E c o n o m i c s   B u l l e t i n

Some sufficient conditions for the existence of a competitive equilibrium in economies with satiated consumers

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Abstract

It is well known that a competitive equilibrium may fail to exist when consumers' preferences are possibly satiated. In this paper, we provide three new sufficient conditions for the existence of a competitive equilibrium in the standard Arrow-Debreu pure exchange economy with satiated consumers. We first consider a condition that restricts the behavior of the excess demand correspondence on the boundary of a certain subset of the price domain. Another two sufficient conditions are obtained by using the existence result under this condition.

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1 Introduction

It is well known that a competitive equilibrium may fail to exist when consumers' preferences are satiated. More precisely, if consumers' preferences are satiated in the individually feasible consumption sets, then, at every price, consumers may choose their optimal consumption bundles in the interior of their budget sets. This leads to a violation of the Walras Law, and hence, to the nonexistence of a competitive equilibrium. [1]

Satiation occurs, for example, when preferences form a complete continuous preorder and consumption sets are compact. Mas-Colell (1992) provides several examples of economies in which consumption sets are naturally compact. Another example of satiation occurs in the capital asset pricing model (CAPM) without a riskless asset (see, for example, Nielsen 1989).

Given satiation in individually feasible consumption sets, many authors investigate the conditions that ensure the existence of a competitive equilibrium. The sufficient conditions proposed in the literature may be classified according to whether they allow consumers to have their satiation points only within the individually feasible consumption sets (Polemarchakis and Siconolfi 1993, Won and Yannelis 2006, etc.), or not (Werner 1987, Sun 1999, Allouch and Le Van 2008, etc.).

Polemarchakis and Siconolfi (1993) consider a certain subset of the price domain in which the aggregate excess demand satisfies the Walras Law, and propose a condition that restricts the behavior of the aggregate excess demand at the prices that lie on the boundary of the subset. Won and Yannelis (2006) study the existence of an equilibrium with satiation in a more general setting: for example, in their model, consumption sets are not necessarily bounded and preferences are allowed to be non-ordered. They propose a condition that may be interpreted as a restriction of consumers' behavior on a certain subset of the price domain (which generally differs from the subset used in Polemarchakis and Siconolfi 1993). Some other sufficient conditions which can be applied to the case in which satiation occurs only within the individually feasible consumption sets, can be found in the literature on the CAPM without a riskless asset, such as Nielsen (1990), Allingham (1991) and Won et al. (2008).

In asset markets with short selling, Werner (1987) introduces a condition that requires each consumer to have a consumption bundle that increases his or her satisfaction when it is added to any given consumption bundle. Sun (1999) provides a similar condition in the standard Arrow-Debreu production economy. It asserts that each consumer has at least one good, consumption of which does not decrease his or her utility. Allouch and Le Van's (2008) condition, which is a generalization of the conditions in Werner (1987) and Sun (1999), asserts that each consumer's satiation area has an intersection outside the individually feasible consumption set. Although these conditions cannot be applied to the case in which there exists a consumer whose satiation area is a subset of the individually feasible consumption set, they are imposed on the primitives of the model and have natural economic interpretations.

In this paper, we provide three new sufficient conditions for the existence of a competitive equilibrium in a pure exchange economy with satiated consumers. We first consider, as in Polemarchakis and Siconolfi (1993), restricting the behavior of the aggregate demand on the boundary of a certain subset of the price domain. Polemarchakis and Siconolfi (1993) assume the strict quasi-concavity of utility functions, which implies that

[1] It is also known that the existence of a competitive equilibrium is ensured if satiation occurs only outside the individually feasible consumption sets (see Bergstrom 1976).

each consumer has a unique satiation point. However, in our analysis, we assume weak convexity of preferences, and consumers may thus have multiple satiation points. As a result, we obtain a condition that contains Polemarchakis and Siconolfi's (1993) condition as a special case (Condition 1). It is worth noting that this condition can be applied to the case in which satiation occurs only within the individually feasible consumption sets. Moreover, by using the existence result under Condition 1, we obtain another two sufficient conditions for the existence of a competitive equilibrium. In contrast to Condition 1, both of these conditions are imposed on the primitives of the model.

This paper is organized as follows. Section 2 describes the model and sets out our assumptions. Section 3 introduces some additional notations. Section 4 presents the new condition (Condition 1), and states the existence of a competitive equilibrium under it. Section 5 presents the other two conditions. In Section 6, we prove that Condition 1 contains Polemarchakis and Siconolfi's (1993) condition as a special case. Section 7 contains our concluding remarks. Some of the proofs are given in the Appendix.

2 Model and Assumptions

We consider a pure exchange economy $\mathcal{E}$ with ℓ commodities and n consumers ($1 \leq \ell, n < \infty$). For convenience, let I be the set of all consumers, that is, $I = \{1, \cdots, n\}$. Each consumer $i \in I$ is characterized by a consumption set $X_i \subset \mathbb{R}^\ell$, [2] an initial endowment $\omega_i \in \mathbb{R}^\ell$, and a preference relation $\succsim_i$ on X_i. Let $X = \prod_{i \in I} X_i$ and let $x = (x_i)_{i \in I}$ denote a generic element of X.

