E C O N O M I C S B U L L E T I N

Efficiency and converse reduction-consistency in collective choice

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Abstract

We consider the problem of selecting a subset of a feasible set over which each agent has a strict preference. We propose an invariance property, converse reduction-consistency, which is the converse of reduction-consistency introduced by Yeh (2006), and study its implications. Our results are two characterizations of the Pareto rule: (1) it is the only rule satisfying efficiency and converse reduction-consistency and (2) it is the only rule satisfying one-agent efficiency, converse reduction-consistency, and reduction-consistency.

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1 Introduction

We consider the problem of selecting alternatives from a set of feasible alternatives over which each agent has a strict preference (no indifference between any two alternatives). Such a problem, called a "collective choice problem," arises for instance when the members of a committee have to elect new members from a slate of candidates. How should the candidate(s) be elected? A "rule" is a mapping that associates with each such problem a non-empty subset of the feasible set.

Our goal here is to propose an invariance property, "converse reduction-consistency," of rules and study its implications. The property is the converse of "reduction-consistency" (Yeh, 2006), which is an application for collective choice problems of a general principle of "consistency."[1] *Reduction-consistency* says the following. Consider a problem and an alternative x chosen by a rule for it. Imagine now that some agents "leave" with the understanding that x would be chosen, and reassess the situation from the viewpoint of the remaining agents. A condition for an alternative to be acceptable as a choice by the remaining agents is that each of the departing agents be indeed guaranteed a welfare level that he was initially promised. The revised preferences of the remaining agents are then obtained by restricting their original preferences to those acceptable alternatives. *Reduction-consistency* requires that x should still be chosen by the rule in the reduced situation just defined.[2] Our main property, *converse reduction-consistency*, says that if an alternative is chosen by a rule for all of its associated reduced situations, then it should be chosen by the rule for the original problem.

We first study the existence of rules that satisfy *converse reduction-consistency*. As we show, the Pareto rule, which chooses all "Pareto-efficient" alternatives for each problem, is *conversely reduction-consistent* (Proposition 1). Moreover, it can be verified that the Pareto rule is also *reduction-consistent*. Is there any rule other than the Pareto rule that satisfies the two properties? The answer is yes. The feasibility rule, which chooses all feasible alternatives for each problem, is another example. However, the rule violates the basic requirement of *efficiency*: if an alternative is chosen, there is no other alternative that all agents strictly prefer.[3] Of course, the Pareto rule is *efficient*. We ask whether there exists any rule other than the Pareto rule satisfying *efficiency*, *reduction-consistency*, and *converse reduction-consistency*. Surprisingly, the answer is no. In fact, a more general result can be proved: *efficiency* and *converse reduction-consistency* are satisfied only by the Pareto rule (Theorem 1). Note that as we show, under *reduction-consistency*, *efficiency* is equivalent to a weaker version of *efficiency*, "one-agent efficiency,"[4] obtained by restricting attention to one-agent situations (Lemma 1). Thus, Theorem 1 and Lemma 1 give us another characterization of the Pareto rule. Namely, it is the only rule satisfying *one-agent efficiency*, *reduction-consistency*, and *converse reduction-consistency* (Theorem 2).

[1] For a comprehensive survey of *consistency* and its converse, see Thomson (2000).

[2] The definition of *reduction-consistency* is motivated by the notion of the property, "separability," which is proposed by Moulin (1984) in the model of choosing a point in an interval over which agents have "single-peaked preferences."

[3] Ju (2005) considers the problem of choosing a subset of alternatives over which each agent has a "trichotomous" or "dichotomous" preference, and bases characterizations of plurality-like social choice rules on *efficiency*.

[4] Ching (1996) refers to it as *individuality*.

2 Notation and definitions

There is an infinite set of "potential" agents, indexed by the natural numbers $\mathbb{N}$. Let $\mathcal{N}$ denote the class of non-empty and finite subsets of $\mathbb{N}$. Let $\mathbb{X}$ be a set of potential alternatives. We assume that $\mathbb{X}$ is countably infinite. Let $\mathcal{X}$ denote the class of non-empty and finite subsets of $\mathbb{X}$. We use $\subset$ for strict set inclusion, and $\subseteq$ for weak set inclusion.

