E C O N O M I C S   B U L L E T I N

Potential games with NM utilities

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Abstract

Given a game admitting an exact potential, affine transformations of utilities are described that do or do not destroy the property. All weighted potentials of a game admitting one are described.

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1 Introduction

Monderer and Shapley (1996a) developed their notion of a potential game in both cardinal and ordinal versions. Most attention was paid to the former; more precisely, to *exact* potentials. However, an obvious objection can be raised against that very notion: a von Neumann–Morgenstern utility is defined up to a strictly increasing affine transformation, and such transformations can easily destroy (or create) an exact potential.

Apparently aware of this objection, Monderer and Shapley also defined a *weighted* potential, which is indeed an attribute of a game with NM utilities (the observation was explicit in Morris and Ui, 2004). Actually, all interesting results on mixed extensions of potential games (Sela, 1992; Monderer and Shapley, 1996b; Huang, 2002; Morris and Ui, 2004) require the presence of a weighted potential.

This note investigates the relationship between cardinal potentials and affine transformations of utilities. Proposition 5 establishes an additive structure of exact potentials, which helps describe transformations that do or do *not* destroy the presence of such potentials (Theorem 1). Similar considerations help us describe all weighted potentials of a game (Theorem 2). The point is Monderer and Shapley showed that two exact potentials of the same game differ in a constant. A weighted potential obviously survives a strictly increasing affine transformation, but it would be generally wrong to assert that only such transformations are allowed.

The next section contains basic definitions and auxiliary results. The additive structure of exact potentials is introduced in Section 3. The main results are in Section 4.

2 Basic notions

A *strategic game* Γ is defined by a finite set of players N (we denote $n = \#N$), and strategy sets X_i and utility functions $u_i\colon X_N \to \mathbb{R}$ for all $i \in N$, where $X_N = \prod_{i\in N} X_i$ is the set of strategy profiles. For each $I \subseteq N$, we denote $X_I = \prod_{i\in I} X_i$; instead of $X_{N\setminus\{i\}}$ and $X_{N\setminus I}$, we write X_{-i} and X_{-I}, respectively. Given $I \subseteq N$ and $x_I \in X_I$, we define a *reduced game* $\Gamma(x_I)$ with $N \setminus I$ as the set of players, the same strategy sets and utilities $u'_i(x_{-I}) = u_i(x_{-I}, x_I)$ for every $i \in N \setminus I$ and $x_{-I} \in X_{-I}$.

A strategic game Γ' is *NM-equivalent* to Γ if $N' = N$ and $X'_i = X_i$ for each $i \in N$, whereas for each $i \in N$ there is a strictly increasing affine transformation, $\lambda_i(v) = a_i v + b_i$ ($a_i > 0$), such that

$$u'_i(x_N) = \lambda_i \circ u_i(x_N)$$

for every $x_N \in X_N$.

Monderer and Shapley (1996a) defined an *exact potential* of a game Γ as a function $P\colon X_N \to \mathbb{R}$ such that

$$u_i(y_N) - u_i(x_N) = P(y_N) - P(x_N) \qquad (1)$$

whenever $i \in N$, $y_N, x_N \in X_N$, and $y_{-i} = x_{-i}$. A *weighted potential* of Γ is a function $P\colon X_N \to \mathbb{R}$ satisfying

$$\forall i \in N\, \exists w_i > 0\, \forall y_N, x_N \in X_N \left[y_{-i} = x_{-i} \Rightarrow u_i(y_N) - u_i(x_N) = w_i \cdot \big(P(y_N) - P(x_N)\big) \right]. \tag{2}$$

Clearly, (2) is invariant under strictly increasing affine transformations of utility functions, which cannot be said about (1).