The pure exchange economy $\mathcal{E}$ is thus summarized by the list

$$\mathcal{E} = \Big(\mathbb{R}^\ell, (X_i, \succsim_i, \omega_i)_{i \in I}\Big).$$

An allocation $x \in X$ is *feasible* if $\sum_{i \in I} x_i = \sum_{i \in I} \omega_i$. Note that we do not allow free disposal. For each $i \in I$, a consumption bundle $x_i \in X_i$ is *individually feasible* if there exists $(x_j)_{j \neq i} \in \prod_{j \neq i} X_j$ such that $x_i + \sum_{j \neq i} x_j = \sum_{j \in I} \omega_j$. Let $\hat{X}_i$ be the set of all individually feasible consumption bundles of consumer i.

We adopt the following standard definition of competitive equilibrium.

Definition 1. *An element $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell$ is a competitive equilibrium of $\mathcal{E}$ if*

(a) for all $i \in I$, $\overline{p} \cdot \overline{x}_i \leq \overline{p} \cdot \omega_i$, *(b) for all $i \in I$, $x_i \succ_i \overline{x}_i$ implies $\overline{p} \cdot x_i > \overline{p} \cdot \omega_i$,*

(c) $\sum_{i \in I} \overline{x}_i = \sum_{i \in I} \omega_i$.

In the definition, $\succ_i$ denotes the strict preference relation corresponding to $\succsim_i$, that is, for any $x_i, y_i \in X_i$, we have $y_i \succ_i x_i$ if and only if $y_i \succsim_i x_i$ and not $x_i \succsim_i y_i$. Note that from Definitions 1 (a) and 1 (c), we have $\overline{p} \cdot \overline{x}_i = \overline{p} \cdot \omega_i$ for each $i \in I$.

We make the following assumptions on the economy $\mathcal{E}$.

Assumption 1. *For each $i \in I$,*

(a) X_i is convex, *(b) X_i is compact,* *(c) $\omega_i \in \operatorname{int} X_i$.*

[2] We shall use the following mathematical notations. The symbol $\mathbb{R}^\ell$ denotes the ℓ-dimensional Euclidean space, and $\mathbb{R}^\ell_+$ denotes the non-negative orthant of $\mathbb{R}^\ell$. For $x, y \in \mathbb{R}^\ell$, we denote by $x \cdot y = \sum_{j=1}^{\ell} x_j y_j$ the inner product, by $\|x\| = \sqrt{x \cdot x}$ the Euclidean norm. Let $B(x_0, r) = \{x \in \mathbb{R}^\ell : \|x - x_0\| < r\}$ denote the open ball centered at x_0 with radius r. For $a \in \mathbb{R} = \mathbb{R}^1$, we denote by $|a|$ the absolute value of a. For $a, b \in \mathbb{R}$ with $a \leq b$, we denote by (a, b) the open interval, and by $[a, b]$ the closed interval between a and b. For a set $A \subset \mathbb{R}^\ell$, we denote by $\operatorname{int} A$ the interior of A in $\mathbb{R}^\ell$, by $\operatorname{bd} A$ the boundary of A in $\mathbb{R}^\ell$ and by $\operatorname{co} A$ the convex hull of A.

Assumption 2. *For each $i \in I$,*

(a) $\succsim_i$ is a complete preorder on X_i,
(b) $\succsim_i$ is continuous, (c) $\succsim_i$ is weakly convex.[3]

These assumptions are fairly standard in the literature, except for Assumption 1 (b), which together with 2 (a) and 2 (b), guarantees that every consumer has at least one satiation point on his or her consumption set.

3 Additional Notations

In order to state our main result, we introduce some additional notations.

For each $p \in \mathbb{R}^\ell$, we define the budget set of consumer $i \in I$ by $B_i(p) = \{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i\}$. For each $i \in I$, let $D_i : \mathbb{R}^\ell \to X_i$ be the demand correspondence of consumer $i \in I$, that is, for each $p \in \mathbb{R}^\ell$,

$$D_i(p) = \{x_i \in X_i : x_i \in B_i(p) \text{ and } x_i \succsim_i y_i \text{ for all } y_i \in B_i(p)\}.$$

Moreover, let $Z : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the excess demand correspondence, that is, for each $p \in \mathbb{R}^\ell$,

$$Z(p) = \sum_{i \in I} D_i(p) - \sum_{i \in I} \omega_i.$$

Note that $\overline{p} \in \mathbb{R}^\ell$ is a competitive equilibrium price if and only if $0 \in Z(\overline{p})$.

Let S_i be the set of all satiation points of consumer $i \in I$, that is,

$$S_i = \{x_i \in X_i : x_i \succsim_i y_i \text{ for all } y_i \in X_i\}.$$

For each $i \in I$ and $s_i \in S_i$, let

$$Q_i(s_i) = \{p \in \mathbb{R}^\ell : p \cdot s_i \geq p \cdot \omega_i\}.$$

Since $S_i \neq \varnothing$ for all $i \in I$ under Assumptions 1 and 2, the set $Q_i(s_i)$ is well defined. Note that $Q_i(s_i)$ is a closed convex cone with vertex 0 for all $i \in I$ and $s_i \in S_i$.

