E C O N O M I C S   B U L L E T I N

The solutions for multi-choice games: TU games approach

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Abstract

In this note we link the solutions for multi-choice games proposed by Hsiao and Raghavan (1992), Derks and Peters (1993), and Peters and Zank (2005) with the Shapley values of some particular TU games.

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1 Introduction

There are four extensions of the Shapley value in the framework of multi-choice games. The four extended solutions are proposed by Hsiao and Raghavan (1992), Derks and Peters (1993), Nouweland et al. (1995) and Peters and Zank (2005), respectively. Calvo and Santos (2000) showed that the solution notion of Nouweland et al. (1995) corresponds to the discrete Aumann-Shapley method proposed by Moulin (1995). Roughly speaking, they introduced the notion of the "replica TU game" for a multi-choice game and showed that the multichoice value, proposed by Nouweland et al. (1995), coincides with the Shapley value of the corresponding replica TU game. That is, the solution of a multi-choice game can be formulated by the Shapley value of the corresponding TU game. In this note, we introduce the notion of TU decomposition games for a multi-choice game and use it to study the solutions for multi-choice games proposed by Hsiao and Raghavan (1992), Derks and Peters (1993), and Peters and Zank (2005), respectively. We show that each of the three extended solutions consists of the Shapley values of the corresponding TU decomposition games.

2 Preliminaries

Let U be the universe of players. Let $N \subseteq U$ be a set of players. For $m = (m_i)_{i\in N}$, where $m_i \in \mathbb{N} \cup \{0\}$ for all $i \in N$, we set $M_i = \{0, 1, \cdots, m_i\}$ as the action space of player i, where the action 0 means not participating, and $M_i^+ = M_i \setminus \{0\}$. For $N \subseteq U$, $N \neq \emptyset$, let $M^N = \prod_{i\in N} M_i$ be the product set of the action spaces for players in N. Denote the zero vector in $\mathbb{R}^N$ by 0_N.

A **multi-choice game** is a triple (N, m, V), where N is a non-empty and finite set of players, m is a vector that describes the number of activity levels for each player, and $V : M^N \to \mathbb{R}$ is a characteristic function which assigns to each action vector $x = (x_i)_{i\in N} \in M^N$ the worth that the players can obtain when each player i plays at activity level $x_i \in M_i$ with $V(0_N) = 0$. If no confusion can arise a game (N, m, V) will sometimes be denoted by its characteristic function V. Denote the class of all multi-choice games by MC.

To state the extended Shapley values, some more notation will be needed. Given $(N, m, V) \in MC$ and $x \in M^N$, we define $\|x\| = \sum\limits_{k\in N} x_k$ and $S(x) = \{k \in N \mid x_k \neq 0\}$.

The analogue of unanimity games for multi-choice games are the **min-**

imal effort games (N, m, U^x), where $x \in M^N$, $x \neq 0_N$, defined as: for all $y \in M^N$,[1]

$$U^x(y) = \begin{cases} 1 & \text{if } y \geq x \ ; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Hsiao and Raghavan (1992) showed that for all $(N, m, V) \in MC$,

$$V = \sum_{\substack{x \in M^N \\ x \neq 0_N}} a^x(V)\ U^x, \tag{2}$$

where [2]

$$a^x(V) = \sum_{S \subseteq S(x)} (-1)^{|S|}\ V(x - e^S). \tag{3}$$

Note that, by Equations (1) and (2), for all $y \in M^N \setminus \{0_N\}$

$$V(y) = \sum_{\substack{x \in M^N \setminus \{0_N\} \\ x \leq y}} a^x(V). \tag{4}$$

Definition 1 *Derks and Peters (1993) proposed a multi-choice Shapley value, the D&P Shapley value. We denote the D&P Shapley value by* Θ*; Peters and Zank (2005) proposed a multi-choice Shapley value, the P&Z Shapley value. We denote the P&Z Shapley value by* Γ*; Hsiao and Raghavan (1992) proposed a multi-choice Shapley value, the H&R Shapley value. We denote the symmetric form of the H&R Shapley value by* Λ*. Formally, the D&P Shapley value* Θ *(the P&Z Shapley value* Γ*; H&R Shapley value* Λ*) is the function on* MC *which associates with each game* (N, m, V) *and each player* $i \in N$ *and each* $j \in M_i^+$ *the value* [3]

- $$\Theta_{i,j}(N, m, V) = \sum_{\substack{x \in M^N \\ x_i \geq j}} \frac{a^x(V)}{\|x\|}.$$

[1] $y \leq x$ means that $y_i \leq x_i$ for all $i \in N$

[2] Given $S \subseteq N$, let $|S|$ be the number of elements in S and let $e^S(N)$ be the binary vector in $\mathbb{R}^N$ whose component $e_i^S(N)$ satisfies

$$e_i^S(N) = \begin{cases} 1 & \text{if } i \in S \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Note that if no confusion can arise $e^S(N)$ will be denoted by e^S.