Proposition 1 (Voorneveld et al., 1999). *A function $P\colon X_N \to \mathbb{R}$ is an exact potential of Γ if and only if for every $i \in N$ there is a function $Q_i\colon X_{-i} \to \mathbb{R}$ such that*

$$u_i(x_N) = P(x_N) + Q_i(x_{-i})$$

for every $x_N \in X_N$.

Proposition 2 (Monderer and Shapley, 1996a). *Let P be an exact potential of a game Γ. Then a function $P'\colon X_N \to \mathbb{R}$ is an exact potential of Γ if and only if there is $C \in \mathbb{R}$ such that $P'(x_N) = P(x_N) + C$ for all $x_N \in X_N$.*

Proposition 3 (Monderer and Shapley, 1996a). *A game Γ admits an exact potential if and only if for every $i, j \in N$ ($i \neq j$), $x'_i, x''_i \in X_i$, $x'_j, x''_j \in X_j$, and $x_{-ij} \in X_{-\{i,j\}}$, there holds*

$$u_i(x''_i, x'_j, x_{-ij}) - u_i(x'_i, x'_j, x_{-ij}) + u_j(x''_i, x''_j, x_{-ij}) - u_j(x''_i, x'_j, x_{-ij}) + \\ u_i(x'_i, x''_j, x_{-ij}) - u_i(x''_i, x''_j, x_{-ij}) + u_j(x'_i, x'_j, x_{-ij}) - u_j(x'_i, x''_j, x_{-ij}) = 0. \tag{3}$$

Proposition 4 (Morris and Ui, 2004). *Let Γ be a game and P be a function $X_N \to \mathbb{R}$. Then P is an exact potential of a game NM-equivalent to Γ if and only if P is a weighted potential of Γ.*

Proof. Indeed, if P satisfies (2), then it satisfies (1) for $u'_i = (1/w_i) \cdot u_i$. Conversely, if P satisfies (1) for $u'_i = a_i \cdot u_i + b_i$, then it satisfies (2) with $w_i = 1/a_i$. □

3 Additive structure

Given a game Γ, we define two binary relations on the set N. First, we say that j *influences* i, and denote the fact $j \Vdash i$, if $i = j$ or there are $x'_i, x''_i \in X_i$, $x'_j, x''_j \in X_j$, and $x_{-ij} \in X_{-\{i,j\}}$ such that

$$u_i(x''_i, x'_j, x_{-ij}) - u_i(x'_i, x'_j, x_{-ij}) \neq u_i(x''_i, x''_j, x_{-ij}) - u_i(x'_i, x''_j, x_{-ij}). \tag{4}$$

Second, we set $j \cong i$ if and only if there are $i_0, i_1, \ldots, i_m \in N$ such that $i_0 = i$, $i_m = j$, and, for each k, either $i_{k+1} \Vdash i_k$ or $i_k \Vdash i_{k+1}$ (i.e., $\cong$ is the symmetric and transitive closure of $\Vdash$). Clearly, $\cong$ is an equivalence relation, hence N is partitioned into equivalence classes;

we denote $\mathcal{N}$ the set of the classes. Whenever $I \in \mathcal{N}$, $i \in I \not\ni j$, $x'_i, x''_i \in X_i$, $x'_j, x''_j \in X_j$, and $x_{-ij} \in X_{-\{i,j\}}$, there holds

$$u_i(x''_i, x'_j, x_{-ij}) - u_i(x'_i, x'_j, x_{-ij}) = u_i(x''_i, x''_j, x_{-ij}) - u_i(x'_i, x''_j, x_{-ij}). \tag{5}$$

It is important for the following to note that both $\Vdash$ and $\cong$ are invariant under strictly increasing affine transformations of utilities, i.e., both relations and the partition $\mathcal{N}$ are the same in NM-equivalent games.

When Γ admits an exact potential, (3) and (4) together imply that the relation $\Vdash$ is symmetric. In particular, if $u_i = \text{const}$ (player i is a dummy), then $\{i\} \in \mathcal{N}$.

