E C O N O M I C S B U L L E T I N

A Note on Nonlinear Income Taxes and the Utility Possibility Set

Tommy Andersson
Department of Economics, Lund University

Abstract

This note analyzes constrained Pareto efficient nonlinear income tax schedules that are monotonic chains to the left. The main result demonstrates that if all individuals have a positive consumption at the tax schedule that maximizes the utility of the worst-off individual, then the constrained utility possibility set is convex. As a consequence, all constrained Pareto efficient nonlinear income tax schedules that are monotonic chains to the left can be identified by maximizing a weighted summation of net utilities.

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1. Introduction

A commonly used procedure to identify constrained Pareto efficient nonlinear income tax schedules is to maximize a welfare function, by choosing a consumption-income bundle for each individual, subject to a budget constraint and a set of self-selection constraints, which guarantee that no individual prefers some other bundle to his/her own bundle. In a model with a continuum of individuals, Lollivier and Rochet (1983) demonstrated that if the utility functions are quasi-linear, then the solution to the optimal income tax problem in Mirrlees (1971) can be determined by solving a reduced-form problem.[1] In this reduced-form problem a weighted summation of net utilities is maximized, by choosing only a consumption level for each individual, subject to that the consumption level is non-negative and non-decreasing in skill level. The weights in the objective function reflect the redistributional objectives of the social welfare maximizer and the constraint in the maximization problem is a necessary condition for self-selection. The maximization problem identifies an optimal consumption vector and the income for each individual is expressed as a function of this consumption vector. Weymark (1986a) derived this reduced-form problem in a finite economy, and investigated the set of nonlinear income tax schedules that are a monotonic chains to the left, i.e., the set of tax schedules where each individual is indifferent between receiving his/her own consumption and the consumption of the next-lowest skilled individual. Bunching and comparative static properties are analyzed in Weymark (1986b,1987).

As described above, in the reduced-form problem, the income for each individual is expressed as a function of the consumption vector. However, there is no guarantee that the reduced-form individual utility functions (i.e. individual utility as a function of only the consumption vector) are concave. If this is not the case, then we cannot be certain that the constrained utility possibility set is convex, and, as a consequence, we cannot be certain that all constrained Pareto efficient nonlinear income taxes can be identified by solving the reduced-form problem, see, e.g., Mas-Colell et al. (1995,pp.560). This is a problem since the weights in the objective function represent the redistributional objectives of the social welfare maximizer, so if not all constrained Pareto efficient tax schedules can be identified, then it is not possible to capture all redistributional aspects of nonlinear income taxation, by solving the reduced-form nonlinear income tax problem. In this paper, we explore this problem in an economy with a finite number of individuals.

We consider the reduced-form nonlinear income tax problem from Weymark (1986a) and investigate, as, e.g., Guesnerie and Seade (1982) and Weymark (1986a,1986b,1987), the set of nonlinear income tax schedules that are a monotonic chains to the left. More explicitly, we examine properties of the constrained utility possibility set, given the above type of income tax schedules. As it turns out, the tax schedule that maximizes the utility of the worst-off individual reveals important properties of the constrained utility possibility set. Our main result demonstrates that if all consumers have a

[1] See also Boadway et al. (2000) and Ebert (1992).

positive consumption level at this schedule, then the constrained utility possibility set is convex, in the reduced-form problem, and, as a consequence, all constrained Pareto efficient nonlinear income tax schedules that are monotonic chains to the left can be identified by solving the reduced-form problem, for some selection of weights, by the supporting hyperplane theorem.

2. The Model and Basic Definitions

We consider an economy consisting of $n \geq 2$ individuals indexed by $i = 1, ..., n$. Individuals have preferences over consumption and labor and we shall denote individual i's consumption and labor supply by c_i and l_i, respectively. Individuals differ in ability but have the same quasi-linear utility function $\widetilde{u} : \mathbb{R}^2 \rightarrow \mathbb{R}$, represented by: $\widetilde{u}(c_i, l_i) = v(c_i) - l_i$, where $v(c_i)$ is a continuous function with: $v(0) = 0$, $v'(c_i) > 0$, $v''(c_i) < 0$, $\lim_{z \rightarrow 0} v'(z) \rightarrow \infty$ and $\lim_{z \rightarrow \infty} v'(z) \rightarrow 0$. The ability of individual i is normalized to equal i's fixed wage rate, w_i, and we shall assume, without loss of generality, that: $0 < w_1 < ... < w_n$. Income is given by: $y_i = w_i l_i$. In terms of consumption and income, the utility function $u_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by:

$$u_i(c_i, y_i) = v(c_i) - \frac{y_i}{w_i}. \tag{1}$$

As observed by Weymark (1986a), the utility functions defined in equation (1) are the appropriate cardinalizations for making interpersonal comparisons.

