E C O N O M I C S   B U L L E T I N

Do investors dislike kurtosis?

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Abstract

We show that decreasing absolute prudence implies kurtosis aversion. The ``proof'' of this relation is usually based on the identification of kurtosis with the fourth centered moment of the return distribution and a Taylor approximation of the utility function. A more sound analysis is required, however, as such heuristic arguments have been shown to be logically flawed.

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1 Introduction

Empirical finance has shown that the distribution of returns on financial assets sampled at a daily, weekly or monthly frequency is not well described by a normal distribution. Most notably, the empirical return distributions tend to be leptokurtic, that is, they are more peaked and fatter tailed than the normal distribution. This leads to considerable interest in the implications of non–Gaussian distributional shape characteristics for investor's welfare and portfolio choice. As the mean and the variance are given by the first two moments of a distribution, an often adopted approach, inspired by traditional mean–variance theory, is to identify preferences for moments higher than the second, and in particular the third and fourth standardized moments, which are frequently associated with skewness and kurtosis (e.g., Scott and Horvath, 1980; Dittmar, 2002; de Athayde and Flôres Jr., 2004; Vinod, 2004; Brandt et al., 2005; Guidolin and Timmermann, 2005; and Jondeau and Rockinger, 2005, 2006).[1]

With regard to kurtosis, it is often argued in an expected utility framework that decreasing absolute prudence, which requires a negative fourth derivative of the expected utility function (Kimball, 1990), implies kurtosis aversion. However, the argument is based on a Taylor approximation of the utility function along with the identification of kurtosis with the centered fourth moment of the return distribution.

It is well–known that such arguments have several weaknesses (e.g., Brockett and Garven, 1998). Briefly, a given value of a particular moment, computed by averaging over the whole support of a random variable, can correspond to very different distributional shapes. Thus, while investors have preferences over different shapes of the distribution of future wealth, their ordering of distributions need not correspond to the ordering with respect to particular moments. Consequently, it is not possible to establish any general connection between the signs of the derivatives of the utility function and some kind of "moment preferences". However, it is shown in this paper that decreasing absolute prudence implies kurtosis aversion when kurtosis is appropriately defined as a comparative shape property, i.e., as peakedness and tailedness.

2 Kurtosis

We assume that the random variables involved are continuous with existing fourth moment. The analysis makes use of a ceteris paribus assumption insofar as kurtosis is the only shape property considered. In particular, all density functions are symmetric. Moreover, we assume that the random variables are supported on the whole real line, but the derivations are easily modified for bounded variables. Zero means can be assumed without loss of generality.

Kurtosis of a density g relative to a density f can then be defined as follows (e.g., Finucan, 1964).

[1] The book edited by Jurczenko and Maillet (2006) is entirely devoted to the moment–based approach to portfolio selection and asset pricing. A partial review of earlier papers arguing in this direction can be found in Brockett and Kahane (1992) and Brockett and Garven (1998).

Definition 1 *If there are two distributions with densities f and g, each being symmetrical with mean zero and common variance $\sigma^2 = \sigma_f^2 = \sigma_g^2$, then g is said to have higher kurtosis than f, if there are numbers a and b such that*

$$g(x) < f(x) \quad for \quad a < |x| < b,$$

while

$$g(x) > f(x) \quad for \quad |x| < a \quad or \quad |x| > b.$$

The above definition of kurtosis captures the pattern which is usually reported in the empirical finance literature since the publication of the seminal papers of Mandelbrot (1963) and Fama (1965).

Note that kurtosis according to Definition 1 means that g has more probability weight in the tails *and* is more concentrated about the mean, so that we cannot say that it is more dispersed than f, and thus kurtosis does not fit into the Rothschild and Stiglitz (1970) framework of increasing risk. An example of two densities satisfying the conditions of Definition 1 is shown in Figure 1. Both densities are symmetric around $\mu = 5$ and have the same variance $\sigma^2 = 2.8$. However, while $g(x)$ has fatter tails, it has also much more weight in the center.[2] Thus, if you have to invest a certain amount of money in one of two portfolios, with return densities given by those in Figure 1, it is not immediately obvious which one to choose. Loosely speaking, while risk aversion implies that investors dislike large losses more than they like large profits, kurtosis aversion requires that they dislike fat tails more than they like high peaks.

Finucan (1964) has shown that, if g has greater kurtosis than f according to Definition 1, then the fourth moment is greater for the g distribution than for the f distribution.[3] The reverse, however, is not true, and a counterexample was already presented by Kaplansky (1945), viz a density less peaked but with a higher fourth moment than the normal.