[3] Peters and Zank (2005) defined the P&Z Shapley value by fixing its values on minimal effort games and imposing Linearity. Also, they named it the egalitarian solution. In this paper, we define the P&Z Shapley value based on the dividends. On the other hand, we define the H&R Shapley value in terms of the dividends. Hsiao and Raghavan (1992) provided an alternative formula of the H&R Shapley value.

- $$\Gamma_{i,j}(N,m,V) = \sum_{\substack{x\in M^N\\ x_i=j}} \frac{a^x(V)}{|S(x)|}.$$

- $$\Lambda_{i,j}(N,m,V) = \sum_{\substack{x\in M^N\\ 0<x_i\leq j}} \frac{a^x(V)}{|S(x)|}.$$

Note that the P&Z Shapley value is a subdivision of the H&R Shapley value. It is known that for each $(N,m,V) \in MC$, $\sum_{i\in N} \Lambda_{i,m_i}(N,m,V) = V(m)$, $\sum_{i\in N}\sum_{j=1}^{m_i} \Gamma_{i,j}(N,m,V) = V(m)$, and $\sum_{i\in N}\sum_{j=1}^{m_i} \Theta_{i,j}(N,m,V) = V(m)$.

3 Main Results

For $x \in \mathbb{R}^N$, we write x_S to be the restriction of x at S for each $S \subseteq N$. Let $N \subseteq U$, $i \in N$ and $x \in \mathbb{R}^N$, for convenience we introduce the substitution notation x_{-i} to stand for $x_{N\setminus\{i\}}$ and let $y = (x_{-i}, j) \in \mathbb{R}^N$ be defined by $y_{-i} = x_{-i}$ and $y_i = j$.

A **transferable utility game (TU game)** [4] is a pair (N, v) where N is a coalition and v is a mapping such that $v : 2^N \longrightarrow \mathbb{R}$ and $v(\emptyset) = 0$. Denote the class of all TU games by G.

The unanimity games for TU games, (N, u^S), where $S \subseteq N, S \neq \emptyset$, are defined by for all $T \subseteq N$,

$$u^S(T) = \begin{cases} 1 & \text{if } S \subseteq T\ ; \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

It is well-known that for all $(N, v) \in G$,

$$v = \sum_{\substack{S\subseteq N\\ S\neq\emptyset}} a^S(v)\ u^S, \tag{6}$$

where

$$a^S(v) = \sum_{T\subseteq S} (-1)^{|S|-|T|}\ v(T). \tag{7}$$

[4] For convenience, we will use lower case letter v and capital letter V to denote the characteristic functions of TU games and multi-choice games, respectively.

Note that, by Equations (5) and (6), for all $T \subseteq N, T \neq \emptyset$,

$$v(T) = \sum_{\substack{S \subseteq T \\ S \neq \emptyset}} a^S(v). \tag{8}$$

Given $(N, m, V) \in MC$. Without loss of generality, suppose that $m_i > 0$ for all $i \in N$. Let $i \in N$ and $j \in M_i^+$. We define the **P&Z-TU decomposition game** $(N, v_{PZ}^{i,j})$ with respect to (i, j) by for all $S \subseteq N$,

$$v_{PZ}^{i,j}(S) = V\big((m_{-i}, j)_S, 0_{N \setminus S}\big) - V\big((m_{-i}, j-1)_S, 0_{N \setminus S}\big). \tag{9}$$

We are now ready for the main theorem.

