well known. They are simply related to the fourth one.

Proposition 1 *For any consumption profile, we have:*

$$\frac{1}{\sigma_{i,j}}(1 + m_{i,j}\frac{c_i}{c_j}) = (r_i + \frac{c_i}{c_j}m_{i,j}r_j) - \rho_{i,j}c_i(1 + m_{i,j}) \tag{8}$$

Proof. Taking the inverse of (6) and multiplying both the numerator and the denominator by $c_i\frac{\partial U}{\partial c_i}$, (8) follows. ■

The above result holds for any twice continuously differentiable multi-attribute utility function. Thus far, the fact that the attributes c_i are consumptions in subsequent periods of time had absolutely no importance. However, if we consider $i = t$ and $j = t + 1$, two successive periods, and if we think of very short periods of time, in practice, it is often the case that $c_t \simeq c_{t+1}$, $m_{t,t+1} \simeq 1$ and $r_t \simeq r_{t+1}$. Equation (8) leads then to:

$$\frac{1}{\sigma_{t,t+1}} \simeq r_t - c_t\rho_{t,t+1} \tag{9}$$

The scalar $\sigma_{t,t+1}$is generally referred to as the IES. In a similar way, we can call $\rho_{t,t+1}$ the index of intertemporal correlation aversion. We can see from (9) that the difference between the coefficient of RRA and the inverse of the IES is roughly given by the product of the index of intertemporal correlation aversion and consumption. The relation is not exact when consumption, marginal utility of consumption or risk aversion with respect to instantaneous consumption are not the same in periods t and $t+1$. However, in practice, if we consider very short periods of time and smooth consumption profiles, this difference vanishes. The intuition can be formalized by looking at the continuous time model.

4 The continuous time model

Now assume that preferences are defined over a set of smooth consumption profiles $C^{\infty}(\mathbb{R}^+, \mathbb{R}^+)$ and are represented by a functional U :

$$c \in C^{\infty}(\mathbb{R}^+, \mathbb{R}^+) \rightarrow U(c) \in R$$

The definitions that we gave in the discrete time model can be simply rewritten in the continuous time framework by making use of Volterra derivatives[2]. Namely, we define:

1. The marginal rate of substitution between consumption at time t_1 and consumption at time t_2:
$$m_{t_1,t_2} = \frac{\frac{\partial U(c)}{\partial c(t_1)}}{\frac{\partial U(c)}{\partial c(t_2)}} \tag{10}$$

2. The RRA with respect to consumption at time t:
$$r_t = -c(t)\frac{\frac{\partial^2 U(c)}{(\partial c(t))^2}}{\frac{\partial U(c)}{\partial c(t)}} \tag{11}$$

3. The inverse of the direct elasticity of substitution between consumption at time t_1 and consumption at time t_2:[3]
$$\frac{1}{\sigma_{t_1,t_2}} = \frac{-\frac{\frac{\partial^2 U(c)}{(\partial c(t_2))^2}}{\left(\frac{\partial U(c)}{\partial c(t_2)}\right)^2} + 2\frac{\frac{\partial^2 U(c)}{\partial c(t_2)\partial c(t_1)}}{\frac{\partial U(c)}{\partial c(t_2)}\frac{\partial U(c)}{\partial c(t_1)}} - \frac{\frac{\partial^2 U(c)}{(\partial c(t_1))^2}}{\left(\frac{\partial U(c)}{\partial c(t_1)}\right)^2}}{\frac{1}{c(t_2)\frac{\partial U(c)}{\partial c(t_2)}} + \frac{1}{c(t_1)\frac{\partial U(c)}{\partial c(t_1)}}} \tag{12}$$

4. The coefficient of correlation aversion with respect to consumption at time t_1 and consumption at time t_2:
$$\rho_{t_1,t_2} = -2\frac{\frac{\partial^2 U(c)}{\partial c(t_1)\partial c(t_2)}}{\frac{\partial U(c)}{\partial c(t1)} + \frac{\partial U(c)}{\partial c(t_2)}} \tag{13}$$

[2]These derivatives were developed by Volterra (1913) and used in several economic papers, such as Ryder and Heal (1973). In short, the Volterra derivative of U with respect to consumption at time t, which we note as $\frac{\partial U(c)}{\partial c(t)}$, is such that $\frac{\partial U(c)}{\partial c(t)}dcdt$ measures the impact on U of an increase in the consumption of dc during dt periods around time t.

[3]It is only possible to define the inverse of the elasticity of substitution, because the second order Volterra derivatives may include Dirac delta functions whose inverse are not defined.

In the continuous time model, we are no longer constrained by the length of the time period, and we can define the IES and the index of intertemporal correlation aversion as follows:

1. The inverse of the IES at time t is defined by:

$$\frac{1}{\sigma_t} = \lim_{\varepsilon \to 0, \varepsilon \neq 0} \frac{1}{\sigma_{t,t+\varepsilon}} \tag{14}$$

2. The index of intertemporal correlation aversion at time t is defined by:

$$\rho_t = \lim_{\varepsilon \to 0, \varepsilon \neq 0} \rho_{t,t+\varepsilon} \tag{15}$$

We have the following result:

Theorem 1 *If the function U is twice continuously Volterra differentiable, then for any consumption profile in $C^\infty(\mathbb{R}^+, \mathbb{R}^+)$:*

$$r_t - \frac{1}{\sigma_t} = c(t)\rho_t \tag{16}$$

Proof. Analogously to (8), we first derive:

$$\frac{1}{\sigma_{t,t+\varepsilon}}(1 + \frac{c(t)}{c(t+\varepsilon)} m_{t,t+\varepsilon}) = (r_t + \frac{c(t)}{c(t+\varepsilon)} m_{t,t+\varepsilon} r_{t+\varepsilon}) - \rho_{t,t+\varepsilon} c(t)(1 + m_{t,t+\varepsilon}) \tag{17}$$

which at the limit $\varepsilon \to 0$ gives (16). ■

Theorem 1 shows that in the continuous time model, the relation (9) becomes exact. The difference between the coefficient of RRA and the inverse of the IES is equal to the product of the index of intertemporal correlation aversion and instantaneous consumption.