3 Decreasing prudence implies kurtosis aversion

3.1 Definitions and assumptions

To show that decreasing absolute prudence implies kurtosis aversion, we employ the concept of stochastic dominance, which was introduced into economics by Hadar and Russell (1969) and Hanoch and Levy (1969). This theory deals with the development of conditions for a given distribution to be preferred over an alternative from the point of view of each and every individual, when all individuals' utility functions, U, are assumed

[2] Density f is the normal density function with mean 5 and variance 2.8, while g is a *discrete scale normal mixture* with $g(x) = 0.8 \times \phi(x; 5, 1) + 0.2 \times \phi(x; 5, 10)$, where $\phi(x; \mu, \sigma^2) = (2\pi\sigma^2)^{-1/2} \exp\{-(x-\mu)^2/(2\sigma^2)\}$ is the density of a normal distribution with mean μ and variance σ^2.

[3] In the literature, "kurtosis" often refers to the standardized fourth moment. However, we will use this term to allude to the (comparative) shape property in Definition 1, because it is exactly this property for which the (standardized) fourth moment is often believed to be a reasonable proxy.

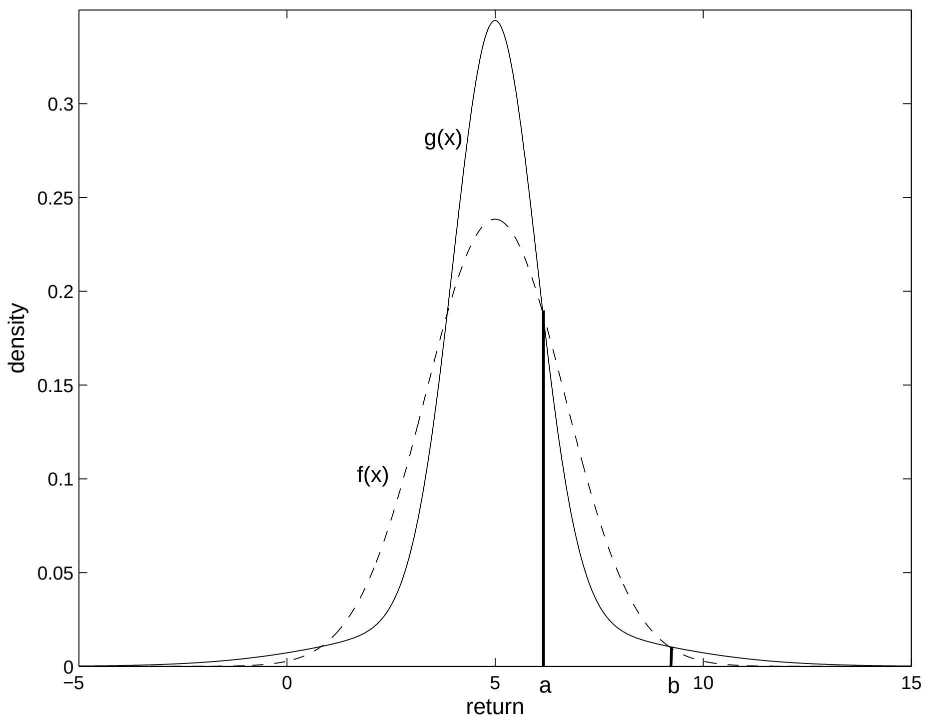

Figure 1: Illustration of Definition 1.

to belong to a given class of functions (see Levy, 1992, for an overview). The stochastic dominance rule of order n identifies the preferred distributions for utility functions in $\mathcal{U}_n$, where

$$\mathcal{U}_n = \{U : (-1)^i U^{(i)}(x) \leq 0, \quad \forall x \in \mathbb{R}, \quad i = 1, \ldots, n\}, \tag{1}$$

and $U^{(i)}(x)$ is the ith derivative of U at x. That is, random variable X is said to nth–order stochastically dominate random variable Y if and only if $\mathrm{E}[U(X)] \geq \mathrm{E}[U(Y)]$ for all $U \in \mathcal{U}_n$. It is often argued that, for a typical investor, $U \in \mathcal{U}_4$, where the signs follow from non–satiation, risk aversion, prudence (or decreasing absolute risk aversion), and decreasing absolute prudence, respectively. We will not make an attempt to justify these assumptions. We only note that they are frequently deemed plausible and show that they indeed imply kurtosis aversion.