For each $s \in S = \prod_{i \in I} S_i$, let

$$Q(s) = \bigcap_{i \in I} Q_i(s_i).$$

It is clear that $Q(s)$ is a closed convex cone with vertex 0 for all $s \in S$.

At the price $p \in Q(s)$, the value of s_i is greater than or equal to that of ω_i for every consumer $i \in I$. Note that if $p \notin Q(s)$ for all $s \in S$, this p is not an equilibrium price. Indeed, if $p \notin Q(s)$ for all $s \in S$, then there exists $i \in I$ such that $p \cdot s_i < p \cdot \omega_i$ for all $s_i \in S_i$, and thus, at the price p, consumer i's optimal consumption bundles must be in the interior of his or her budget set (recall that at an equilibrium price, each consumer's budget constraint must be held with equality). Therefore, if there exists an equilibrium price, it must be in the set $\cup_{s \in S} Q(s)$.

Let $Q^0(s)$ denote the polar of $Q(s)$, that is,

$$Q^0(s) = \{q \in \mathbb{R}^\ell : q \cdot p \leq 0 \text{ for all } p \in Q(s)\}.$$

[3]A preference relation $\succsim_i$ is *weakly convex* if $y_i \succsim_i x_i$ implies that $ty_i + (1-t)x_i \succsim_i x_i$ for all $t \in (0,1)$.

The set $Q^0(s)$ is also a closed convex cone with vertex 0 for all $s \in S$.

4 Main Result

We now propose the following sufficient condition for the existence of a competitive equilibrium.

Condition 1. *There exists $s \in S$ satisfying*
(a) $\operatorname{int} Q(s) \neq \varnothing$,
(b) for all $p \in \operatorname{bd} Q(s) \setminus \{0\}$, if there exists $z \in Z(p) \cap Q^0(s)$ such that $p \cdot z = 0$, then $0 \in Z(p)$.

Condition 1 (a) asserts that the set $Q(s)$ is sufficiently large. Condition 1 (b) restricts the behavior of aggregate demand on the boundary of $Q(s)$.

It is worth noting that this condition allows S_i to be contained in the individually feasible set, unlike the conditions proposed by Werner (1987), Sun (1999), and Allouch and Le Van (2008). Moreover, Condition 1 is an extension of the condition introduced in Polemarchakis and Siconolfi (1993), where each consumer's preference is assumed to be strictly convex, [4] and thus, S_i is a singleton for all $i \in I$. In section 6, we prove that Condition 1 is equivalent to Polemarchakis and Siconolfi's (1993) condition under the strict convexity of consumers' preferences.

The first and main result of this paper is the following proposition, the proof of which is given in the Appendix.

Proposition 1. *Under Assumptions 1 and 2 and Condition 1, there exists a competitive equilibrium.*

5 Another two sufficient conditions

By using the main result, we can obtain another two sufficient conditions for the existence of a competitive equilibrium. Unlike Condition 1, these conditions are imposed on the primitives of the model. The proofs of the propositions stated in this section are provided in the Appendix.

We first provide a condition that implies Condition 1 under our assumptions.

For arbitrarily chosen $s \in S$, consider n vectors $s_1 - \omega_1, \cdots, s_n - \omega_n$. Each vector $s_i - \omega_i$ indicates the direction from consumer i's initial endowment to his or her satiation point s_i. Condition 1 (a) may be violated for this s when the vectors $s_1 - \omega_1, \cdots, s_n - \omega_n$ point in widely different directions. Indeed, s violates Condition 1 (a) if for some $i, j \in I$, two vectors $s_i - \omega_i$ and $s_j - \omega_j$ point in the opposite directions. Conversely, however, if all the vectors $s_1 - \omega_1, \cdots, s_n - \omega_n$ point in the same direction, s satisfies not only Condition 1 (a) but also 1 (b).

More precisely, the following condition implies Condition 1 under our assumptions, and is therefore another sufficient condition for the existence of a competitive equilibrium.

Condition 2. *There exist $s \in S$ and $v \in \mathbb{R}^\ell$ with $\|v\| = 1$ such that*

$$s_i = \omega_i + \alpha_i v \quad \textit{for all} \quad i \in I,$$

where $\alpha_i = \|s_i - \omega_i\| \geq 0$ for each $i \in I$.

[4] A preference relation $\succsim_i$ is *strictly convex* if $y_i \succsim_i x_i$ implies that $ty_i + (1-t)x_i \succ_i x_i$ for all $t \in (0,1)$.

Condition 2 asserts that every consumer $i \in I$ has at least one satiation point in the direction v, which is common to all the consumers, from his or her initial endowment. This condition is imposed on the primitives of the model and provides an intuitive interpretation of Condition 1.

Proposition 2. *Under Assumptions 1 (a) and 2 (c), Condition 2 implies Condition 1.*

We next provide a condition that generalizes Bergstrom's (1976) nonsatiation condition. The existence of a competitive equilibrium under the condition is shown by using Proposition 1.

