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The Stability of Dynamic Contests with Asymmetric and Endogenous Prizes

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Abstract

The paper develops a simple theoretical framework for analyzing repeated contests. At each stage of the infinitely repeated game, a Tullock contest is played by two players. We consider local stability of the Nash equilibrium with respect to adjustment speed and the level of the prize. The model is extended to an asymmetric valuation of the prize and to the case with an endogenous prize, where the level of the prize is influenced by the investments of the players.

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1 Introduction

In various circumstances, contests are used to model the rivalry for a resource.[1] There are many examples where the contest is not only organized once. For example, many military conflicts endure for long time periods. Furthermore, duopolists compete for customers every season just like politicians or lobbyists, who campaign for a political rent repeatedly. Therefore, we consider a dynamic contest, where a Tullock contest is played every period. We assume the players to adjust their expenditures according to their marginal profits and we analyze the local asymptotic stability of the Nash equilibrium. We solve the simplest possible case with an exogenous prize in the contest. In this case, our contest model is algebraically isomorphic to a duopoly model, with an iso-elastic demand function. The supplied quantities of the duopolists map to the contest investments, with the prize representing the market size.[2] Our finding is that the contest is asymptotically stable if the prize is high enough or the adjustment speeds are low enough. The adjustment speed can be interpreted as aggressiveness of bidding in our contest setting. Thus, the contest will not be stable if the players use a very aggressive adjustment strategy for their bids.

As there are many contests where the prize is not valued equally by the players, we will generalize the game regarding asymmetries. In addition to the above findings, we show that a sufficient asymmetric valuation has a destabilizing effect.

If we consider the above-mentioned examples, we observe that contested prizes are often endogenous. In a war, the contested land will be damaged by the struggle for it. Thus, we have a negative externality of the contest investments. Otherwise, there are cases where we have positive externalities on the prize. We can use contests to model advertising campaigns in oligopoly markets. Here, investments would have positive effects on market size. Furthermore, in innovation contests, the prize can rise with investments as a superior innovation yields a higher reward. This can also be true in a rent seeking or lobbying setting, where a higher rent is assigned if the lobbying efforts are rising.[3] Therefore, we extend the endogenous prize model of Shaffer (2006) to our dynamical setting. We find that a positive externality has a positive impact on stability, whereas negative externalities have destabilizing effects. In this case, a higher basic contested prize only has a stabilizing effect if the externality is positive or negative and sufficiently small-sized.

Finite repeated contests have been modeled as elimination tournaments, see the seminal paper of Rosen (1986) and e. g. the more recent paper of Groh et al. (2003). The optimal number of stages in a contest is considered by Gradstein and Konrad (1999) and Moldovanu and Sela (2006). Contrary to elimination tournaments, we consider a constant number of participants and an infinite number of stages.

[1] For a recent survey containing an overview over examples of use see Konrad (2007).

[2] There are various examples for dynamic duopoly models. The first consideration of stability of oligopolies is Theocharis (1960). For surveys of the topic see Okuguchi (1976) as well as Gandolfo (1996). See also Puu (1993), pp. 205-217, who considers the iso-elastic demand function with adaptive players.

[3] See Chung (1996) for a rent seeking contest with an endogenous prize, which is increasing in aggregate investments of the contestants.

A dynamic contest is modeled by Xu and Szidarovszky (1999) which develop a model of dynamic rent-seeking games under continuous and discrete time scales. They use a general contest success function[4] with the probability that player i will win the rent given as $p_i = \frac{f_i(x_i)}{\sum\limits_j f_j(x_j)}$. They focus on the "production functions"[5] $f_i(x_i)$ in the contest and assume the prize of the contest to be 1. Furthermore, in the discrete part of their model, the players have naive expectations. Whereas we use a much simpler contest success function, allowing for an analysis of the case with players adjusting their investments according to marginal profits and a variable prize. We do not consider different production functions in the contest. Instead we analyze situations with different valuations of the prize and an endogenous prize.[6]

The remainder of the paper is organized as follows: Section 2 presents the basic model of an infinitely repeated contest. Section 3 extends this model to the cases with different valuation of the prize and an endogenous prize, and Section 4 concludes.