Theorem 1 *Let $(N, m, V) \in MC$, $i \in N$, $j \in M_i^+$ and $(N, v_{PZ}^{i,j})$ be the P&Z-TU decomposition game with respect to (i, j). Then*

$$\Gamma_{i,j}(N, m, V) = \phi_i(N, v_{PZ}^{i,j}), \tag{10}$$

where ϕ is the Shapley value on TU games.
In particular, for all $S \subseteq N, S \neq \emptyset$,

$$a^S(v_{PZ}^{i,j}) = \sum_{\substack{x \leq (m_{-i}, j) \\ x_i = j, S(x) = S}} a^x(V). \tag{11}$$

Proof. Let $(N, m, V) \in MC$, $i \in N$, $j \in M_i^+$ and $S \subseteq N, S \neq \emptyset$. By Equation (4),

$$\begin{aligned} & V\big((m_{-i}, j)_S, 0_{N \setminus S}\big) - V\big((m_{-i}, j-1)_S, 0_{N \setminus S}\big) \\ = & \sum_{\substack{x \in M^N \setminus \{0_N\} \\ x \leq \left((m_{-i}, j)_S, 0_{N \setminus S}\right)}} a^x(V) - \sum_{\substack{x \in M^N \setminus \{0_N\} \\ x \leq \left((m_{-i}, j-1)_S, 0_{N \setminus S}\right)}} a^x(V) \\ = & \sum_{\substack{x \leq \left((m_{-i}, j)_S, 0_{N \setminus S}\right) \\ x_i = j}} a^x(V) \\ = & \sum_{\substack{T \subseteq S \\ T \neq \emptyset}} \sum_{\substack{x \leq (m_{-i}, j) \\ x_i = j, S(x) = T}} a^x(V). \end{aligned} \tag{12}$$

Combining Equation (12) with Equations (8) and (9),

$$\begin{aligned} \sum_{\substack{T \subseteq S \\ T \neq \emptyset}} a^T(v_{PZ}^{i,j}) = & \; v_{PZ}^{i,j}(S) \quad \text{(by Equation (8))} \\ = & \; V\big((m_{-i}, j)_S, 0_{N \setminus S}\big) - V\big((m_{-i}, j-1)_S, 0_{N \setminus S}\big) \quad \text{(by Equation (9))} \\ = & \sum_{\substack{T \subseteq S \\ T \neq \emptyset}} \sum_{\substack{x \leq (m_{-i}, j) \\ x_i = j, S(x) = T}} a^x(V). \quad \text{(by Equation (12))} \end{aligned}$$

Hence, for all $S \subseteq N, S \neq \emptyset$, $a^S(v^{i,j}_{PZ}) = \sum_{\substack{x \leq (m_{-i}, j) \\ x_i = j, S(x) = S}} a^x(V)$.

Finally, using the fact that $\phi_i(N, v^{i,j}_{PZ}) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{a^S(v^{i,j}_{PZ})}{|S|}$, we have that

$$
\begin{aligned}
\Gamma_{i,j}(N, m, V) &= \sum_{\substack{x \in M^N \\ x_i = j}} \frac{a^x(V)}{|S(x)|} \\
&= \sum_{\substack{S \subseteq N \\ i \in S}} \sum_{\substack{x \in M^N \\ x_i = j, S(x) = S}} \frac{a^x(V)}{|S(x)|} \\
&= \sum_{\substack{S \subseteq N \\ i \in S}} \sum_{\substack{x \leq (m_{-i}, j) \\ x_i = j, S(x) = S}} \frac{a^x(V)}{|S(x)|} \\
&= \sum_{\substack{S \subseteq N \\ i \in S}} \frac{a^S(v^{i,j}_{PZ})}{|S|} \qquad \text{(by Equation (11) and } S(x) = S) \\
&= \phi_i(N, v^{i,j}_{PZ}).
\end{aligned}
$$

■

Remark 1 *In Theorem 4.1 of Peters and Zank (2005), they provided a probabilistic interpretation of the P&Z Shapley value closely related to the well-known interpretation of the Shapley value, with the understanding that players (other than player i) inside the formed coalition are active on their highest level. This representation implies that the P&Z Shapley value only uses a restricted subset of the information present in the multi-choice game v: we only need to know the worths of those coalitions where at most one player is active on a level different from 0 and m. This result coincides with our Theorem 1.*

Remark 2 *Let $i \in N$ and $j \in M_i^+$. We define the* **H&R-TU decomposition game** *$(N, v^{i,j}_{HR})$ with respect to (i, j) by for all $S \subseteq N$,*

$$v^{i,j}_{HR}(S) = V\big((m_{-i}, j)_S, 0_{N \setminus S}\big). \tag{13}$$

Then, each component of the H&R Shapley value of a multi-choice game is the Shapley value of the corresponding H&R-TU decomposition game, i.e.,

$$\Lambda_{i,j}(N, m, V) = \phi_i(N, v^{i,j}_{HR}). \tag{14}$$

In particular, for all $S \subseteq N, S \neq \emptyset$,

$$a^S(v^{i,j}_{HR}) = \sum_{\substack{x \leq (m_{-i}, j) \\ S(x) = S}} a^x(V). \tag{15}$$

Next, we offer an analogue of Theorem 1 wit respect to the D&P Shapley value.

