E C O N O M I C S B U L L E T I N

On integrability and aggregation of superior demand functions

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Abstract

When each of the members of a collective displays a demand behavior that is consistent with a homogeneous of degree one in income demand, it is well known that some properties carry over to the aggregate representative consumer. We investigate those issues when the components of the society are allowed to behave in agreement with less restrictive demand patterns, namely superior demand functions.

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1. Introduction

Many authors have studied conditions ensuring that aggregate demand behavior has properties that are desirable for individual demand. This enables us to attach the aggregate behavior to a *fictitious representative consumer*. A common assumption has inveterately been that the distribution of income be fixed. Eisenberg (1961) and Chipman (1974) showed that if the individual demands are representable by homogeneous and concave utilities and satisfy the budget equality, then so is the aggregate. Shafer (1977) adopted a revealed preference position, arguing that to prove properties of aggregate demand is a problem of algebraic nature. Thus he stated that if the individual demand functions are homogeneous of degree one in income and satisfy the budget equality, and each of them satisfies the Strong Axiom of Revealed Preference (SARP), then their aggregate demand satisfies a strengthened version of the Strong Axiom.

We intend to investigate to which extent Shafer's algebraic technique can yield new conditions ensuring that aggregate demand can be generated by utilities, in the sense of e.g., Chipman (1974). We consider the larger class of superior demand functions when the distribution of income remains fixed. In Section 2 we make our setting explicit and recall in passing the solution available for the individual homogeneous case, which permits to infer our motivating aggregation result. Along Section 3 we explore the main issue. Subsection 3.1 provides examples that show that extending classical results by Shafer is feasible. In Subsection 3.2 we prove that the Strong Axiom is not aggregable in the whole class of superior demand functions, even when rather restrictive requirements are imposed on the individual demand functions and the distribution of income remains fixed. All these results help to interpret correctly the meaning of a widely spread but discouraging comment by Shafer on the little chances of aggregating the Strong Axiom except for the tight realm of homogeneous demand functions. We gather our conclusions in Section 4.

2. Integrability and the aggregation of homogeneous demand

Let $h^1, ..., h^m : P \times M \longrightarrow X$ be demand functions associated with a population of m consumers and consumption set $X \subseteq \mathbb{R}^n_+$. We assume $h^i(\alpha p, \alpha w) = \alpha h^i(p, w)$ for each $(p, w) \in P \times M$ and $\alpha > 0$. Unless otherwise stated, we also assume $P \times M = \mathbb{R}^n_{++} \times \mathbb{R}_{++}$.

The demand function h^i is *representable* if there is a function $u^i : X \longrightarrow \mathbb{R}$ such that $h^i(p, w) = \{x \in B(p, w) : u^i(x) \geqslant u^i(y) \text{ for all } y \in B(p, w)\}$, for any $(p, w) \in P \times M$, where $B(p, w) = \{x \in X : px \leqslant w\}$. We also write that u^i generates h^i instead.

We now consider the group of consumers as a social aggregate, and denote their demand function by $H(p, w)$. Should the collective have w at their disposal, once that income has been splitted among them ($w = w_1 + ... + w_m$ with $w_i \geqslant 0$ for all i) then $H(p, w) = h^1(p, w_1) + ... + h^m(p, w_m)$. Under fixed distribution of income, it is assumed that there are prefixed $\delta_1, ..., \delta_m$ strictly positive and summing up to 1 in such way that $w_i = \delta_i w$ for each i. Now the question arises as to when we can ensure the integrability of $H(p, w)$, that is, when there is a sufficiently nice function $u : X \longrightarrow \mathbb{R}$ such that $H(p, w) = \{x \in B(p, w) : u(x) \geqslant u(y) \text{ for all } y \in B(p, w)\}$, for any $(p, w) \in P \times M$.