2 The basic model

In this study, we consider two players in a dynamic setting, where a Tullock contest[7] takes place at discrete-time periods $t = 0, 1, 2, \ldots$. We assume the information in the market to be incomplete so that players cannot play their optimal responses immediately but have to adjust their investments proportionally to their local marginal profits. This adjustment process is called boundedly rational in oligopoly theory.[8] The player i who behaves in this manner makes its decision on contest investments $x_i \in \mathbb{R}_+$ based on a local estimate of the marginal profit $\frac{\partial \Pi_i}{\partial x_i}$. A player decides to increase its contest investment if it has a positive marginal profit, or decreases its investment if the marginal profit is negative. Then the dynamical system of the two players is described by the two-dimensional map $C : \mathbb{R}^2_+ \mapsto \mathbb{R}^2_+$

$$C : \begin{cases} x_1(t+1) = x_1(t) + \alpha \, \frac{\partial \Pi_1}{\partial x_1(t)} \\ x_2(t+1) = x_2(t) + \beta \, \frac{\partial \Pi_2}{\partial x_2(t)}, \end{cases} \tag{1}$$

where α, β are positive parameters representing the adjustment speed or aggressiveness of player 1 and 2, respectively.[9]

[4]See Chiarella and Szidarovszky (2002) for a dynamic model with a standard Tullock contest success function but different cost functions.

[5]They assume $f_i(0) = 0$, $f_i'(x_i) > 0$, and $f_i''(x_i) < 0$ for all i.

[6]For a model which considers global stability instead of local stability but does not examine the cases of asymmetric and endogenous prizes, see Okuguchi and Yamazaki (2007).

[7]See Tullock (1980) for the first use in a rent seeking setting and Skaperdas (1996) as well as Kooreman and Schoonbeek (1997) for its axiomatisation.

[8]Boundedly rational duopolists are considered by Zhang et al. (2007), Agiza and Elsadany (2007) and Agiza and Elsadany (2003). All this articles focus on different costs of the duopolist, which we do not consider at all. See also Chiarella and Szidarovszky (2002) for a related behavior in contests.

[9]We get qualitatively equivalent results if we use a relative adjustment process with the adjustment speed being linear in investments, i. e. $\alpha = a\,x_1(t)$; $\beta = b\,x_2(t)$; $a, b > 0$ as in e. g. Zhang et al.

In each time period, a contest takes place, so that the profit of player i in period t is given by[10]

$$\Pi_i(t) = \begin{cases} P\,\frac{x_i(t)}{x_i(t)+x_j(t)} - x_i(t) & \text{if } \max\{x_i(t), x_j(t)\} > 0 \\ \frac{1}{2}P & \text{otherwise,} \end{cases} \tag{2}$$

$i \in \{1,2\}, \quad i \neq j$, where P is the prize in the contest.

Exogenous prize. We first consider the case of an exogenous given prize P in the contest and discuss the local stability of the linearization of system (1) in the Nash equilibrium. The Nash equilibrium is the fixed point of the two-dimensional map, which is obtained by setting $x_i(t+1) = x_i(t)$ for both players, or equivalently $\frac{\partial \Pi_i}{\partial x_i} = 0$. This can be calculated to be

$$x_i^N = \frac{P}{4}, \qquad i \in \{1,2\}. \tag{3}$$

As we are interested in the local stability of the Nash equilibrium, we have to calculate the eigenvalues of the Jacobian matrix of the map C. The Jacobian matrix at the state (x_1, x_2) has the form

$$J(x_1,x_2) = \begin{pmatrix} 1 - \frac{2\,P\,x_2\,\alpha}{(x_1+x_2)^3} & \frac{P\,(x_1-x_2)\,\alpha}{(x_1+x_2)^3} \\ \frac{P\,(x_2-x_1)\,\beta}{(x_1+x_2)^3} & 1 - \frac{2\,P\,x_1\,\beta}{(x_1+x_2)^3} \end{pmatrix}. \tag{4}$$