To this end, define the repeated integrals

$$F_n(t) = \int_{-\infty}^{t} F_{n-1}(x)dx, \quad n \geq 1, \quad \text{where} \quad F_0(t) = \int_{-\infty}^{t} f(x)dx \tag{2}$$

is the cumulative distribution function associated with density f. Also, let

$$\Delta_n(t) = F_n(t) - G_n(t), \tag{3}$$

where $G_n(t)$ is defined analogous to (2). The f distribution stochastically dominates the g distribution to the nth order if and only if (Jean, 1980; Ingersoll, 1987, p. 138)

$$\Delta_{n-1}(t) \leq 0, \quad \forall t \in \mathbb{R}, \quad \text{and} \quad \lim_{t\to\infty} \Delta_k(t) \leq 0, \quad k = 1, \ldots, n-2. \tag{4}$$

We wish to establish that an ordering of distributions with respect to kurtosis implies an expected utility ordering for all $U \in \mathcal{U}_4$. The definition of kurtosis in (1) implies that f and g have the same first and second moments. Thus, using (cf. Rao, 1973)

$$\mu_f = \int_0^\infty [1 - F_0(x) - F_0(-x)]dx, \quad \text{and} \quad \sigma_f^2 = 2\int_0^\infty [F_1(x) - x + \mu_f + F_1(-x)]dx - \mu_f^2,$$

where μ_f is the mean associated with density f, we have $\Delta_1(t) \to 0$ and $\Delta_2(t) \to 0$ as $t \to \infty$, so that the f distribution fourth–order stochastically dominates the g distribution if and only if $\Delta_3(t) \leq 0$ for all t.

We also need a relation between $F_n(t)$ defined in (2) and the lower partial moment of order n of f, defined by

$$\mathrm{LPM}_n^f(t) = \int_{-\infty}^t (t-x)^n f(x)dx. \tag{5}$$

We have (Bawa 1975; Ingersoll, 1987)

$$\mathrm{LPM}_n^f(t) = n!F_n(t). \tag{6}$$

3.2 Establishing kurtosis aversion

Marsaglia, Marshall and Proschan (1965) provided a generalization of Finucan's (1964) result concerning the relation between kurtosis and the standardized fourth moment, which, as it will serve as an input in our subsequent analysis, will be stated as Lemma 1. To do so, we adopt the notation

$$\nu_s^f = \int_{-\infty}^\infty |x|^s f(x)dx, \quad \nu_s^g = \int_{-\infty}^\infty |x|^s g(x)dx. \tag{7}$$

Lemma 1 *(Marsaglia, Marshall and Proschan, 1965) Let f and g satisfy the assumptions of Definition 1, then*

$$\nu_s^f > \nu_s^g \quad \text{for} \quad 0 < s < 2.$$

If $s < 0$ or $s > 2$ and ν_s^g is finite, then ν_s^f is finite and

$$\nu_s^f < \nu_s^g.$$

We also observe the following symmetry property of the functions defined in (3).

Lemma 2 *For symmetric densities f and g with mean zero and common positive integer even moments up to the N^{th} power, the quantities (3) satisfy $\Delta_n(t) = (-1)^{n+1}\Delta_n(-t)$, $n = 1, \ldots, N+1$.*

Proof. For even n, we have, using (6),

$$\begin{aligned} n![\Delta_n(-t) + \Delta_n(t)] &= \int_{-\infty}^{-t} (t+x)^n [f(x) - g(x)]dx + \int_{-\infty}^t (t-x)^n [f(x) - g(x)]dx \\ &= \int_{-\infty}^\infty (t-x)^n [f(x) - g(x)]dx = 0, \end{aligned}$$

where we have used symmetry of the densities involved, i.e., $f(x) = f(-x)$.

For n odd, we have

$$\begin{aligned} LPM_n^f(t) - LPM_n^f(-t) &= \int_{-\infty}^{t} (t-x)^n f(x)dx - \int_{-\infty}^{-t} (-t-x)^n f(x)dx \\ &= \int_{-\infty}^{t} (t-x)^n f(x)dx + \int_{-\infty}^{-t} (t+x)^n f(x)dx \\ &= \int_{-\infty}^{-t} [(t-x)^n + (t+x)^n] f(x)dx + \int_{-t}^{t} (t-x)^n f(x)dx. \end{aligned} \tag{8}$$

Using the binomial formula, we find

$$(t-x)^n + (t+x)^n = \sum_{i=0}^{n} \binom{n}{i} t^{n-i}(-1)^i x^i + \sum_{i=0}^{n} \binom{n}{i} t^{n-i} x^i = 2 \sum_{i=0}^{(n-1)/2} \binom{n}{2i} t^{n-2i} x^{2i},$$

and

$$\int_{-t}^{t} (t-x)^n f(x)dx = \sum_{i=0}^{n} \binom{n}{i} t^{n-i}(-1)^i \int_{-t}^{t} x^i f(x)dx = 2 \sum_{i=0}^{(n-1)/2} \binom{n}{2i} t^{n-2i} \int_{-t}^{0} x^{2i} f(x)dx,$$

using that, by symmetry, $\int_{-t}^{t} x^i f(x)dx = 0$ for i odd. Thus, (8) becomes

$$\begin{aligned} LPM_n^f(t) - LPM_n^f(-t) &= 2 \sum_{i=0}^{(n-1)/2} \binom{n}{2i} t^{n-2i} \int_{-\infty}^{0} x^{2i} f(x)dx \\ &= \sum_{i=0}^{(n-1)/2} \binom{n}{2i} t^{n-2i} \int_{-\infty}^{\infty} x^{2i} f(x)dx = \sum_{i=0}^{(n-1)/2} \binom{n}{2i} t^{n-2i} \mathrm{E}(X^{2i}). \end{aligned}$$