Take any demand function h. If $y \neq x$ and there is $(p, w) \in P \times M$ such that $y \in B(p, w)$ and $x \in h(p, w)$, then we say that x is *revealed preferred to* y, denoted $x\, S\, y$. Also, h satisfies the *Weak Axiom of Revealed Preference* (WARP) if S is asymmetric, and h satisfies the *Strong Axiom of Revealed Preference* (SARP) if S is acyclic[1].

In Fuchs-Seliger (1981) a list of axioms is provided in order to ensure that a demand function is generated by S^* (cf. Theorem 1 below).

DI. $h : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \rightarrow \mathbb{R}^n_{++}$ is a continuous demand function.

DII. The range of h is $\mathbb{R}^n_{++}$.

DIII. h satisfies the budget equality $p \cdot h(p, w) = w,\ \forall (p, w) \in \mathbb{R}^{n+1}_{++}$.

DIV. Lipschitz condition with respect to w: for any $p^0, p^1 \in \mathbb{R}^n_{++}$ there is $K \in \mathbb{R}_{++}$ such that for all $w, w' \geqslant 0$ and for all $t \in [0, 1]$: $\|h(p^t, w) - h(p^t, w')\| \leqslant K|w - w'|$, where $p^t = tp^0 + (1 - t)p^1$.

DV. S is acyclic (i.e., $xS^*y \Rightarrow \neg(ySx)$, for all x, y in the range of h).

DVI. h is homogeneous of degree one in income.

The system of axioms DI – DV is a modification of a former one due to Uzawa (1971). As he pointed out, under these conditions an upper semicontinuous utility functions generating the given demand function exists. In the current axiomatics, DIII is redundant as stems from (8) in Alcantud *et al.* (2006), and so is DIV too. We keep mentioning Walras' law in the statements for the sake of the tradition.

Define the binary relation $\overline{S^*}$ by means of the expression: $x\overline{S^*}y$ if and only if yS^*x is false. We say that it is *monotone* if $x \geqslant y$ plus $x \neq y$ entail $x\overline{S^*}y$, $\forall x, y \in X$. It is *homothetic* if $x\overline{S^*}y$ yields $\lambda x\overline{S^*}\lambda y, \forall x, y \in X, \forall \lambda > 0$. Then (cf. Fuchs-Seliger, 1981):

THEOREM 1 *Under hypotheses DI, DII, DIII, DV and DVI, $\overline{S^*}$ is transitive, complete, strictly convex, monotone, homothetic and continuous. Therefore, there exists a utility function which is homogeneous of degree one, continuous, strictly quasi-concave and monotone, representing $\overline{S^*}$ and generating the given demand function.*

The next immediate though nonetheless original consequence may be seen as an axiomatization leading to the setting that is presumed in Chipman (1974), Theorem 4, but without any reference to differentiability restrictions (as in Shafer and Sonnenschein, 1982, Theorem 3).

COROLLARY 1 *If each of $h^1, \ldots, h^m$ satisfies DI, DII, DIII, DV and DVI and we have fixed distribution of income, then their aggregate demand function $H : P \times M \longrightarrow \mathbb{R}^n_+$ derives from maximization of a continuous, monotone, homogeneous of degree 1, and strictly quasi-concave utility function $u : \mathbb{R}^n_{++} \longrightarrow \mathbb{R}_+$, when the range of H is $\mathbb{R}^n_{++}$.*

[1]There are alternative statements for the strong axiom, e.g., that the transitive hull S^* of S be asymmetric (see Uzawa, 1971, Fuchs-Seliger, 1980) or irreflexive (Hurwicz and Richter, 1971).