Similarly, let $(N,m,V) \in MC$, $i \in N$ and $j \in M_i^+$. We define [5] the **D&P-TU decomposition game** $(N, v_{DP}^{i,j})$ with respect to (i,j) by for all $S \subseteq N$,

$$v_{DP}^{i,j}(S) = \sum_{\substack{T \subseteq S \\ T \neq \emptyset}} |T| \sum_{\substack{x \in M^N \\ x_i \geq j, S(x)=T}} \frac{\sum_{K \subseteq S(x)} (-1)^{|K|} V(x - e^K)}{\|x\|}. \tag{16}$$

Theorem 2 *Let $(N,m,V) \in MC$, $i \in N$, $j \in M_i^+$ and $(N, v_{DP}^{i,j})$ be the D&P-TU decomposition game with respect to (i,j). Then*

$$\Theta_{i,j}(N,m,V) = \phi_i(N, v_{DP}^{i,j}), \tag{17}$$

where ϕ is the Shapley value on TU games.
In particular, for all $S \subseteq N, S \neq \emptyset$,

$$\frac{a^S(v_{DP}^{i,j})}{|S|} = \sum_{\substack{x \leq m \\ x_i \geq j, S(x)=S}} \frac{a^x(V)}{\|x\|}. \tag{18}$$

Proof. By Equations (3), (8) and (16),

$$\begin{aligned}
\sum_{\substack{T \subseteq S \\ T \neq \emptyset}} a^T(v_{DP}^{i,j}) &= v_{DP}^{i,j}(S) \qquad \text{(by Equation (8))} \\
&= \sum_{\substack{T \subseteq S \\ T \neq \emptyset}} |T| \sum_{\substack{x \in M^N \\ x_i \geq j, S(x)=T}} \frac{\sum_{K \subseteq S(x)} (-1)^{|K|} V(x - e^K)}{\|x\|} \qquad \text{(by Equation (16))} \\
&= \sum_{\substack{T \subseteq S \\ T \neq \emptyset}} |T| \sum_{\substack{x \in M^N \\ x_i \geq j, S(x)=T}} \frac{a^x(V)}{\|x\|}. \qquad \text{(by Equation (3))}
\end{aligned}$$

Hence, for all $S \subseteq N, S \neq \emptyset$, $\frac{a^S(v_{DP}^{i,j})}{|S|} = \sum_{\substack{x \leq m \\ x_i \geq j, S(x)=S}} \frac{a^x(V)}{\|x\|}$.

[5] By Equation (3), $a^x(V) = \sum_{K \subseteq S(x)} (-1)^{|K|} V(x - e^K)$. It is not difficult to derive an alternative formula of the P&Z-TU decomposition game $(N, v_{PZ}^{i,j})$ with respect to (i,j) as follows: for all $S \subseteq N$,

$$v_{PZ}^{i,j}(S) = \sum_{\substack{T \subseteq S \\ T \neq \emptyset}} |T| \sum_{\substack{x \in M^N \\ x_i = j, S(x)=T}} \frac{\sum_{K \subseteq S(x)} (-1)^{|K|} V(x - e^K)}{|S(x)|}.$$

Finally, using the fact that $\phi_i(N, v_{DP}^{i,j}) = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{a^S(v_{DP}^{i,j})}{|S|}$, we have that

$$\begin{aligned}
\Theta_{i,j}(N,m,V) &= \sum_{\substack{x\in M^N\\ x_i\geq j}} \frac{a^x(V)}{\|x\|} \\
&= \sum_{\substack{S\subseteq N\\ i\in S}} \sum_{\substack{x\in M^N\\ x_i\geq j, S(x)=S}} \frac{a^x(V)}{\|x\|} \\
&= \sum_{\substack{S\subseteq N\\ i\in S}} \sum_{\substack{x\leq m\\ x_i\geq j, S(x)=S}} \frac{a^x(V)}{\|x\|} \\
&= \sum_{\substack{S\subseteq N\\ i\in S}} \frac{a^S(v_{DP}^{i,j})}{|S|} \qquad \text{(by Equation (18))} \\
&= \phi_i(N, v_{DP}^{i,j}).
\end{aligned}$$

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