A bundle is a vector $x_i = (c_i, y_i) \in \mathbb{R}^2$ and an allocation or, equivalently, a nonlinear income tax schedule $x = (x_1, ..., x_n) \in \mathbb{R}^{2n}$ is a vector of n bundles. The marginal rate of substitution for individual i at bundle $x_i = (c_i, y_i)$ is given by:

$$\text{MRS}_i(c_i, y_i) = \frac{1}{w_i v'(c_i)}. \tag{2}$$

The assumption of quasi-linearity implies that the slopes of the indifference curves are independent of income. Moreover, individuals with higher ability have flatter indifference curves and, as a consequence, the single-crossing property holds. We shall also, without loss of generality, normalize the price of the consumption good to one. There is a constant-returns-to-scale technology specifying feasible allocations:

$$\sum_{i=1}^{n} (y_i - c_i) \geq 0. \tag{3}$$

As is standard in the literature, we shall suppose that the social planner knows the distribution of wages and the functional form of the utility function. However, neither w_i nor l_i is separately observable but the (pretax) income y_i is observable for each individual. Consequently, to induce individuals to report their ability truthfully, the social planner offers each individual a choice from the same set of bundles, where each bundle is intended for a particular individual and where the bundles must satisfy the self-selection constraint: $u_i(x_i) \geq u_i(x_j)$ for all i, j. However, since the single-crossing condition is satisfied, it is easy to demonstrate that self-selection is satisfied for each individual i if (see, e.g., Cooper, 1984):

$$w_i v(c_i) - y_i \geq w_i v(c_{i-1}) - y_{i-1} \text{ for all } i \neq 1, \tag{4}$$
$$w_i v(c_i) - y_i \geq w_i v(c_{i+1}) - y_{i+1} \text{ for all } i \neq n. \tag{5}$$

Henceforth, we shall refer to constraints (4) and (5) as the downward SS constraint and the upward SS constraint, respectively. A tax schedule that satisfies the feasibility and the SS constraints is said to be a constrained tax schedule. A constrained tax schedule x is said to be constrained Pareto efficient if there is no other constrained tax schedule x' where $u_i(x_i') \geq u_i(x_i)$ for all i with $u_i(x_i') > u_i(x_i)$ for some i. The social welfare function $W : \mathbb{R}^{2n} \to \mathbb{R}$ is given by:

$$W(x) = \sum_{i=1}^{n} \alpha_i u_i(x_i). \tag{6}$$

In the above specification, the vector $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$ represents the welfare weights and we shall suppose, without loss of generality, that: $\sum_{i=1}^{n} \alpha_i = 1$. It is well-known that the problem for the social planner can be thought of as choosing the allocation directly, rather than indirectly through the choice of a nonlinear income tax schedule, see, e.g., Stiglitz (1982) or Weymark (1986a,1986b,1987). Hence, the nonlinear taxation problem for the social planner is:

Problem 1 *Choose an allocation x to maximize the social welfare function* (6) *subject to the feasibility constraint* (3) *and the SS constraints* (4) *and* (5).

3. The Reduced-form Problem

In this section, we derive a reduced-form nonlinear income tax problem, which involves only the consumption vector.[2] We first state a few well-known results that will be useful for future reference.

Lemma 1 *For any solution to Problem* 1 *and $i \neq n$ it is true that:* (i) *the feasibility constraint is binding,* (ii) $(c_{i+1}, y_{i+1}) \geq (c_i, y_i)$ *and* (iii) $u_n(x_n) > ... > u_1(x_1)$.

Since the above results are well-known, we shall omit their proofs. From Part (ii) of Lemma 1, it is clear that the consumption vector must belong to:

$$C = \{c \in \mathbb{R}_+ \mid 0 \leq c_1 \leq ... \leq c_n\}.$$

[2] The reduced-form income tax problem was first investigated by Lollivier and Rochet (1983) in an economy with a continuum of individuals and by Weymark (1986a) in a finite economy.

Lemma 2 *C is a convex set.*

Proof. Suppose that $c \in C$ and $c' \in C$, then C is a convex set if $c'' = (1-\lambda)c + \lambda c' \in C$ for all $\lambda \in [0,1]$. Since $c \in C$ and $c' \in C$, it must be true that $c_{i-1} \leq c_i$ and $c'_{i-1} \leq c'_i$ for all $i \neq 1$. But then $(1-\lambda)c_{i-1} \leq (1-\lambda)c_i$ and $\lambda c'_{i-1} \leq \lambda c'_i$ for all $\lambda \in [0,1]$. Adding these inequalities yields: $c''_{i-1} = (1-\lambda)c_{i-1} + \lambda c'_{i-1} \leq (1-\lambda)c_i + \lambda c'_i = c''_i$ for all $i \neq 1$ and all $\lambda \in [0,1]$. Hence $c'' = (1-\lambda)c + \lambda c' \in C$ for all $\lambda \in [0,1]$. ■