3. Integrability and aggregation properties without homogeneity of the demand

One can infer Corollary 1 because the aggregate demand satisfies the same list of axioms and thus Theorem 1 applies. The only relevant issue in its proof is that an appeal to Theorem 2 in Shafer (1977) is needed in order to aggregate the Strong Axiom. The fundamental mechanisms involved in the sufficiently "nice" differentiable instance can be now explained in view of subsequent achievements as follows. Suppose that we consider a list $h^1, ..., h^m$ of individual demand functions of $\mathcal{C}^1$ satisfying the budget equality and being defined for each positive price-income pair. Now, assume that they satisfy the Strong Axiom and are homogeneous of degree 1 in income (for example, because each h^i derives from a homothetic preference as in Chipman, 1974). Then, the Slutsky matrix of each demand function is symmetric and negative semidefinite (everywhere on its domain), by Uzawa (1971). Following the proof of Chipman (1974), Theorem 4, their aggregate demand by any fixed distribution of income has symmetric and negative semidefinite Slutsky matrix too. Therefore, such aggregate demand function can be derived from a homogeneous of degree 1 utility function (Chipman, 1974, Theorem 2, also Richter, 1979, Theorems 10 and 12).

In the literature on the topic this argument is presumed to suppose a bound as to possible extensions of our initial result, on the basis of Shafer's (1977) argument (pp. 1177-8). Quoting from p. 1178, "there is little hope of obtaining SARP (...) in the aggregate without the assumption of homogeneity of degree -1 in p" (or homogeneity of degree 1 in income, which is equivalent under DIII). Because aggregate demand functions generated by utilities forcefully satisfy the Strong Axiom, such restriction must apply to them.

However, Shafer's argument focuses on certain *differentiable* demand functions. So, the question remains: Can we get aggregate demand functions *other than homogeneous of degree 1 in income* that are generated by utilities? We proceed to show that the answer is positive in the next Subsection. In Subsection 3.2 we wonder if this can be achieved when we consider superior demand functions. The answer to that question is negative.

3.1 Some clarifying examples

We describe some classes of situations that illustrate the feasibility of obtaining SARP in the aggregate without any homogeneity assumption on the demand functions involved. Example 1 below puts forward a whole collection of such situations. As a straightforward example, we pick the demand function h^0 associated with the utility function $u^0(x_1, \ldots, x_n) = (x_1 + a)x_2 \cdots x_n$ (where $a > 0$ is fixed, and $x \in \mathbb{R}^n_+$). Then h^0 satisfies the Strong Axiom DV since it is mandatory for representable demand functions. Nonetheless, h^0 is not homogeneous in income, despite the many nice properties of u^0. The explicit expression of h^0 is

$$h^0(p,w) = \begin{cases} \left(\frac{w-(n-1)p_1 a}{np_1}, \frac{w+p_1 a}{np_2}, ..., \frac{w+p_1 a}{np_n}\right) \text{ for } w > (n-1)ap_1 \\ \left(0, \frac{w}{(n-1)p_2}, ..., \frac{w}{(n-1)p_n}\right) \text{ for } w \leqslant (n-1)ap_1. \end{cases}$$

Deviations from Shafer's prediction increase as long as we move away from the domain restriction in DI.

In the first place, if we are not bounded by DI then we can get a subclass of *quasi-homothetic* and differentiable demand functions that preserve SARP under aggregation by a fixed distribution of income. We use the term quasi-homothetic for a demand function h when it has the functional form $h(p,w) = A(p)w + B(p)$, where $A : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ and $B : \mathbb{R}^n_{++} \to \mathbb{R}^n$. Observing that the demand function

$$\tilde{h}(p,w) = \left(\frac{w-(n-1)a_1p_1 + a_2p_2 + ... + a_np_n}{np_1}, \frac{w + a_1p_1 - (n-1)a_2p_2 + ... + a_np_n}{np_2}, ..., \frac{w + a_1p_1 + a_2p_2 + ... - (n-1)a_np_n}{np_n} \right),$$

for $(p,w) \in P \times M \subseteq \mathbb{R}^n_{++} \times \mathbb{R}_+$ adequate (we need to request $w - (n-1)a_1p_1 + a_2p_2 + ... + a_np_n \geqslant 0, ..., w + a_1p_1 + a_2p_2 + ... - (n-1)a_np_n \geqslant 0$), one can see that it derives from the utility assignment $\tilde{u}(x_1,...,x_n) = (x_1 + a_1) \cdots (x_n + a_n)$ for $x_j \geqslant -a_j$, $x_j \geqslant 0$ for any $j \leqslant n$. In the following example we will assume that every individual possesses such a demand function.