For each $i \in I$, let $\operatorname{int}_{X_i} \hat{X}_i$ denote the interior of $\hat{X}_i$ in the relative topology of X_i, that is, $x_i \in \operatorname{int}_{X_i} \hat{X}_i$ if and only if there exists an open ball $B(x_i, r)$ centered at x_i with radius r such that $B(x_i, r) \cap X_i \subset \hat{X}_i$. Let $x_i \in \hat{X}_i$ and $x_i + \sum_{j \neq i} x_j = \sum_{j \in I} \omega_j$ for $(x_j)_{j \neq i} \in \prod_{j \neq i} X_j$. Then, it is easy to check that $x_i \in \operatorname{int}_{X_i} \hat{X}_i$ if $x_j \in \operatorname{int} X_j$ for some $j \neq i$. Note also that since X_i is connected and $\operatorname{int}_{X_i} \hat{X}_i \neq \varnothing$ for all $i \in I$ under our assumptions, if $X_i \neq \hat{X}_i$, the set $\operatorname{int}_{X_i} \hat{X}_i$, which is open in X_i, does not coincide with $\hat{X}_i$, which is closed in X_i.

Consider the following condition.

Condition 3. $S_i \cap \operatorname{int}_{X_i} \hat{X}_i = \varnothing$ *for all* $i \in I$.

Condition 3 asserts that each consumer's preference is nonsatiated within the interior (in the above sense) of the individually feasible consumption set. This condition generalizes Bergstrom's (1976) nonsatiation condition, which requires that $S_i \cap \hat{X}_i = \varnothing$ for all $i \in I$, because our condition allows consumer i's satiation points to be individually feasible as long as they lie on the boundary of $\hat{X}_i$ in X_i (i.e., $\hat{X}_i \setminus \operatorname{int}_{X_i} \hat{X}_i$).

Allouch and Le Van's (2008) nonsatiation condition, which requires that $S_i \cap (X_i \setminus \hat{X}_i) \neq \varnothing$ for all $i \in I$, also generalizes Bergstrom's condition. While their condition allows S_i to intersect with $\operatorname{int}_{X_i} \hat{X}_i$, Condition 3 does not. However, Condition 3 allows S_i to be fully contained in the boundary of $\hat{X}_i$ in X_i, while their condition does not.

We now state the existence of a competitive equilibrium under Condition 3. [5]

Proposition 3. *Under Assumptions 1 and 2 and Condition 3, there exists a competitive equilibrium.*

6 Relation to Polemarchakis and Siconolfi's Result

Polemarchakis and Siconolfi (1993) investigate economies with satiated consumers and propose a sufficient condition similar to Condition 1. However, their condition cannot be applied to the case in which consumers have multiple satiation points, because in their analysis, they assume the strict quasi-concavity of each consumer's utility function, which corresponds to the strict convexity of $\succsim_i$. In contrast, in Proposition 1 of this paper, we only assume the weak convexity of preferences, which allows consumers' demand functions to be multivalued. Moreover, Condition 1 contains Polemarchakis and Siconolfi's (1993)

[5] Condition 3 does not imply Condition 1 under our assumptions. Consider the following economy with $\ell = n = 2$. Let $X_1 = X_2 = \mathbb{R}^2_+$ and $\omega_1 = \omega_2 = (1, 1)$. Consumers' preferences are represented by the utility functions $u_1(x_1) = -(x_{11} - 2)^2 - x_{12}^2$ and $u_2(x_2) = -x_{21}^2 - (x_{22} - 2)^2$. Then, $s_1 = (2, 0)$ and $s_2 = (0, 2)$ are the unique satiation points of each consumer. It is easy to check that $s_i \notin \operatorname{int}_{X_i} \hat{X}_i$ for each $i \in I$. However, for $s = (s_1, s_2) \in S$, the set $Q(s)$ is a one-dimensional linear subspace of $\mathbb{R}^2$, and thus, $\operatorname{int} Q(s) = \varnothing$.

condition as a special case; in other words, Condition 1 and Polemarchakis and Siconolfi's (1993) condition are equivalent under the strict convexity of preferences. We prove this fact in this section.

We first replace Assumption 2 (c) with

2 (c') $\succsim_i$ is strictly convex.

Under Assumption 2 (c'), the excess demand correspondence $Z : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is single-valued and S_i is a singleton for each $i \in I$. Let s_i denote consumer i's unique satiation point, and let $s = (s_i)_{i \in I}$.

For each $p \in \mathbb{R}^\ell$, define a subset $N(p)$ of $\mathbb{R}^\ell$ by

$$N(p) = \{\textstyle\sum_{i \in I} \lambda_i (s_i - \omega_i) \in \mathbb{R}^\ell : \lambda_i \leq 0 \text{ and } \lambda_i p \cdot (s_i - \omega_i) = 0 \text{ for all } i \in I\}.$$

Polemarchakis and Siconolfi (1993) propose the following sufficient condition for the existence of a competitive equilibrium.

Condition PS.
(a) $\operatorname{int} Q(s) \neq \varnothing$,[6]
(b) for all $p \in \operatorname{bd} Q(s) \setminus \{0\}$, *if* $Z(p) \in N(p)$, *then* $Z(p) = 0$.