Given $N \in \mathcal{N}$, $X \in \mathcal{X}$, and $i \in N$, **agent i's preference relation on X**, denoted by R_i, is a binary relation on X. We assume that R_i satisfies the following two conditions. We say that R_i is **complete** if for each $\{x, y\} \subseteq X$, we have either $x\ R_i\ y$ or $y\ R_i\ x$. Thus, completeness implies that for each $x \in X$, $x\ R_i\ x$. We say that R_i is **transitive** if for each $\{x, y, z\} \subseteq X$, $x\ R_i\ y$ and $y\ R_i\ z$ together imply $x\ R_i\ z$. Throughout our presentation, we restrict attention to preference relations for which distinct alternatives are never indifferent. Formally, R_i is **strict** if for each $\{x, y\} \subseteq X$, $x\ R_i\ y$ and $y\ R_i\ x$ together imply $x = y$. Let P_i denote the strict preference relation derived from R_i. Let $\mathcal{P}(X)$ be the class of all strict preference relations on X. A **preference profile on X** is a list $P \equiv (P_i)_{i\in N}$ such that for each $i \in N$, $P_i \in \mathcal{P}(X)$. A choice problem for N, or simply a **problem for N**, is a pair (X, P) such that $X \in \mathcal{X}$ and $P \in \mathcal{P}^N(X)$.[5] Let $\mathcal{D}^N$ denote the class of all problems for N, and $\mathcal{D} \equiv \bigcup_{N\in\mathcal{N}} \mathcal{D}^N$. A choice rule on $\mathcal{D}$, or simply a **rule on $\mathcal{D}$**, is a correspondence that associates with each $N \in \mathcal{N}$ and each $(X, P) \in \mathcal{D}^N$ a non-empty subset of X. Our generic notation for rules is φ.

We now introduce the Pareto rule. It is the rule that chooses all "Pareto-efficient" alternatives.

Pareto rule, ***PE***: For each $N \in \mathcal{N}$ and each $(X, P) \in \mathcal{D}^N$,

$$PE\,(X, P) \equiv \{x \in X \mid \nexists\ y \in X \setminus \{x\} \quad \text{such that} \quad y\ P_i\ x\ \forall i \in N\}\,.$$

The Pareto rule satisfies the following properties informally defined in the introduction.

Efficiency: For each $N \in \mathcal{N}$, each $(X, P) \in \mathcal{D}^N$, and each $x \in \varphi(X, P)$, there is no $y \in X \setminus \{x\}$ such that for each $i \in N$, $y\ P_i\ x$.

One-agent efficiency: For each $N \in \mathcal{N}$ with $|N| = 1$, each $(X, P) \in \mathcal{D}^N$, and each $x \in \varphi(X, P)$, there is no $y \in X \setminus \{x\}$ such that for each $i \in N$, $y\ P_i\ x$.

Clearly, *efficiency* implies *one-agent efficiency*. As we show in the next section, *one-agent efficiency*, when imposed in conjunction with the following invariance property introduced by Yeh (2006), implies *efficiency*. To define the property, we introduce the following notation. Let $N \in \mathcal{N}$ with $|N| > 1$, $(X, P) \in \mathcal{D}^N$ with $|X| > 1$, $x \in X$, and $N' \in \mathcal{N}$ with $N' \subset N$. Let $X' \equiv \{y \in X \mid y\ P_i\ x\ \forall i \in N \setminus N'\ \} \bigcup \{x\}$. For each $i \in N'$, let $P_i|_{X'}$ denote the restriction of P_i to X'. Formally, for each $\{y, z\} \subseteq X'$, $y\ P_i\ z$ if and only if $y\ P_i|_{X'}\ z$. Then, the **reduced problem of (X, P) relative to x and N'**, denoted $r^x_{N'}(X, P)$, is defined by

$$r^x_{N'}(X, P) \equiv \left(X', (P_i|_{X'})_{i\in N'}\right).$$

[5] $\mathcal{P}^N(X)$ means the Cartesian product of $|N|$ copies of $\mathcal{P}(X)$, indexed by the elements of N. Similar expressions in the rest of the paper should be interpreted in the same manner.

Reduction-consistency: For each $N \in \mathcal{N}$, each $(X, P) \in \mathcal{D}^N$, each $x \in \varphi(X, P)$, and each $N' \in \mathcal{N}$ with $N' \subset N$, we have $r^x_{N'}(X, P) \in \mathcal{D}^{N'}$ and $x \in \varphi(r^x_{N'}(X, P))$.[6]

Next is the converse of *reduction-consistency*, which is central to our analysis.