EXAMPLE 1 *Consider the demand functions*

$$h^i(p,w) = \left(\frac{w-(n-1)a^i_1p_1 + a^i_2p_2 + ... + a^i_np_n}{np_1}, \ ..., \frac{w + a^i_1p_1 + a^i_2p_2 + ... - (n-1)a^i_np_n}{np_n} \right)$$

for $(p,w) \in P^i \times M$ adequate, then they satisfy SARP by the comment above. We aggregate these demand functions under fixed distribution of income $(\delta_1,...,\delta_m)$.

Let $\widetilde{P^i} = \{(p,\delta_i w) : (p,w) \in P^i \times M\}$, and $A_j = \sum_{i=1}^m a^i_j$ for each j. Because the aggregate of $h_1,...,h_m$ by a fixed distribution of income can be defined on $\bigcap_{i=1,...,m} \widetilde{P^i}$ by

$$H(p,w) = \left(\frac{w-(n-1)A_1p_1 + A_2p_2 + ... + A_np_n}{np_1}, \ ..., \frac{w + A_1p_1 + A_2p_2 + ... - (n-1)A_np_n}{np_n} \right)$$

it satisfies SARP due to the comment preceding Example 1. Hence, the individual demand functions and their aggregate satisfy the Strong Axiom although they are not homogeneous of degree one in income.

In the second place, the next example shows that, within the class of demand function that satisfy only $w > w' \Rightarrow h(p,w) \geqslant h(p,w')$ and $h(p,w) \neq h(p,w')$, the Strong Axiom can be aggregated when DI is not mandatory[2].

EXAMPLE 2 *Consider two individuals with the same utility function $u^i : \mathbb{R}^3_+ \to \mathbb{R}$, $u^i(x) = x_1x_2 + x_3$ for $i = 1,2$.*

The demand function generated by u^i is:

$$h^i(p,w^i) = \begin{cases} \left(\frac{p_2}{p_3}, \frac{p_1}{p_3}, \frac{w^ip_3 - 2p_1p_2}{p_3^2}\right) & \text{for } w^i > \frac{2p_1p_2}{p_3} \\ \left(\frac{w^i}{2p_1}, \frac{w^i}{2p_2}, 0\right) & \text{for } w^i \leqslant \frac{2p_1p_2}{p_3}. \end{cases}$$

[2]This is in accordance with a more general framework, as we can find in e.g., Sonnenschein (1971), where the domain of h may be a special subset of $P \times M$ and the range of h may be a subset of $\mathbb{R}^n$.

Inquiring the Engel curve $\{x|x = h(\bar{p}, w^i), \forall w^i \in \mathbb{R}_+\}$, *for any fixed* $\bar{p} \in \mathbb{R}^3_{++}$, *we realize that it is increasing in* w, *and it is kinked at* $w^i = \frac{2\bar{p}_1\bar{p}_2}{p_3}$. *Thus, the partial derivatives of* $h(\bar{p}, w^i)$ *with respect to* w^i *do not exist at that point.*

Assume the fixed distribution of income $(\delta, 1-\delta)$, *where* $\delta \leqslant \frac{1}{2}$ *without loss of generality. Let* $D = \{(p, w) : p \in \mathbb{R}^3_{++}, w > \frac{2p_1p_2}{\delta p_3}\}$, *It is simple to check that the aggregate demand function on* D *has the form* $H(p, w) = (\frac{2p_2}{p_3}, \frac{2p_1}{p_3}, \frac{wp_3 - 4p_1p_2}{p_3^2})$. *Therefore, as one can realize, such* $H(p, w)$ *is generated by the utility function* $u(x) = x_1x_2 + 2x_3$.