In order to consider stability at the Nash equilibrium, we estimate the eigenvalues of the Jacobian at (x_1^N, x_2^N), which are

$$\lambda_1 = 1 - \frac{4\,\alpha}{P}, \quad \lambda_2 = 1 - \frac{4\,\beta}{P}. \tag{5}$$

Obviously, both eigenvalues are real and less than 1. As for stability the eigenvalues must be located inside the unit circle of the complex plane, we can calculate the value of α, β, where the system loses stability, labeled $\overline{\alpha}$.

$$1 - \frac{4\,\overline{\alpha}}{P} = -1 \quad \Leftrightarrow \quad \overline{\alpha} = \frac{1}{2}P. \tag{6}$$

Thus, we have proven the following proposition

Proposition 1
The Nash equilibrium of the dynamic contest with exogenous prize is locally stable provided that $\{\alpha, \beta\} \in \left]0, \frac{1}{2}P\right[$.

(2007), who model a duopoly.

[10]There are two interpretations of the contest success function. The first is that each player i receives a fraction $\frac{x_i}{x_i+x_j}$ of the contested prize. Or we might have a winner-take-all contest with risk-neutral contestants and the winning probability being given by $\frac{x_i}{x_i+x_j}$.

We have obtained the result that the Nash equilibrium is stable as long as the adjustment speeds of the players are small enough or the contested prize is high enough. We can say that an increase of the speeds of adjustment or a decrease of the prize has a destabilizing effect.[11] The intuition behind this result is that if one player is to aggressive, it overshoots the equilibrium point which leads to less stability. This overshooting is attenuated by a higher absolute value of the prize. Thus, aggressiveness in contests has a negative impact on stability of the equilibrium.

3 Asymmetric valuation and endogenous prize

Asymmetric valuation of the prize. In the following we consider the case where the prize in the contest does not have the same value to the two contestants. We assume that player 1 values the prize with $k\,P$ ($k \in]0, \infty[$), so that we have the profits

$$\begin{aligned} \Pi_1(t) &= P\,k\,\frac{x_1(t)}{x_1(t)+x_2(t)} - x_1(t), \\ \Pi_2(t) &= P\,\frac{x_2(t)}{x_1(t)+x_2(t)} - x_2(t). \end{aligned} \tag{7}$$

In this case the Nash equilibrium is

$$x_1^N = \frac{k^2\,P}{(1+k)^2}, \qquad x_2^N = \frac{k\,P}{(1+k)^2}. \tag{8}$$

The players use the same adjustment strategy as above but we use the assumption that both players are equal aggressive, i. e. $\beta = \alpha$.
The Jacobian matrix at the Nash equilibrium takes the form

$$J(x_1^N, x_2^N) = \begin{pmatrix} \frac{k\,P-2\,\alpha\,(1+k)}{k\,P} & \frac{(k^2-1)\,\alpha}{k\,P} \\ \frac{(1-k^2)\,\alpha}{k^2\,P} & \frac{k\,P-2\,\alpha\,(1+k)}{k\,P} \end{pmatrix}. \tag{9}$$

The two eigenvalues are

$$\begin{aligned} \lambda_1^k &= 1 - \frac{2\,\alpha\,(1+k)}{k\,P} - \imath\,\frac{\sqrt{k}\,|1-k^2|\,\alpha}{k^2\,P}, \\ \lambda_2^k &= 1 - \frac{2\,\alpha\,(1+k)}{k\,P} + \imath\,\frac{\sqrt{k}\,|1-k^2|\,\alpha}{k^2\,P}, \end{aligned} \tag{10}$$

with $\imath$ representing the imaginary unit. As the two eigenvalues are complex conjugates and have the same absolute value, we can calculate the unique value of α where this absolute value equals 1 and stability disappears. This value will be labeled $\overline{\alpha}^k$:

$$\overline{\alpha}^k = \frac{4\,k^2\,P}{(1+k)^3}. \tag{11}$$

[11] This result is just in line with the outcome of the above mentioned duopoly models with the prize representing the size of the market.