Converse reduction-consistency: For each $N \in \mathcal{N}$, each $(X, P) \in \mathcal{D}^N$, and each $x \in X$, if for each $N' \subset N$, $r^x_{N'}(X, P) \in \mathcal{D}^{N'}$ and $x \in \varphi(r^x_{N'}(X, P))$, then $x \in \varphi(X, P)$.[7]

3 Results

We present two characterizations of the Pareto rule and start with the following fact, which is used to prove the existence parts of our results.

Proposition 1 The Pareto rule satisfies *converse reduction-consistency.*

Proof. Let $N \in \mathcal{N}$, $(X, P) \in \mathcal{D}^N$, and $x \in X$. Let $N' \in \mathcal{N}$ with $N' \subset N$, $r^x_{N'}(X, P) \in \mathcal{D}^{N'}$, and $x \in PE(r^x_{N'}(X, P))$. We show that $x \in PE(X, P)$. Suppose, by contradiction, that $x \notin PE(X, P)$. Then, there is $y \in X \backslash \{x\}$ such that for each $i \in N$, $y \; P_i \; x$. Thus, (i) for each $i \in N'$, $y \; P_i \; x$ and (ii) for each $i \in N \backslash N'$, $y \; P_i \; x$. It follows that $x \notin PE(r^x_{N'}(X, P))$, a contradiction. *Q.E.D.*

Thanks to Proposition 1, we are now ready to prove our first characterization of the Pareto rule.[8]

Theorem 1 The Pareto rule is the only rule satisfying *efficiency* and *converse reduction-consistency.*

Proof. Clearly, the Pareto rule satisfies *efficiency.* As shown in Proposition 1, the rule satisfies *converse reduction-consistency.* Conversely, let φ be a rule satisfying the two properties. Let $N \in \mathcal{N}$ and $(X, P) \in \mathcal{D}^N$. We show that $PE(X, P) = \varphi(X, P)$. The proof is by induction on $|N|$.

Case 1: $|N| = 1$. Since there is only one agent and his preference relation is strict, $|PE(X, P)| = 1$. By *efficiency,* $\varphi(X, P) \subseteq PE(X, P)$. Thus, $PE(X, P) = \varphi(X, P)$.

Case 2: $|N| > 1$. The induction hypothesis is that for each $N' \in \mathcal{N}$ with $N' \subset N$ and $1 \leq |N'| < |N|$, and each $(X', P') \in \mathcal{D}^{N'}$, $PE(X', P') = \varphi(X', P')$. We show that $PE(X, P) = \varphi(X, P)$. By *efficiency,* $\varphi(X, P) \subseteq PE(X, P)$. We show next that $PE(X, P) \subseteq \varphi(X, P)$. Let $x \in PE(X, P)$. Note that the Pareto rule is *reduction-consistent.* It follows that for each $N' \in \mathcal{N}$ with $N' \subset N$ and each

[6] Alternatively, we can define *reduction-consistency* as follows: for each $N \in \mathcal{N}$, each $(X, P) \in \mathcal{D}^N$, each $x \in \varphi(X, P)$, and each $N' \in \mathcal{N}$ with $N' \subset N$, if $r^x_{N'}(X, P) \in \mathcal{D}^{N'}$, then $x \in \varphi(r^x_{N'}(X, P))$. Our results do not change essentially even if we use this alternative definition.

[7] Alternatively, we can define *converse reduction-consistency* by writing the hypothesis only for all subsets of size 2. Formally, for each $N \in \mathcal{N}$, each $(X, P) \in \mathcal{D}^N$ and each $x \in X$, if for each $N' \subset N$ with $|N'| = 2$, $r^x_{N'}(X, P) \in \mathcal{D}^{N'}$ and $x \in \varphi(r^x_{N'}(X, P))$, then $x \in \varphi(X, P)$. However, if we use this alternative definition, the uniqueness parts of our main results are not guaranteed. For detailed discussions, see the concluding remarks.

[8] The proof of Case 2 of Theorem 1 is an application of the "Elevator Lemma" (Thomson, 2000), which states that if a rule φ is *consistent*, φ' is *conversely consistent*, and $\varphi \subseteq \varphi'$ in the two-agent case, then $\varphi \subseteq \varphi'$ in general.

$r^x_{N'}(X,P) \in \mathcal{D}^{N'}$, $x \in PE\left(r^x_{N'}(X,P)\right)$. Since $|N'| < |N|$, by induction hypothesis, $\varphi\left(r^x_{N'}(X,P)\right) = PE\left(r^x_{N'}(X,P)\right)$. Thus, $x \in \varphi\left(r^x_{N'}(X,P)\right)$. By *converse reduction-consistency*, $x \in \varphi(X,P)$. *Q.E.D.*

Our second characterization of the Pareto rule makes use of the following logical relation between *one-agent efficiency*, *reduction-consistency*, and *efficiency*.[9]

Lemma 1 If a rule satisfies *one-agent efficiency* and *reduction-consistency*, then it satisfies *efficiency*.