A tax schedule where all downward SS constraints are binding is refereed to by Guesnerie and Seade (1982) as a monotonic chain to the left. In the remaining part of the paper, we limit our attention to allocations that are monotonic chains to the left, i.e., allocations where all $n-1$ downward SS constraints are binding. Since the feasibility constraint also is binding, by Part (i) of Lemma 1, we can, as demonstrated by Weymark (1986a), use these n binding constraints in order to solve for the n incomes, which yields:

$$y_1(c) = \frac{1}{n}\left(\sum_{j=1}^{n} c_j - \sum_{j=1}^{n-1}(n-j)w_{j+1}(v(c_{j+1}) - v(c_j))\right), \tag{7}$$

$$y_i(c) = y_{i-1} + w_i(v(c_i) - v(c_{i-1})) \text{ for all } i \neq 1. \tag{8}$$

From the above equations, it is clear that the incomes can be expressed as a function of the consumption vector, c. But then we can also express the consumption bundles, the tax schedule and the individual utilities as a function of the consumption vector, i.e., $x_i(c) = (c_i, y_i(c))$, $x(c) = (c, y(c))$ and $u_i(c) = u_i(x_i(c)) = u_i(c_i, y_i(c))$. Note also that if the incomes are given by the above equations and $c \in C$, then tax schedule $x(c)$ satisfies the feasibility requirement and the SS constraints, so it is a constrained tax schedule. By substituting equations (7) and (8) into the welfare function (6), we obtain the reduced-form social welfare function $\widetilde{W} : C \to \mathbb{R}$, i.e.:

$$\widetilde{W}(c) = \frac{1}{n}\sum_{i=1}^{n}(\beta_i v(c_i) - \rho c_i), \tag{9}$$

where:

$$\beta_i = (w_i - (n-i)(w_{i+1} - w_i))\sum_{j=1}^{i}\frac{\alpha_j}{w_j} + (iw_{i+1} - (i-1)w_i)\sum_{j=i+1}^{n}\frac{\alpha_j}{w_j} \text{ for all } i, \tag{10}$$

$$\rho = \sum_{j=1}^{n}\frac{\alpha_j}{w_j}, \tag{11}$$

and w_{n+1} is an arbitrary real number.[3] The constant β_i is refereed to by Weymark (1986a,p.209) as an adjusted wage rate. The reduced-form problem is the following:

[3]Note that $\beta_n = \rho w_n$ for all $w_{n+1} \in \mathbb{R}$. Weymark (1986a) assumes that the welfare weights sum to n and works with a monotone transformation of the utility functions, given by: $\widehat{u}(c_i, y_i) = w_i v(c_i) - \gamma y_i$ for all i. These observations account for the differences between conditions (9) and (10) in this paper and conditions (23) and (24) in Weymark (1986a).

Problem 2 *Choose a consumption vector c to maximize the reduced-form social welfare function* (9)*, subject to $c \in C$.*

Weymark (1986a) demonstrated that the solution to Problem 1 can be found by solving for the optimal consumption vector in Problem 2 and substitute the optimal consumption vector into equations (7) and (8) to find the optimal incomes. The following result relates the adjusted wage rates to the optimal consumption vector.

Lemma 3 (Weymark, 1986b) *Suppose that $\widehat{c} \in C$ is an optimal solution to Problem* 2 *and that $\beta_i > 0$ for all i, then* (i) *$\widehat{c}_i > 0$ for all i and* (ii) *$\widehat{c}_i = \widehat{c}_{i+1} > 0$ if $\beta_i \geq \beta_{i+1}$ for some $i \neq n$.*

4. The Constrained Utility Possibility Set

We next investigate the constrained utility possibility set (U, henceforth). At the constrained tax schedule $x(c)$, the utility for individual i is given by $u_i = u_i(c)$, and the vector of utilities is given by $u = (u_1, ..., u_n) \in \mathbb{R}^n$. Formally, U is defined as:

$$U = \{u \in \mathbb{R}^n \mid u_i \leq u_i(c') \text{ for all } i \text{ and some constrained tax schedule } x(c')\}.$$

By definition of constrained Pareto efficiency, the vector of utility values u of a constrained Pareto efficient allocation must belong to the boundary of the constrained utility possibility set (U^0, henceforth), i.e.:

$$U^0 = \{u \in \mathbb{R}^n \mid \text{there is no } u' \in U \text{ such that } u'_i \geq u_i \text{ for all } i \text{ with } u'_i > u_i \text{ for some } i\}.$$

It is well-known that if U is a convex set, then all constrained Pareto efficient allocations can be identified by maximizing a weighted summation of net utilities for some selection of welfare weights, by the supporting hyperplane theorem, see, e.g., Negishi (1960) or Mas-Colell et al. (1995,pp.560). In the remaining part of this paper, we investigate under which circumstances that U is a convex set.