Proposition 5. *Let P be an exact potential of a game Γ; then for each $I \in \mathcal{N}$ there is a function $P_I \colon X_I \to \mathbb{R}$ such that*

$$P(x_N) = \sum_{I \in \mathcal{N}} P_I(x_I) \text{ for all } x_N \in X_N. \tag{6}$$

If there is also a list of functions $P'_I \colon X_I \to \mathbb{R}$, $I \in \mathcal{N}$, such that (6) holds when P_I are replaced with P'_I, then there are constants $C_I \in \mathbb{R}$ such that $\sum_{I \in \mathcal{N}} C_I = 0$, and $P'_I(x_I) = P_I(x_I) + C_I$ for all $I \in \mathcal{N}$ and $x_I \in X_I$.

Proof. Let $I \in \mathcal{N}$ and $\#I = m$. We fix one-to-one mappings $\sigma \colon \{1, \dots, m\} \to I$ and $\sigma' \colon \{1, \dots, n-m\} \to N \setminus I$. Combining (1) and (5) for $i = \sigma(k)$ ($k = 1, \dots, m$) and $j = \sigma'(h)$ ($h = 1, \dots, n-m$), we obtain

$$\begin{aligned}
&P(x''_I, x'_{-I}) - P(x'_I, x'_{-I}) = \\
&\sum_{k=1}^{m} \big[P(x''_{\sigma(\{1,\dots,k\})}, x'_{\sigma(\{k+1,\dots,m\})}, x'_{-I}) - P(x''_{\sigma(\{1,\dots,k-1\})}, x'_{\sigma(\{k,k+1,\dots,m\})}, x'_{-I})\big] = \\
&\sum_{k=1}^{m} \big[u_{\sigma(k)}(x''_{\sigma(\{1,\dots,k\})}, x'_{\sigma(\{k+1,\dots,m\})}, x'_{-I}) - u_{\sigma(k)}(x''_{\sigma(\{1,\dots,k-1\})}, x'_{\sigma(\{k,k+1,\dots,m\})}, x'_{-I})\big] = \\
&\sum_{k=1}^{m} \big[u_{\sigma(k)}(x''_{\sigma(\{1,\dots,k\})}, x'_{\sigma(\{k+1,\dots,m\})}, x''_{\sigma'(1)}, x'_{\sigma'(\{2,\dots,n-m\})}) - \\
&u_{\sigma(k)}(x''_{\sigma(\{1,\dots,k-1\})}, x'_{\sigma(\{k,k+1,\dots,m\})}, x''_{\sigma'(1)}, x'_{\sigma'(\{2,\dots,n-m\})})\big] = \dots = \\
&\sum_{k=1}^{m} \big[u_{\sigma(k)}(x''_{\sigma(\{1,\dots,k\})}, x'_{\sigma(\{k+1,\dots,m\})}, x''_{\sigma'(\{1,\dots,h\})}, x'_{\sigma'(\{h+1,\dots,n-m\})}) - \\
&u_{\sigma(k)}(x''_{\sigma(\{1,\dots,k-1\})}, x'_{\sigma(\{k,k+1,\dots,m\})}, x''_{\sigma'(\{1,\dots,h\})}, x'_{\sigma'(\{h+1,\dots,n-m\})})\big] = \dots = \\
&\sum_{k=1}^{m} \big[u_{\sigma(k)}(x''_{\sigma(\{1,\dots,k\})}, x'_{\sigma(\{k+1,\dots,m\})}, x''_{-I}) - u_{\sigma(k)}(x''_{\sigma(\{1,\dots,k-1\})}, x'_{\sigma(\{k,k+1,\dots,m\})}, x''_{-I})\big] = \\
&\sum_{k=1}^{m} \big[P(x''_{\sigma(\{1,\dots,k\})}, x'_{\sigma(\{k+1,\dots,m\})}, x''_{-I}) - P(x''_{\sigma(\{1,\dots,k-1\})}, x'_{\sigma(\{k,k+1,\dots,m\})}, x''_{-I})\big] = \\
&P(x''_I, x''_{-I}) - P(x'_I, x''_{-I})
\end{aligned} \tag{7}$$