As the moments involved are identical for the densities f and g, we conclude that $LPM_n^f(t) - LPM_n^f(-t) = LPM_n^g(t) - LPM_n^g(-t) \Leftrightarrow \Delta_n(t) = \Delta_n(-t)$. ■

Now we can show that kurtosis aversion holds within all utility functions $U \in \mathcal{U}_4$.

Proposition 3 *The f distribution fourth–order stochastically dominates the g distribution if g has more kurtosis than f according to Definition 1.*

Proof. We will successively elaborate the graphs of the functions $\Delta_0(t)$, $\Delta_1(t)$, $\Delta_2(t)$, and $\Delta_3(t)$, in order to show that $\Delta_3(t) < 0$ for all t.

A zero of a function $x \mapsto h(x)$ will be called *nodal* if the function changes sign as x passes through this point. A function h has a local extremum at a point ξ if and only if ξ is a nodal zero of its derivative h'.

(1) Let c be defined by $\int_0^c f(x)dx = \int_0^c g(x)dx$. This fixes a unique value $c \in (a, b)$. The equation $\Delta_0(t) = 0$ has three roots, namely $-c$, 0, and c, all of which are nodal zeros. That is $\Delta_0(t) < 0$ for $t < -c$ or $0 < t < c$, and $\Delta_0(t) > 0$ for $-c < t < 0$ or $t > c$.

(2) As $\Delta_0(t) = \Delta_1'(t)$, and the zeros of $\Delta_0(t)$ are nodal, $\Delta_1(t)$ has three local extrema, located at $-c$, 0, and c. Also, $\lim_{t\to-\infty} \Delta_1(t) =: \Delta_1(-\infty) = 0$ and $\Delta_1'(t) < 0$ for $t < -c$. This implies $\Delta_1(t) < 0$ for $t \leq -c$. However, $\Delta_1(t)$ has a nodal zero in $(-c, 0)$, because, from Lemma 1 and (6),

$$\Delta_1(0) = -\int_{-\infty}^{0} x\{f(x) - g(x)\}dx = \frac{1}{2}\int_{-\infty}^{\infty} |x|\{f(x) - g(x)\}dx = (\nu_1^f - \nu_1^g)/2 > 0.$$

By symmetry (Lemma 2), then, $\Delta_1(t)$ has exactly two zeros, which are nodal and equal in absolute value, and which we denote by $-d$ and d, with $d \in (0, c)$. Also, $\Delta_1(t) < 0$ for $t < -d$ or $t > d$, and $\Delta_1(t) > 0$ for $-d < t < d$.

(3) As $\Delta_1(t) = \Delta_2'(t)$, and the zeros of $\Delta_1(t)$ are nodal, $\Delta_2(t)$ has two local extrema, located at $-d$ and d. Also, $\Delta_2(-\infty) = 0$ and $\Delta_2'(t) < 0$ for $t < -d$. In addition, we know that $\Delta_2(0) = (\sigma_f^2 - \sigma_g^2)/2 = 0$. Thus, by Lemma 2, we have $\Delta_2(t) < 0$ for $t < 0$ and $\Delta_2(t) > 0$ for $t > 0$.

(4) $\Delta_3(t)$ has only one extremal point, located at $t = 0$. By Lemma 1 and (6), $\Delta_3(0) = (\nu_3^f - \nu_3^g)/2 < 0$. Then, from $\Delta_3(-\infty) = 0$, $\Delta_3'(t) < 0$ for $t < 0$ and Lemma 2, we have $\Delta_3(t) < 0 \ \forall t \in \mathbb{R}$.

■

4 Conclusions

We have shown that decreasing absolute prudence implies kurtosis aversion. Kurtosis was defined to be a property of the entire distributional shape rather than just a particular moment, as is required for establishing stochastic dominance relationships. The usefulness of the result derives from the growing interest in the implications of distributional properties such as skewness and kurtosis for the welfare and the portfolio decisions of economic agents. For example, if we accept the argument that the special features of financial return data do allow for the use of particular moments as proxies for characteristic shape properties, then the result of the present paper may provide a partial theoretical justification for the moment–based approach to portfolio choice, which assumes that agents have preferences over a few moments of the portfolio return distribution, such as mean, variance, and moment–based *proxies* for skewness and kurtosis.

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