5 Two examples of preferences with intertemporal correlation aversion

The first example we will consider involves preferences *à la* Kihlstrom and Mirman (1974) represented by von Neumann-Morgenstern utility functions of the form:

$$U_1(c) = f\left(\int_0^{+\infty} u(c(t)) e^{-\beta t} dt\right) \tag{18}$$

where f is increasing. Such preferences are ordinally equivalent to those of the standard additively separable life cycle model. Concavity in f introduces some risk aversion.

With such preferences, we have:

$$\frac{\partial U_1(c)}{\partial c(t_1)} = u'(c(t_1)) e^{-\beta t_1} f'\left(\int_0^{+\infty} u(c(t)) e^{-\beta t} dt\right) \tag{19}$$

and for $t_2 \neq t_1$:

$$\frac{\partial^2 U_1(c)}{\partial c(t_1)\partial c(t_2)} = u'(c(t_1))u'(c(t_2))e^{-\beta(t_1+t_2)}f''\left(\int_0^{+\infty} u(c(t))e^{-\beta t}dt\right) \tag{20}$$

The index of intertemporal correlation aversion is thus given by:

$$\rho_t = -u'(c(t))e^{-\beta t}\frac{f''}{f'}\left(\int_0^{+\infty} u(c(t))e^{-\beta t}dt\right) \tag{21}$$

Therefore, in the case where f is linear, $\rho_t = 0$ and $\frac{1}{\sigma_t} = r_t$. This corresponds to the well known relation between RRA and IES in the additively separable model. However, as soon as f is strictly concave, the index of intertemporal correlation aversion is positive and, therefore, the coefficient of RRA is greater than the inverse of the IES.

The second example is provided by the class of recursive utility functions studied in Epstein (1983):

$$U_2(c) = \int_0^{+\infty} u(c(t))\exp\left(-\int_0^t v(c(\tau))d\tau\right)dt \tag{22}$$

This is the general form of recursive preferences over infinitely long consumption paths. Such preferences are stationary.

We have:

$$\frac{\partial U_2(c)}{\partial c(t_1)} = \exp\left(-\int_0^{t_1} v(c(\tau))d\tau\right)\left[u'(c(t_1)) - v'(c(t_1)\int_{t_1}^{+\infty} u(c(t))\exp\left(-\int_{t_1}^{t} v(c(\tau))d\tau\right)dt\right] \tag{23}$$

and for any $t_2 < t_1$:

$$\frac{\partial^2 U_2(c)}{\partial c(t_2)\partial c(t_1)} = -v'(c(t_2))\frac{\partial U_2(c)}{\partial c(t_1)} \tag{24}$$

Thus, the index of intertemporal correlation aversion is simply given by:

$$\rho_t = v'(c(t)) \tag{25}$$

Again, when v is constant, as in the standard additively separable model, the index of intertemporal correlation aversion equals zero. When the function v is increasing, the index of intertemporal correlation aversion is positive, and thus, the coefficient of RRA is greater than the inverse of the IES.

6 Conclusion

Intertemporal correlation aversion is an intuitive concept indicating whether an individual prefers lotteries concerning consumption at different moments in time to be positively or negatively correlated. I show that the difference between the coefficient of RRA and the inverse of the IES equals the product of the instantaneous consumption by the index of intertemporal correlation aversion. This latter has a simple interpretation in terms of the premium that would compensate for a positive correlation between a lottery on consumption at a given moment in time and a lottery on consumption at another moment in time.

References

[1] de Finetti, B., 1952, “Sulla preferibilità”, *Giornale degli Economisti e Annali di Economia*, 11:685-709.

[2] Epstein, L.G, 1983, “Stationary Cardinal Utility and Optimal Growth Under Uncertainty”, *Journal of Economic Theory*, 31:133-52.

[3] Epstein, L. G., 1992, “Behavior Under Risk: Recent Developments in Theory and Applications”. In J.-J. Laffont (ed), *Advances in Economic Theory*, Cambridge University Press.

[4] Epstein, L.G, and S. M. Tanny, 1980, “Increasing Generalized Correlation: A Definition and Some Economic Consequences”, *The Canadian Journal of Economics*, 13(1):16-34.

[5] Epstein, L. G. and S. E. Zin, 1989, “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework”, *Econometrica*, 57(4):937-69.

[6] Finkelshtain, I., Kella, O. and M. Scarsini, 1999, On risk aversion with two risks. *Journal of Mathematical Economics*, 31(2):239-50.

[7] Kihlstrom, R.E. and L.J. Mirman, 1974, “Risk aversion with many commodities”, *Journal of Economic Theory*, 8:361-88.

[8] Richard, S. F., 1975, “Multivariate Risk Aversion, Utility Independence and Separable Utility Functions”, *Management Science*, 22(1):12-21.

[9] Ryder, H. E. and G. M. Heal, 1973, “Optimal Growth with Intertemporally Dependent Preferences”, *The Review of Economic Studies*, 40(1):1-31.

[10] Volterra, V., 1913, *Leçons sur les équations intégrales et les équations intégro-différentielles* (Paris: Gauthier-Villars).

[11] Weil, P., 1990, “Nonexpected Utility in Macroeconomics”, *The Quarterly Journal of Economics*, 105(1):29-42.