for all $x''_I, x'_I \in X_I$ and $x''_{-I}, x'_{-I} \in X_{-I}$. Fixing $x^0_N \in X_N$, we define

$$P_I(x_I) = P(x_I, x^0_{-I}) - \frac{\#\mathcal{N} - 1}{\#\mathcal{N}} P(x^0_N) \quad \text{and} \quad Q(x_N) = \sum_{I \in \mathcal{N}} P_I(x_I). \tag{8}$$

Now for every $i \in I \in \mathcal{N}$ and $y_N, x_N \in X_N$ such that $x_{-i} = y_{-i}$ (hence $x_{-I} = y_{-I}$) we have $u_i(y_N) - u_i(x_N) = P(y_N) - P(x_N) = P(y_I, x^0_{-I}) - P(x_I, x^0_{-I}) = P_I(y_I) - P_I(x_I) = Q(y_N) - Q(x_N)$, the first equality following from (1), the second from (7), the last two from (8). Therefore, Q is an exact potential of Γ; by Proposition 2, $Q(x_N) - P(x_N) = \text{const}$. Since $Q(x^0_N) = P(x^0_N)$, we have $Q(x_N) = P(x_N)$ for all $x_N \in X_N$.

To prove the second statement, we notice that if P is an exact potential of Γ and (6) is valid, then each P_I is an exact potential of the reduced game $\Gamma(x^0_{-I})$ (regardless of the choice of $x^0_N \in X_N$): If $i \in I \in \mathcal{N}$, $y_N, x_N \in X_N$ and $x_{-i} = y_{-i}$ then $u_i(y_I, x^0_{-I}) - u_i(x_I, x^0_{-I}) = P(y_I, x^0_{-I}) - P(x_I, x^0_{-I}) = P_I(y_I) - P_I(x_I)$. Therefore, $P'_I(x_I) - P_I(x_I) = \text{const}$ by Proposition 2 applied to $\Gamma(x^0_{-I})$. □

4 Main results

Theorem 1. *Let Γ be a game admitting an exact potential P, and Γ' be a game NM-equivalent to Γ (with strictly increasing affine transformations $\lambda_i(v) = a_i v + b_i$, $i \in N$). Then Γ' admits an exact potential if and only if $a_i = a_j$ whenever $j \cong i$. If the condition is satisfied, then $P'(x_N) = \sum_{I \in \mathcal{N}} a_I \cdot P_I(x_I)$ is an exact potential of Γ' whenever* (6) *holds (a_I, naturally, denotes the common value of a_i for $i \in I$).*

Proof. If the condition on λ_i's is satisfied and a representation (6) is given, we define $P'(x_N) = \sum_{I \in \mathcal{N}} a_I \cdot P_I(x_I)$ for all $x_N \in X_N$. Let $i \in I \in \mathcal{N}$, $y_N, x_N \in X_N$, and $y_{-i} = x_{-i}$; then $u'_i(y_N) - u'_i(x_N) = a_I\big(u_i(y_N) - u_i(x_N)\big) = a_I\big(P_I(y_N) - P_I(x_N)\big) = P'(y_N) - P'(x_N)$, i.e., P' is an exact potential of Γ'.