Theorem 1. *Under Assumptions 1, 2 (a), 2 (b), and 2 (c') and Condition PS, there exists a competitive equilibrium.*

We now prove the equivalence between Condition 1 and Condition PS under the strict convexity of preferences. Note that under Assumption 2 (c'), Condition 1 (b) is restated as

for all $p \in \operatorname{bd} Q(s) \setminus \{0\}$, *if* $Z(p) \in Q^0(s)$ *and* $p \cdot Z(p) = 0$, *then* $Z(p) = 0$.

Proposition 4. *Under Assumption 2 (c'), Condition 1 is equivalent to Condition PS.*

Proof. We first prove that Condition PS implies Condition 1. Let $p \in \operatorname{bd} Q(s) \setminus \{0\}$ and $Z(p) \in Q^0(s)$ with $p \cdot Z(p) = 0$. We prove that $Z(p) \in N(p)$; then, from Condition PS, we have $Z(p) = 0$.

From Florenzano and Le Van (2001) [Corollary 2.3.1, p.31] and the definition of $Q^0(s)$, we obtain

$$Q^0(s) = \{\textstyle\sum_{i \in I} \lambda_i (s_i - \omega_i) \in \mathbb{R}^\ell : \lambda_i \leq 0 \text{ for all } i \in I\}. \tag{1}$$

Thus, there exist n non-positive real numbers $\lambda_1, \cdots, \lambda_n$ such that

$$Z(p) = \sum_{i \in I} \lambda_i (s_i - \omega_i).$$

Since $p \in Q(s)$ and $p \cdot Z(p) = 0$, if $\lambda_i < 0$, then $p \cdot (s_i - \omega_i) = 0$. Thus, we obtain $\lambda_i p \cdot (s_i - \omega_i) = 0$ for all $i \in I$, which implies that $Z(p) \in N(p)$.

The converse is immediate since $N(p) \subset Q^0(s)$ by equation (1) and $p \cdot y = 0$ for all $y \in N(p)$. □

[6]To be precise, Polemarchakis and Siconolfi (1993) impose not $\operatorname{int} Q(s) \neq \varnothing$ but $Q(s) \neq \{0\}$. Although Proposition 1 is still true even if we adapt $Q(s) \neq \{0\}$ instead of $\operatorname{int} Q(s) \neq \varnothing$, when $\operatorname{int} Q(s) = \varnothing$, imposing Conditions 1 (b) or PS (b) can be equivalent to imposing the existence of a competitive equilibrium itself (consider the case when $Q(s)$ is a one-dimensional linear subspace of $\mathbb{R}^\ell$).

7 Concluding Remarks

Throughout the paper, we assume the compactness of each consumer's consumption set, which guarantees that every consumer's preference has at least one satiation point. If there exists a consumer whose consumption set is unbounded and whose preference is never satiated, then Condition 1, as well as the condition proposed by Polemarchakis and Siconolfi (1993), cannot be applied because we cannot define the set $Q_i(\cdot)$ for such a consumer. To handle this case, we need to extend the condition further.

Moreover, our analysis is limited to the pure exchange economy. Whether our results can be extended to production economies is another question.

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Appendix

In the proof of Proposition 1, we use the following theorem by Debreu (1955).

Theorem 2. *Let C be a closed, convex cone with vertex 0 in $\mathbb{R}^\ell$, which is not a linear manifold; let C^0 be its polar. If the correspondence ζ from $C \cap S(0,1)$*[7] *to $\mathbb{R}^\ell$ is upper semicontinuous and bounded, and if for every $p \in C \cap S(0,1)$, the set $\zeta(p)$ is nonempty, convex, and satisfies $p \cdot \zeta(p) \leq 0$, then there is a p in $C \cap S(0,1)$ such that $\zeta(p) \cap C^0 \neq \varnothing$.*

Proof of Proposition 1. Let $s \in S$ be the element satisfying Condition 1. If $\sum_{i \in I} s_i = \sum_{i \in I} \omega_i$, then $(s, 0) \in X \times \mathbb{R}^\ell$ is a competitive equilibrium. Thus, we may assume $\sum_{i \in I} s_i \neq \sum_{i \in I} \omega_i$. This implies that $s_i \neq \omega_i$ for some $i \in I$, and thus, $Q(s) \neq \mathbb{R}^\ell$. In addition, $Q(s)$ is not a linear manifold of $\mathbb{R}^\ell$ by Condition 1 (a).

Let $\beta_i : S(0,1) \to X_i$ be a correspondence defined by

$$\beta_i(p) = \{x_i \in X_i : p \cdot x_i = p \cdot \omega_i\}, \quad p \in S(0,1).$$

For each $i \in I$, let $\varphi_i : S(0,1) \to X_i$ be a correspondence defined by

$$\varphi_i(p) = \{x_i \in X_i : x_i \in \beta_i(p) \text{ and } x_i \succsim_i y_i \text{ for all } y_i \in \beta_i(p)\}, \quad p \in S(0,1).$$

Under Assumptions 1 and 2, for each $i \in I$, the correspondence φ_i is upper semicontinuous and nonempty compact convex valued (see Montesano 2001, Propositions 1 and 2).