Proof. Let φ be a rule satisfying *one-agent efficiency* and *reduction-consistency*. We show that φ is *efficient*. Suppose, by contradiction, that φ is not *efficient*. Then, there is $N \in \mathcal{N}$, $(X,P) \in \mathcal{D}^N$, and $x \in \varphi(X,P)$ in which there is $y \in X\backslash\{x\}$ such that for each $i \in N$, $y \; P_i \; x$. Let $N' \subset N$ with $|N'| = 1$. By *reduction-consistency*, $x \in \varphi\left(r^x_{N'}(X,P)\right)$. Since for each $i \in N$, $y \; P_i \; x$, it follows that x is not the most preferred alternative in the reduction problem $r^x_{N'}(X,P)$. Thus, it contradicts *one-agent efficiency*. *Q.E.D.*

Our second characterization of the Pareto rule is an immediate consequence of Lemma 1 and Theorem 1. We omit its proof.

Theorem 2 The Pareto rule is the only rule satisfying *one-agent efficiency*, *reduction-consistency*, and *converse reduction-consistency*.

We now show that the properties listed in Theorems 1 and 2 are logically independent. For this purpose, we introduce additional rules. The first rule, Z, chooses all feasible alternatives. Formally, for each $N \in \mathcal{N}$ and each $(X,P) \in \mathcal{D}^N$, $Z(X,P) \equiv X$.

Next is the family of fixed-order rules introduced by Yeh (2006).[10] Formally, let $P_0 \in \mathcal{P}(\mathbb{X})$ be a strict preference relation on $\mathbb{X}$. Then, the fixed-order rule relative to P_0, F^{P_0}, chooses the most preferred alternative according to P_0 from the set of Pareto-efficient alternatives. Formally, for each $N \in \mathcal{N}$ and each $(X,P) \in \mathcal{D}^N$, $F^{P_0}(X,P) \equiv \{x \in PE(X,P) \,|\, \text{ for each } \; y \in PE(X,P) \setminus \{x\}, \; x \; P_0 \; y\}$.

The last rule is a modification of the feasibility rule. To define it, we introduce the top rule, Top, which chooses the alternatives most preferred by at least one agent. Formally, for each $N \in \mathcal{N}$, each $(X,P) \in \mathcal{D}^N$, each $x \in X$, and each $i \in N$, if x is the most preferred alternative according to P_i, then $t(x,P_i) \equiv 1$; otherwise, $t(x,P_i) \equiv 0$. Then, $Top(X,P) \equiv \{x \in X \,|\, \exists i \in N \text{ such that } \; t(x,P_i) = 1\}$. Thus, our modified feasibility rule, Z^*, is defined as follows. If there is only one agent, it chooses the most preferred alternative of that agent; otherwise, it chooses all feasible alternatives. For each $N \in \mathcal{N}$ and each $(X,P) \in \mathcal{D}^N$, if $|N| = 1$, then $Z^*(X,P) \equiv Top(X,P)$; otherwise, $Z^*(X,P) \equiv Z(X,P)$.

The feasibility rule satisfies *converse reduction-consistency* but violates *efficiency*. The fixed-order rules satisfy *efficiency* but violate *converse reduction-consistency*.

[9] Similar results have also been obtained in the theory of TU games. For references, see Lemma 5.4 in Peleg (1985), Lemma 5.5 in Peleg (1986), and Lemma 1 in Tadenuma (1992).

[10] This family of rules is inspired by the family of "target rules" studied by Ching and Thomson (1992) in the context of choosing a point in an interval over which each agent has a "single-peaked preference." Given a point or a target in an interval, the associated target rule is described as follows: if the target is "Pareto-efficient," then the rule chooses this point; otherwise, it chooses the point in the set of Pareto-efficient points that is closest to the target.

Thus, the properties listed in Theorem 1 are logically independent. Note that the feasibility rule satisfies *reduction-consistency* but violates *one-agent efficiency*. The fixed-order rules satisfy *one-agent efficiency* and *reduction-consistency*, but violate *converse reduction-consistency*. The modified feasibility rule satisfies *one-agent efficiency* and *converse reduction-consistency*, but violates *reduction-consistency*. Thus, the properties listed in Theorem 2 are independent.