3.2 Limits of aggregation in case of superior demand functions

We now turn to a class wider than homogeneous of degree one demand functions, namely superior demand functions. For bundles $x, y \in \mathbb{R}^n$, $x > y$ means $x_i > y_i$ $\forall i = 1, ..., n$. Then, superiority of h amounts to:

DVII. $w > w' \Rightarrow h(p, w) > h(p, w')$, $\forall w, w' > 0$, $\forall p \in \mathbb{R}^n_{++}$.

Under mild differentiablity assumptions, demand functions h satisfying DVII (and the budget equivalence) can be characterized by: for each $j = 1, ..., n$, $\frac{\partial h_j}{\partial w}(p, w) \geqslant 0$ for all (p, w). DIII prevents a demand function satisfying this latter requirement –and, thus, having non-decreasing demand of good j for fixed prices p– from displaying $w > w'$ with $h(p, w) = h(p, w')$.

By a proof which is very similar to the proof of Theorem 1 one gets:

THEOREM 2 (Fuchs-Seliger, 2003) *Let* $H(p, w)$ *satisfy DI – DIII, DV and DVII. Then,* $\bar{S}^*_H$ *is transitive, complete, monotone, strictly convex and continuous, and therefore it can be represented by a continuous, strictly quasiconcave utility function.*

In Theorem 2 we consider a class of possibly "kinked" demand functions for which SARP holds without being necessarily homogeneous in income (for instance, the demand function h^0 in Subsection 3.1 satisfies DI–DV according to Uzawa, 1971, and also DVII, for $w > (n-1)ap_1$, but it is not homogeneous in income). For each H under the hypotheses of Theorem 2, any choice of $k = 2, 3, ...$ gives place to a new demand function, namely $h_k(p, w) = \frac{1}{k}H(p, kw)$, that also satisfies these properties though it is homogeneous in income if and only if so is H (and, in this event, $H = h_k$). Moreover, H is the aggregation of k individual demand functions of type h_k under proportional distribution of income ($\delta_1 = ... = \delta_k = \frac{1}{k}$) since $H(p, w) = kh_k(p, \frac{1}{k}w)$.

Is SARP aggregable for superior demand functions? If yes, that would permit to use the corresponding result at individual level –i.e., Theorem 2– exactly as Shafer did, in order to get an analogue to Corollary 1 for superior aggregate demand functions. Unfortunately, the answer to that question is negative, even when some heavy additional restricions are imposed: the next example [3] proves that *the analogue to Corollary 1 for superior demand functions does not hold true:*

[3]Example 3 adds information to the comment in Example 5 of Shafer and Sonnenschein (1982) with respect to the aggregability properties of a very specific functional form.

EXAMPLE 3 *Let us define* $h^1, h^2 : \mathbb{R}^3_{++} \times \mathbb{R}_{++} \to \mathbb{R}^2_{++}$ *continuous demand functions as follows. The demand* $h^1(p, w)$ *is derived from the utility function* $u_1(x, y, z) = log(x) + log(y) + \sqrt{z}$*. Some of its properties are analyzed e.g., in Shafer (1977), p. 1178 (see also Shafer and Sonnenschein, 1982, Example 5). Thus we are aware that DIII and DV are satisfied. Letting* $\varphi(t) = \left(\sqrt{4+t} - 2\right)^2$*, we obtain*

$$h^1(p,w) = \left(\frac{1}{2p_1}(w - p_3\varphi(\frac{w}{p_3})),\ \frac{1}{2p_2}(w - p_3\varphi(\frac{w}{p_3})),\ \varphi(\frac{w}{p_3})\right)$$

or, alternatively,

$$h^1(p,w) = \left(\frac{2p_3}{p_1}\sqrt{\varphi(\frac{w}{p_3})},\ \frac{2p_3}{p_2}\sqrt{\varphi(\frac{w}{p_3})},\ \varphi(\frac{w}{p_3})\right)$$

thus it is easy to check that DII and DVII (therefore DIV) hold. Clearly, DVI does not.