We now define the correspondence $\zeta : Q(s) \cap S(0,1) \to \mathbb{R}^\ell$ by

$$\zeta(p) = \sum_{i \in I} \varphi_i(p) - \sum_{i \in I} \omega_i, \quad p \in Q(s) \cap S(0,1).$$

By this definition, it is clear that ζ is upper semicontinuous, nonempty compact convex valued on $Q(s) \cap S(0,1)$ and that $p \cdot \zeta(p) = 0$ for all $p \in Q(s) \cap S(0,1)$.

Applying Theorem 2 as $C = Q(s)$, we obtain an element $\overline{p} \in Q(s) \cap S(0,1)$ such that $\zeta(\overline{p}) \cap Q^0(s) \neq \varnothing$.

Let $z \in \zeta(\overline{p}) \cap Q^0(s)$. We prove that $z \in Z(\overline{p})$. From the definition of ζ, there exists $\overline{x} \in X$ such that

$$\overline{x}_i \in \varphi_i(\overline{p}) \quad \text{for all } i \in I \quad \text{and} \quad z = \sum_{i \in I} \overline{x}_i - \sum_{i \in I} \omega_i.$$

For each $i \in I$, from the definition of φ_i, we have $\overline{p} \cdot \overline{x}_i = \overline{p} \cdot \omega_i$, and $\overline{p} \cdot x_i \neq \overline{p} \cdot \omega_i$ for all $x_i \succ_i \overline{x}_i$. Moreover, since $\overline{p} \in Q(s)$, we have $\overline{p} \cdot s_i \geq \overline{p} \cdot \omega_i$. Together with Assumptions 2 (a) and 2 (c), this implies that either $\overline{x}_i \in S_i$ or $\overline{p} \cdot x_i > \overline{p} \cdot \omega_i$ for all $x_i \succ_i \overline{x}_i$. Thus, we obtain $\overline{x}_i \in D_i(\overline{p})$ for all $i \in I$, which implies that $z = \sum_{i \in I} \overline{x}_i - \sum_{i \in I} \omega_i \in Z(\overline{p})$.

We now prove that $\overline{p} \in Q(s)$ is an equilibrium price.

Since $z \in Z(\overline{p}) \cap Q^0(s)$ and $\overline{p} \cdot z = 0$, if $\overline{p} \in \operatorname{bd} Q(s)$, we have $0 \in Z(\overline{p})$ by Condition 1 (b).

Suppose that $\overline{p} \in \operatorname{int} Q(s)$. If $z \neq 0$, there exists a ℓ-dimensional vector $q \in \mathbb{R}^\ell$ such that $q \cdot z > 0$ and $q \in Q(s)$. Indeed, let $q(t) = tz + (1-t)\overline{p}$ for $t \in (0,1)$. Since $\overline{p} \cdot z$=0, we have $q(t) \cdot z > 0$ for all $t \in (0,1)$. We also have $q(t) \in Q(s)$ for t sufficiently close to 0 since $\overline{p} \in \operatorname{int} Q(s)$. Thus, for t sufficiently close to 0, we have $q(t) \cdot z > 0$ and $q(t) \in Q(s)$. However, this contradicts the fact that $z \in Q^0(s)$. Thus, $z = 0 \in Z(\overline{p})$. □

[7] The symbol $S(0,1)$ denotes the $(\ell - 1)$ dimensional unit sphere, that is, $S(0,1) = \{q \in \mathbb{R}^\ell : \|q\| = 1\}$.

Proof of Proposition 2. Let $s \in S$ and $v \in \mathbb{R}^\ell$ with $\|v\| = 1$ be the elements satisfying Condition 2. We prove that s satisfies Conditions 1 (a) and 1 (b).

If $s = \omega$, since $Q(\omega) = \mathbb{R}^\ell$ and $\operatorname{bd} Q(\omega) = \varnothing$, Conditions 1 (a) and 1 (b) trivially hold.

In the following, we suppose that $s \neq \omega$. Note that this implies that $\alpha_i = \|s_i - \omega_i\| > 0$ for at least one $i \in I$.

From the definitions of s and v, it is clear that

$$Q(s) = \{p \in \mathbb{R}^\ell : p \cdot v \geq 0\}.$$

Therefore, $\operatorname{int} Q(s) \neq \varnothing$. Note that from Florenzano and Le Van (2001) [Corollary 2.3.1, p.31], we obtain

$$Q^0(s) = \{\beta v \in \mathbb{R}^\ell : \beta \leq 0\}. \tag{2}$$

Suppose now that for some $p \in \operatorname{bd} Q(s) \setminus \{0\}$, there exists $z \in Z(p) \cap Q^0(s)$ with $p \cdot z = 0$ and $z \neq 0$. We prove that $0 \in Z(p)$.

From (2), there exists a negative real number $\beta < 0$ such that $z = \beta v$. Since $p \cdot z = 0$, we have $p \cdot v = 0$, which implies

$$p \cdot (s_i - \omega_i) = 0 \quad \text{for all} \quad i \in I.$$

Thus, $s_i \in D_i(p)$ for all $i \in I$ and $\sum_{i\in I} s_i - \sum_{i \in I} \omega_i \in Z(p)$.