4 Concluding remarks

We proposed an invariance property, *converse reduction-consistency*, which is the converse of *reduction-consistency* (Yeh, 2006), and studied its implications. We showed that the Pareto rule is the only *efficient* rule satisfying the property (Theorem 1), suggesting that *converse reduction-consistency* is quite demanding. In the theory of TU games, Peleg (1985, 1986) and Tadenuma (1992) showed that "individual rationality" together with "consistency" implies "Pareto-optimality." We obtained a similar result for the model under consideration. Namely, *one-agent efficiency* together with *reduction-consistency* implies *efficiency* (Lemma 1). Exploiting Theorem 1 and Lemma 1 gives us another characterization of the Pareto rule: it is the only rule satisfying *one-agent efficiency*, *reduction-consistency*, and *converse reduction-consistency*. This result is parallel to Peleg and Tijs (1996)'s characterization of Nash equilibrium solution for games in strategic form.

One may wonder whether it is crucial for the results to write the hypothesis of *converse reduction-consistency* for all subsets of the set of agents rather than for subsets of size 2. The answer is yes. The following rule, R^*, shows that if the hypothesis of *converse reduction-consistency* is made only for subsets of size 2, the Pareto rule is not the only rule satisfying *efficiency* and the new version of *converse reduction-consistency* just defined. The rule R^* is defined as follows: for each $N \in \mathcal{N}$ and each $(X, P) \in \mathcal{D}^N$, if $|N| = 2$, then $R^*(X, P) \equiv Top(X, P)$; otherwise, $R^*(X, P) \equiv PE(X, P)$.

In Section 3, we introduced the three rules, Z, F^{P_0}, Z^*, to illustrate that the properties listed in each of our characterizations are logically independent. In Section 4, we introduced the rule, R^*, to illustrate that the Pareto rule is not the only rule satisfying efficiency and the converse reduction-consistency for subsets of size 2. To summarize these results, we make the following table.

Property \ Rule	Z	F^{P_0}	Z^*	R^*
one-agent efficiency	No	Yes	Yes	Yes
efficiency	No	Yes	No	Yes
reduction-consistency	Yes	Yes	No	No
converse reduction-consistency	Yes	No	Yes	No
converse reduction-consistency for subsets of size 2	Yes	No	Yes	Yes

Table 1: The notation "Yes" ("No") means that a certain rule satisfies (violates) a certain property.

Appendix

In the text, we claim that the fixed-order rules and the rule R^* violate *converse reduction-consistency*. Here are proofs.

Claim 1 The fixed order rules violate *converse reduction-consistency.*

Proof. The proof is by means of an example. Let $N \equiv \{1, 2, 3\}$, $X \equiv \{x, y, z\}$, and $y\ P_0\ x\ P_0\ z$. Consider the following preference profile:

P_1	P_2	P_3
x	x	y
y	z	z
z	y	x

Consider $x \in X$. The reduced problems with respect to x and $N' \subset N$ with $|N'| = 2$ are the followings:

$P_1\|_{\{x,y,z\}}$	$P_2\|_{\{x,y,z\}}$
x	x
y	z
z	y

$P_1\|_{\{x\}}$	$P_3\|_{\{x\}}$
x	x

$P_2\|_{\{x\}}$	$P_3\|_{\{x\}}$
x	x

The reduced problems with respect to x and $N' \subset N$ with $|N'| = 1$ are the followings:

$P_1\|_{\{x\}}$
x

$P_2\|_{\{x\}}$
x

$P_3\|_{\{x\}}$
x

Clearly, for each $N' \subset N$, $\{x\} = F^{P_0}(r^x_{N'}(X, P))$. However, $\{y\} = F^{P_0}(X, P)$. *Q.E.D.*

Claim 2 The rule R^* violates *converse reduction-consistency.*

Proof. The proof is by means of an example. Let $N \equiv \{1, 2\}$ and $X \equiv \{x, y, z\}$. Consider the following preference profile:

P_1	P_2
x	z
y	y
z	x

The reduced problems with respect to y and $N' \subset N$ with $|N'| = 1$ are the followings:

$P_1\|_{\{x\}}$
y
z

$P_2\|_{\{x\}}$
y
x

Clearly, for each $N' \subset N$, $\{y\} = R^*(r^x_{N'}(X, P))$. However, $R^*(X, P) \equiv \{x, z\}$. *Q.E.D.*

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