Let the condition be violated: there are $i', i'' \in N$ such that $i' \cong i''$, but $a_{i'} \neq a_{i''}$. By the definition of $\cong$, there are $i, j \in N$, $x'_i, x''_i \in X_i$, $x'_j, x''_j \in X_j$, and $x_{-ij} \in X_{-\{i,j\}}$ such that (4) holds, but $a_i \neq a_j$. By Proposition 3, (3) holds; it can be rewritten as

$$\begin{aligned} u_i(x''_i, x'_j, x_{-ij}) - u_i(x'_i, x'_j, x_{-ij}) + u_i(x'_i, x''_j, x_{-ij}) - u_i(x''_i, x''_j, x_{-ij}) = \\ - [u_j(x''_i, x''_j, x_{-ij}) - u_j(x''_i, x'_j, x_{-ij}) + u_j(x'_i, x'_j, x_{-ij}) - u_j(x'_i, x''_j, x_{-ij})], \end{aligned}$$

which is not 0 by (4). Multiplying u_i by a_i and u_j by a_j, and taking into account that $a_i \neq a_j$, we immediately see that (3) is violated for u'_i and u'_j. By Proposition 3, Γ' cannot admit an exact potential. □

Proposition 6. *For every game Γ, the following statements are equivalent.*

1. *Every game NM-equivalent to Γ admits an exact potential.*

2. *Each $I \in \mathcal{N}$ is a singleton.*

3. *For each $i \in N$, there are functions $P_i \colon X_i \to \mathbb{R}$ and $Q_i \colon X_{-i} \to \mathbb{R}$ such that $u_i(x_N) = P_i(x_i) + Q_i(x_{-i})$ for every $x_N \in X_N$.*

Proof. The equivalence Statement 1 $\iff$ Statement 2 immediately follows from Theorem 1. If Statement 3 holds, then we obtain Statement 1 by the sufficiency part of Proposition 1. Finally, if Statement 2 holds, then there is an exact potential by Statement 1, hence Proposition 5 and the necessity part of Proposition 1 imply Statement 3. $\square$

Theorem 2. *Let P be a weighted potential of a game Γ, and P' be a function $X_N \to \mathbb{R}$. Then the following statements are equivalent.*

1. *P' is a weighted potential of Γ.*

2. *There is a representation* (6) *and a strictly increasing affine transformation $\lambda_I \colon \mathbb{R} \to \mathbb{R}$ for every $I \in \mathcal{N}$ such that*

$$P'(x_N) = \sum_{I \in \mathcal{N}} \lambda_I \circ P_I(x_I) \text{ for all } x_N \in X_N. \tag{9}$$

3. *For every representation* (6)*, there are strictly increasing affine transformations $\lambda_I \colon \mathbb{R} \to \mathbb{R}$ ($I \in \mathcal{N}$) such that* (9) *holds.*

Proof. By Proposition 4, we may assume that P is an exact potential of Γ.

Let Statement 1 hold; by Proposition 4, P' is an exact potential of a game Γ' NM-equivalent to Γ. Given a representation (6), we apply Theorem 1, obtaining that $a_i = a_j$ whenever $j \cong i$. The second statement of Theorem 1 implies that $P''(x_N) = \sum_{I \in \mathcal{N}} a_I \cdot P_I(x_I)$ is also an exact potential of Γ'. By Proposition 2, there is $C \in \mathbb{R}$ such that $P'(x_N) = P''(x_N) + C$ for all $x_N \in X_N$. Picking a list of $\langle b_I \rangle_{I \in \mathcal{N}}$ such that $\sum_{I \in \mathcal{N}} b_I = C$ and defining $\lambda_I(v) = a_I \cdot v + b_I$, we obviously obtain (9). Therefore, Statement 3 holds.

If Statement 2 holds, then the second statement of Theorem 1 immediately implies that P' is an exact potential of a game NM-equivalent to Γ, hence is a weighted potential of Γ by Proposition 4. $\square$

Corollary. *Let Γ be a game in which $\mathcal{N}$ is a singleton, P be a weighted potential of Γ, and P' be a function $X_N \to \mathbb{R}$. Then P' is a weighted potential of Γ if and only if there is a strictly increasing affine transformation λ such that $P'(x_N) = \lambda \circ P(x_N)$ for all $x_N \in X_N$.*

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