Let $z' = \sum_{i\in I} s_i - \sum_{i\in I} \omega_i = \sum_{i\in I}(s_i - \omega_i)$. Then, $z' = \gamma v$, where $\gamma = \sum_{i \in I} \alpha_i > 0$.

Let

$$\lambda = \frac{\gamma}{|\beta| + \gamma}.$$

Then, $\lambda \in (0, 1)$, and

$$\lambda z + (1 - \lambda) z' = \lambda(\beta v) + (1 - \lambda)(\gamma v) = 0.$$

Moreover, since $Z(p)$ is convex under Assumptions 1 (a) and 2 (c),

$$\lambda z + (1 - \lambda) z' \in Z(p).$$

Thus, $0 \in Z(p)$. □

Proof of Proposition 3. By Assumptions 1 (b), 2 (a) and 2 (b), $S \neq \varnothing$. If there exists $s \in S$ with $\sum_{i\in I} s_i = \sum_{i\in I} \omega_i$, then, $(s, 0) \in X \times \mathbb{R}^\ell$ is a competitive equilibrium. Thus, we may suppose without loss of generality that any $s \in S$ is not feasible.

For each $i \in I$, take an arbitrary $s_i \in S_i$ and let $s = (s_i)_{i\in I} \in S$. In the following, we prove that Conditions 1 (a) and 1 (b) hold for this s.

Let

$$t_i = s_i - \omega_i.$$

By Assumption 1 (c) and Condition 3, we have $t_i \neq 0$ for all $i \in I$. Note also that

$$Q(s) = \{p \in \mathbb{R}^\ell : p \cdot t_i \geq 0 \text{ for all } i \in I\}.$$

Step 1: We first prove that s satisfies Condition 1 (a). For this, it suffices to show that

$$0 \notin \operatorname{co}\{t_1, \cdots, t_n\}.$$

Indeed, if the convex hull of $\{t_1, \cdots, t_n\}$ does not contain the origin, then, from the strict separation theorem, there exists a ℓ-dimensional vector $\overline{p} \in \mathbb{R}^\ell \setminus \{0\}$ such that

$$\overline{p} \cdot z > 0 \quad \text{for all} \quad z \in \operatorname{co}\{t_1, \cdots, t_n\}.$$

Especially,

$$\overline{p} \cdot t_i > 0 \quad \text{for all} \quad i \in I,$$

and thus, $\overline{p} \in \operatorname{int} Q(s)$.

Suppose now that $0 \in \operatorname{co}\{t_1, \cdots, t_n\}$. Then, there exist n real numbers $\lambda_1, \cdots, \lambda_n$ such that

$$\sum_{i \in I} \lambda_i = 1, \quad \lambda_i \in [0,1] \quad \text{for all} \quad i \in I$$

and

$$\sum_{i \in I} \lambda_i t_i = 0. \tag{3}$$

Without loss of generality, we may assume

$$\lambda_1 > 0 \quad \text{and} \quad \lambda_1 \geq \lambda_i \quad \text{for all} \quad i \neq 1.$$

Then, from equation (3),

$$\begin{aligned} t_1 + \sum_{i=2}^{n} \frac{\lambda_i}{\lambda_1} t_i = 0 & \\ \Leftrightarrow \quad (s_1 - \omega_1) + \sum_{i=2}^{n} \frac{\lambda_i}{\lambda_1}(s_i - \omega_i) = 0 & \\ \Leftrightarrow \quad s_1 + \sum_{i=2}^{n} \left[\frac{\lambda_i}{\lambda_1} s_i + \left(1 - \frac{\lambda_i}{\lambda_1}\right) \omega_i \right] = \sum_{i \in I} \omega_i. & \end{aligned} \tag{4}$$

Since X_i is convex for all $i \in I$ (by Assumption 1 (a)) and

$$\frac{\lambda_i}{\lambda_1} \in [0,1] \quad \text{for all} \quad i \neq 1,$$

we have

$$\frac{\lambda_i}{\lambda_1} s_i + \left(1 - \frac{\lambda_i}{\lambda_1}\right) \omega_i \in X_i \quad \text{for all} \quad i \neq 1.$$

If $\lambda_i = \lambda_1$ for all $i \in I$, equation (4) implies that $s \in S$ is feasible, which contradicts our supposition that any element of S is not feasible. Therefore, $\lambda_i < \lambda_1$ for some $i \neq 1$. However, by Assumptions 1 (a) and 1 (c), for this i,

$$\frac{\lambda_i}{\lambda_1} s_i + \left(1 - \frac{\lambda_i}{\lambda_1}\right) \omega_i \in \operatorname{int} X_i.$$

Then, equation (4) implies

$$s_1 \in \operatorname{int}_{X_1} \hat{X}_1,$$

which contradicts Condition 3.

Thus, we conclude that $0 \notin \operatorname{co}\{t_1, \cdots, t_n\}$.