The Nash equilibrium is stable for adjustment speeds low enough, i. e. $\alpha \in]0, \overline{\alpha}^k[$. It is easy to see that

$$\frac{\partial \overline{\alpha}^k}{\partial k} = \frac{4(2-k)k\,P}{(1+k)^4} \begin{cases} > 0 & \text{for } k \in]0,2[\\ < 0 & \text{for } k \in]2,\infty[\end{cases}, \qquad \frac{\partial \overline{\alpha}^k}{\partial P} > 0 \tag{12}$$

holds. If the valuation of player 1 increases, the contest gets less asymmetrical for $k \in]0,1[$ and more asymmetrical for $k \in]1,\infty[$. A higher valuation of player 1 has a stabilizing effect as long as the contest is not too asymmetrical ($k < 2$). Thus, the stability region increases if the contest is enough symmetrical and the overall valuation of the prize rises. By (11) and (12), we have Proposition 2 about local stability of the Nash equilibrium in the asymmetrical case.

Proposition 2
If the two players have different valuations of the prize, the Nash equilibrium of the dynamic contest is locally stable as long as the aggressivenesses of the players satisfy $\alpha \in \left]0, \frac{4\,k^2\,P}{(1+k)^3}\right[$. A lower prize or a sufficient asymmetrical valuation has a destabilizing effect.

Endogenous prize. In the following, we extend the analysis of Shaffer (2006) on an endogenous prize to a dynamic contest. In this case, the prize of each period is influenced by the players' investments in this period by means of the parameter γ. As an example of a negative externality, i. e. $\gamma < 0$, we can consider war, where parts of the contested resource (e. g. land) are destroyed in the course of the struggle for its ownership. On the other hand, we could have positive externalities, i. e. $\gamma > 0$, if we look at an advertising contest or an innovation contest, where the contested prize rises because of a higher demand in the market or a more valuable innovation. We use the contest with linear externalities, where the prize takes the shape of

$$P = P_0\,(1 + \gamma\,x_1(t) + \gamma\,x_2(t)). \tag{13}$$

Here, $P_0 > 0$ is the basic prize and the value of γ determines whether the prize is enhanced ($\gamma > 0$) or abased ($\gamma < 0$) by the investments of the players. We assume $\gamma < \frac{1}{P_0}$ to assure the equilibrium investments in the contest to be non-negative.[12] The Nash equilibrium can be calculated to be

$$x_i^N = \frac{P_0}{4\,(1 - P_0\,\gamma)} \qquad i \in \{1,2\}. \tag{14}$$

As the players behave according to system (1), we can again calculate the Jacobian in the Nash equilibrium and find the eigenvalues:

$$\begin{aligned} \lambda_1^\gamma &= 1 - 4\,\alpha\left(\frac{1}{P_0} - 2\,\gamma + P_0\,\gamma^2\right) \\ \lambda_2^\gamma &= 1 - 4\,\beta\left(\frac{1}{P_0} - 2\,\gamma + P_0\,\gamma^2\right). \end{aligned} \tag{15}$$

[12]This is also a sufficient condition to assure the endogenous prize P to be positive.

In our model, both eigenvalues are real and less than 1. Thus, the system shows stability as long as $\{\lambda_1^\gamma\,,\,\lambda_2^\gamma\} \in]-1,1[$ holds. We can calculate the supremum of the stability region for α and β, which will be labeled $\overline{\alpha}^\gamma$:

$$\overline{\alpha}^\gamma = \frac{1}{2\left(\frac{1}{P_0} - 2\,\gamma + P_0\,\gamma^2\right)}. \tag{16}$$

As we have

$$\frac{\partial\,\overline{\alpha}^\gamma}{\partial\,\gamma} = \frac{1 - P_0\,\gamma}{\left(\frac{1}{P_0} - 2\,\gamma + P_0\,\gamma^2\right)^2}, \tag{17}$$