Our first two results demonstrate that if the reduced-form utility function for individual 1 is concave, then the reduced form utility function for all individuals are concave (Proposition 1), and, as a consequence, the constrained utility possibility set, in the reduced-form problem, is convex (Proposition 2).

Proposition 1 *Suppose that the incomes are given by equations* (7) *and* (8) *and that $u_1(c)$ is concave in C, then $u_i(c)$ is concave in C for all i.*

Proof. From the identity (1) and equations (7) and (8), it follows that:

$$u_1(c) = \frac{1}{nw_1}\left(\sum_{j=1}^{n} v(c_j)(w_j - (n-j)(w_{j+1} - w_j)) - c_j\right), \tag{12}$$

$$u_i(c) = \frac{1}{w_i}\left(w_{i-1}u_{i-1}(c) + v(c_{i-1})\left(w_i - w_{i-1}\right)\right) \text{ for all } i \neq 1. \tag{13}$$

Since C is a convex set, by Lemma 2, $v(\cdot)$ is concave and $0 < w_1 < ... < w_n$, it is immediate from condition (13) that $u_i(c)$ is concave in C for all $i \neq 1$ when $u_1(c)$ is concave in C. ■

Proposition 2 *Suppose that the incomes are given by equations* (7) *and* (8) *and that* $u_1(c)$ *is concave in* C*, then* U *is a convex set.*

Proof. Suppose that the constrained tax schedules $x(c)$ and $x(c')$ correspond to utility vectors $u \in U$ and $u' \in U$, respectively. We need to demonstrate that $(1-\lambda)u + \lambda u' \in U$ for all $\lambda \in [0,1]$, i.e., that there is another constrained tax schedule $x(c'')$ where $u_i(c'') \geq (1-\lambda)u_i(c) + \lambda u_i(c'_i)$ for all i and all $\lambda \in [0,1]$. Define now $c'' = (1-\lambda)c + \lambda c'$, and note that $c'' \in C$, by Lemma 2. But since $u_1(c)$ is concave in C, by assumption, the utility functions are concave in C, by Proposition 1, and, therefore, $u_i(c'') \geq (1-\lambda)u_i(c) + \lambda u_i(c'_i)$ for all i and all $\lambda \in [0,1]$. ■

From the above two propositions, we conclude that a sufficient condition for the constrained utility possibility set to be convex is that the reduced-form utility function for individual 1 is concave. Let us now investigate this condition in more detail. Note first that $u_1(c)$ is concave in C if:

$$u_1((1-\lambda)c + \lambda c') \geq (1-\lambda)u_1(c) + \lambda u_1(c') \text{ for } c, c' \in C \text{ and all } \lambda \in [0,1].$$

Using equation (12), this condition reduces to:

$$\frac{1}{nw_1} \sum_{j=1}^{n} \left[w_j - (n-j)(w_{j+1} - w_j)\right] \left(v((1+\lambda)c_j + \lambda c'_j) - (1-\lambda)v(c_j) - \lambda v(c'_j)\right) \geq 0.$$

Since $v(\cdot)$ is a concave function, we see that this inequality is satisfied if:

$$w_j - (n-j)(w_{j+1} - w_j) > 0 \text{ for all } j = 1, ..., n.$$

From the specification of the adjusted wage rates (10), this condition reduces to:

$$\beta_j > 0 \text{ for all } j = 1, ..., n \text{ when } \alpha_1 = 1 \text{ and } \alpha_i = 0 \text{ for all } i \neq 1. \tag{14}$$

But this selection of welfare weights is the selection that maximizes the utility of the worst-off individual, i.e., individual 1, by Part (iii) of Lemma 1. Moreover, if $\beta_j > 0$ for all j at the tax schedule that maximizes the utility of the worst-off individual, then the consumption is positive for all individuals, by Lemma 3. We gather our observations in the following theorem.

Theorem 1 *If condition* (14) *holds or, equivalently, if the consumption vector that maximizes the utility of the worst-off individual is positive, then every constrained Pareto efficient nonlinear tax schedule that is a monotonic chain to the left can be identified by solving Problem* 2*, for some selection of welfare weights.*

Proof. If condition (14) holds, then U is a convex set, by Proposition 2, and the conclusion follows directly from the supporting hyperplane theorem. ■

Note, finally, that in the case when $n \geq 3$, at least one welfare weight must be negative, in order to identify some of the constrained Pareto efficient tax schedules. This result is attributed to Andersson (2007).

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