The demand $h^2(p, w)$ *is derived from the utility function* $u_2(x, y, z) = log(x)+log(z)+\sqrt{y}$*. Obviously, it satisfies DI to DV and DVII as well, and its explicit form is:*

$$h^2(p,w) = \left(\frac{1}{2p_1}(w - p_2\varphi(\frac{w}{p_2})),\ \varphi(\frac{w}{p_2}),\ \frac{1}{2p_3}(w - p_2\varphi(\frac{w}{p_2}))\right)$$

or, alternatively,

$$h^2(p,w) = \left(\frac{2p_2}{p_1}\sqrt{\varphi(\frac{w}{p_2})},\ \varphi(\frac{w}{p_2}),\ \frac{2p_2}{p_3}\sqrt{\varphi(\frac{w}{p_2})}\right).$$

However, consider the aggregate demand by the distribution of income $(\frac{1}{2}, \frac{1}{2})$*, namely,* $H(p, w) = (H_1(p, w), H_2(p, w), H_3(p, w))$*, where*

$$H_1(p,w) = \frac{2p_2}{p_1}\left(\sqrt{4 + \frac{w}{2p_2}} - 2\right) + \frac{2p_3}{p_1}\left(\sqrt{4 + \frac{w}{2p_3}} - 2\right)$$

$$H_2(p,w) = \left(\sqrt{4 + \frac{w}{2p_2}} - 2\right)^2 + \frac{2p_3}{p_2}\left(\sqrt{4 + \frac{w}{2p_3}} - 2\right)$$

$$H_3(p,w) = \frac{2p_2}{2p_3}\left(\sqrt{4 + \frac{w}{2p_2}} - 2\right) + \left(\sqrt{4 + \frac{w}{2p_3}} - 2\right)^2$$

Its Slutsky matrix $(S^H_{ij}(p, w))_{i,j}$ *is not symmetric for each price-income pair. For instance, take* $(p, w) = (1, 2, 3, 4)$*. One can check that*

$$H_1(1,2,3,4) = 2\sqrt{42} + 4\sqrt{5} - 20,\ H_2(1,2,3,4) = 3 - 4\sqrt{5} + \sqrt{42}$$

$$\frac{\partial H_1}{\partial p_2}(1,2,3,4) = \frac{9\sqrt{5}}{5} - 4,\ \ \frac{\partial H_2}{\partial p_1} = 0$$

$$\frac{\partial H_1}{\partial w}(1,2,3,4) = \frac{\sqrt{42}}{28} + \frac{\sqrt{5}}{10},\ \ \frac{\partial H_2}{\partial w}(1,2,3,4) = \frac{\sqrt{42}}{56} - \frac{\sqrt{5}}{10} + \frac{1}{4}$$

Whence, $S^H_{12}(1,2,3,4) = \frac{294\sqrt{5}+15\sqrt{42}-6\sqrt{210}-630}{140}$ *and* $S^H_{21}(1,2,3,4) = \frac{420\sqrt{5}-18\sqrt{210}+20\sqrt{42}-770}{140}$ *differ in* $\frac{-126\sqrt{5}+12\sqrt{210}-5\sqrt{42}+140}{140} \approx -0.001798197876$*. This means that* H *can not satisfy the Strong Axiom DV (Richter, 1979, Theorems 10 and 12; Jerison, 1984, Lemma 1).*

4. Conclusions

Despite a commonly held belief, the question of finding a class of demand functions other than homogeneous of degree one in income for which the Strong Axiom aggregates is meaningful and remains open. We have provided arguments to show that the preliminary restriction posed by Shafer's comment is less stringent than apparent. Particular examples have illustrated that possibility; the specific form of the domain restriction was crucial.

Theorem 2 hinted that superior demand functions might provide a very general answer to that question. However, Example 3 proved that SARP is not aggregable in the class of superior demand functions even when many other relevant restrictions are imposed.

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