$\overline{\alpha}^\gamma$ is rising in γ as long as our assumption $\gamma < \frac{1}{P_0}$ applies. Thus, if the externality becomes more positive, the interval of stability is widened.
The influence of the prize on stability is as follows:

$$\frac{\partial\,\overline{\alpha}^\gamma}{\partial\,P_0} = \frac{\frac{1}{P_0^2} - \gamma^2}{2\left(\frac{1}{P_0} - 2\gamma + P_0\,\gamma^2\right)^2} > 0 \quad \Leftrightarrow \quad |\gamma| < \frac{1}{P_0}. \tag{18}$$

Thus, we can state the following proposition.

Proposition 3
In the case with an endogenous prize in the dynamic contest, we have local stability of the Nash equilibrium for $\{\alpha\,,\,\beta\} \in]0, \frac{1}{2\left(\frac{1}{P_0} - 2\gamma + P_0\,\gamma^2\right)}[$. The region of stability increases if a negative externality has less impact or a positive externality has more impact on the prize. Furthermore, a raising of the basic prize has only a positive effect on stability if the externality is positive or not too largely negative.

We will give an interpretation by applying these results to our examples:
In our example of a positive externality, where two duopolists invest in advertising to increase their market shares, a higher positive impact of these advertising expenditures on the market would have positive impact on the stability of the Nash equilibrium. If the duopolists are very effective in influencing the consumers to buy their products, stability is maintained even with a higher adjustment speed. In this case, a higher basic prize, which is interpreted as initial overall market size, also increases stability.
In the case of a negative externality, we had used war as example. Here, a more destructive war technology has destabilizing effects. Interestingly, in this case, a higher basic prize, i. e. initial contested land, only has a stabilizing effect if the destructive influence is not very large. If the negative externality is very strong it is favorable for stability to only have a small initial contested resource.

4 Conclusion

We have considered stability in a repeated contest with two players. The Nash equilibrium in the dynamic contest with exogenous prize is locally stable provided that the prize is high enough or the aggressivenesses of the players are low enough. In the next step we extended this to the case where the players have different valuations of the prize. In this case, a higher prize also yields to more stability. On the other hand, asymmetries between the players are destabilizing if the asymmetry is severe enough. Thus, very asymmetric contests will not converge to the Nash equilibrium (assuming that adjustment speeds are not too low) and should not be observed very often.

Finally, we considered endogenous prizes. If there are large negative externalities, e. g. war, the Nash equilibrium will not be stable. The Nash equilibrium in a war is only stable if the negative impact of the struggle for land is not very severe. Having said that, a positive externality, e. g. advertising campaigns, leads to higher stability. Here, the higher the impact of advertising on attracting additional consumers to the market the larger is the region of stability. Thus, we should observe many contests with positive externalities.

If we apply this result to innovation contests, we have a desire for productive investments (high γ). As the innovation contests where investments are very productive are especially stable, the intended contests are promoted. Unfortunately, this is not true for rent seeking frameworks with positive externalities, where we desire low rents to be allocated. Here, our findings show that equilibria with severe externalities are supported. Having considered externalities, we can consider the level of the basic prize in these contests. We see that with positive externalities and small negative externalities, a lower prize has destabilizing effects. Thus, if we do not want a repeated rent seeking contest to be stable, we have to lower the possible rent, even if this rent is influenced by the rent seeker. In the case of war, contesting less land only has a negative impact on the stability of the equilibrium if the war technology is not very destructive. Large negative externalities (i. e. destructive technology) are the only cases where a lower initial prize leads to more stability.

As we have used a special function for endogenizing the contested prize, further research should focus on generalizing our results with respect to the kind of influence of the investments. Furthermore, it would be interesting to analyze the case where the contest investments in one period influence the prize in another -e. g. the following- period, as can be imagined in the frame of common pool resources like fishing or pasture.

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