Step 2: We next prove that s satisfies Condition 1 (b). More precisely, we prove that for all $p \in \operatorname{bd} Q(s) \setminus \{0\}$, if $z \in Z(p) \cap Q^0(s)$ and $p \cdot z = 0$, then, $z = 0$.

Suppose that for some $p \in \operatorname{bd} Q(s) \setminus \{0\}$, there exists $z \in Z(p) \cap Q(s)^0$ with $p \cdot z = 0$ and $z \neq 0$. In the following, we prove that for some $i \in I$, there exists a consumption bundle $\overline{x}_i \in X_i$ such that $\overline{x}_i \in S_i \cap \operatorname{int}_{X_i} \hat{X}_i$, which contradicts Condition 3.

First, since $z \in Q^0(s)$, from Florenzano and Le Van (2001) [Corollary 2.3.1, p.31], there exist n non-positive real numbers $\mu_1, \cdots, \mu_n$ such that

$$z = \sum_{i \in I} \mu_i t_i. \tag{5}$$

Since $p \cdot z = 0$ and $p \cdot t_i \geq 0$ for all $i \in I$, if $\mu_i < 0$, then $p \cdot t_i = 0$.

Since $z \neq 0$, there exists at least one $j \in I$ such that

$$\mu_j < 0 \quad \text{and} \quad |\mu_j| \geq |\mu_i| \quad \text{for all} \quad i \neq j.$$

Without loss of generality, we may assume that $j = 1$.

Since $z \in Z(p)$, there exists $x = (x_i)_{i \in I} \in X$ such that

$$x_i \in D_i(p) \quad \text{for all} \quad i \in I$$

and

$$z = \sum_{i \in I} x_i - \sum_{i \in I} \omega_i = \sum_{i \in I} (x_i - \omega_i). \tag{6}$$

Then, from (5) and (6),

$$\sum_{i \in I} (x_i - \omega_i) + \sum_{i \in I} |\mu_i| (s_i - \omega_i) = 0. \tag{7}$$

For each $i \in I$, let $\overline{x}_i$ be the ℓ-dimensional vector defined by

$$\overline{x}_i = \frac{1}{1 + |\mu_1|} x_i + \frac{|\mu_i|}{1 + |\mu_1|} s_i + \frac{|\mu_1| - |\mu_i|}{1 + |\mu_1|} \omega_i. \tag{8}$$

Note that for all $i \in I$,

$$\begin{aligned} \frac{1}{1 + |\mu_1|} &\in (0, 1), \\ \frac{|\mu_i|}{1 + |\mu_1|} &\in [0, 1], \\ \frac{|\mu_1| - |\mu_i|}{1 + |\mu_1|} &\in [0, 1], \\ \frac{1}{1 + |\mu_1|} + \frac{|\mu_i|}{1 + |\mu_1|} + \frac{|\mu_1| - |\mu_i|}{1 + |\mu_1|} &= 1. \end{aligned}$$

Hence, from the convexity of X_i,

$$\overline{x}_i \in X_i \quad \text{for all} \quad i \in I.$$

Moreover, $\overline{x} = (\overline{x}_i)_{i \in I} \in X$ is feasible. Indeed, from (8),

$$(1 + |\mu_1|)(\overline{x}_i - \omega_i) = (x_i - \omega_i) + |\mu_i| (s_i - \omega_i) \quad \text{for all} \quad i \in I. \tag{9}$$

Summing up the equations in (9) over i and using (7), we have

$$(1 + |\mu_1|) \sum_{i \in I} (\overline{x}_i - \omega_i) = \sum_{i \in I} (x_i - \omega_i) + \sum_{i \in I} |\mu_i| (s_i - \omega_i) = 0.$$

Therefore,

$$\sum_{i \in I} (\overline{x}_i - \omega_i) = 0. \tag{10}$$

We claim that for all $i \in I$ with $|\mu_i| = |\mu_1|$, we have $\overline{x}_i \in S_i$ (especially, $\overline{x}_1 \in S_1$). Let $i \in I$ be the consumer with $|\mu_i| = |\mu_1|$. Since $|\mu_i| = |\mu_1| > 0$, we have $p \cdot t_i = 0$, which implies $s_i \in D_i(p)$. Then, from Assumption 2 (c), $x_i \in S_i$. Using Assumption 2 (c) again,

$$\overline{x}_i = \frac{1}{1 + |\mu_1|} x_i + \frac{|\mu_1|}{1 + |\mu_1|} s_i \in S_i.$$

We next claim that from the feasibility of $\overline{x} \in X$, we have $|\mu_i| < |\mu_1|$ for some $i \neq 1$. Indeed, if $|\mu_i| = |\mu_1|$ for all $i \in I$, the previous claim implies that $\overline{x} \in S$, which contradicts the supposition that any element of S is not feasible.

Then, for $i \neq 1$ with $|\mu_i| < |\mu_1|$, from (8) and Assumptions 1 (a) and 1 (c), we have $\overline{x}_i \in \operatorname{int} X_i$. In view of the feasibility of $\overline{x} \in X$, this implies

$$\overline{x}_1 \in \operatorname{int}_{X_1} \hat{X}_1.$$

Since $\overline{x}_1 \in S_1$, we obtain the desired contradiction. □