E C O N O M I C S   B U L L E T I N

# Bankruptcy problems with interval uncertainty

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## *Abstract*

In this paper, bankruptcy situations with interval data are studied. Two classical bankruptcy rules, namely the proportional rule and the rights-egalitarian rule, are extended to the interval setting. It turns out that these bankruptcy interval rules generate elements in the interval core of a related cooperative interval game.

The second author acknowledges the support of TUBITAK (Turkish Scientific and Technical Research Council) and hospitality of Department of Mathematics, University of Genoa, Italy.
**Citation:** Branzei, Rodica and Sirma Zeynep Alparslan Gok, (2008) "Bankruptcy problems with interval uncertainty." *Economics Bulletin,* Vol. 3, No. 56 pp. 1-10
**Submitted:** July 3, 2008. **Accepted:** September 16, 2008.
**URL:** http://economicsbulletin.vanderbilt.edu/2008/volume3/EB-08C70030A.pdf

# 1 Introduction

This paper focuses on bankruptcy situations with interval data and related cooperative interval games. Classical bankruptcy problems and bankruptcy games have been intensively studied. We refer here to O'Neill (1982), Aumann and Maschler (1985), Herrero, Maschler and Villar (1999) and Young (1987). In a classical bankruptcy situation a certain amount of money (estate) has to be divided among some people (claimants) who have individual claims on the estate, and the total claim is weakly larger than the estate.
A bankruptcy situation with set of claimants $N$ is a pair $(E, d)$, where $E \geq 0$ is the estate to be divided and $d \in \mathbb{R}^N_+$ is the vector of claims such that $\sum_{i \in N} d_i \geq E$. We assume without loss of generality that $d_1 \leq d_2 \leq \ldots \leq d_n$ and denote by $BR^N$ the set of bankruptcy situations with player set $N$. The total claim is denoted by $D = \sum_{i \in N} d_i$. A bankruptcy rule is a function $f : BR^N \rightarrow \mathbb{R}^N$ which assigns to each bankruptcy situation $(E, d) \in BR^N$ a payoff vector $f(E, d) \in \mathbb{R}^N$ such that $0 \leq f(E, d) \leq d$ (reasonability) and $\sum_{i \in N} f_i(E, d) = E$ (efficiency). In this paper we are interested in bankruptcy rules that are coordinate-wise (weakly) increasing in $E$. The proportional rule ($PROP$) is one of the most often used in real life. It is defined by $PROP_i(E, d) = \frac{d_i}{D}E$ for all $i \in N$. Another interesting bankruptcy rule is the rights-egalitarian rule, defined by $f_i^{RE}(E, d) = d_i + \frac{1}{n}(E - D)$, for each $(E, d) \in BR^N$ and all $i \in N$. We notice that the rights-egalitarian rule was introduced in Herrero, Maschler and Villar (1999) as a division rule for all circumstances of division problems.
Recall that a cooperative game in coalitional form is a pair $< N, v >$, where $N$ is the set of players, and $v : 2^N \rightarrow \mathbb{R}$ is a function assigning to each coalition $S \in 2^N$ a real number, such that $v(\emptyset) = 0$. We denote by $G^N$ the family of all classical cooperative games with player set $N$. The core (Gillies (1959)) is defined by

$$C(v) = \left\{ x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{for each } S \in 2^N \right\},$$

for each $v \in G^N$. A game $v \in G^N$ is convex if and only if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \in 2^N$.
To each bankruptcy situation $(E, d) \in BR^N$ one can associate a pessimistic bankruptcy game $v_{E,d}$ defined by $v_{E,d}(S) = (E - \sum_{i \in N \setminus S} d_i)_+$ for each $S \in 2^N$, where $x_+ = \max(0, x)$. The game $v_{E,d}$ is convex and the bankruptcy rules

$PROP$ and $f^{RE}$ provide allocations in the core of the game.

Cooperative interval games arising from bankruptcy situations where the claims can vary within closed intervals are introduced and analyzed in Branzei, Dimitrov and Tijs (2003). A bankruptcy situation where the claims are certain but the available estate can vary within a closed interval is used in Alparslan Gök, Miquel and Tijs (2008) to illustrate cores for two-person interval games. This paper considers bankruptcy situations where the estate and (some of the) claims vary within closed intervals, which we call bankruptcy interval situations.

It is important to consider interval claims because in various disputes including inheritance (O'Neill (1982)) claimants face uncertainty regarding their effective rights and, as a result, individual claims can be expressed in the form of closed intervals without any probability distributions attached to them. In such situations our model based on interval claims fits better than the more standard claims approach and, additionally, offers flexibility in conflict resolution under interval uncertainty of the estate at stake. Economic applications of our approach include funds' allocation of a firm among its divisions (Pulido, Sánchez-Soriano and Llorca (2002), Pulido et al. (2008)), priority problems (Moulin (2000)), distribution of penalty costs in delayed projects (Branzei et al. (2002)) and disputes related to cooperation in joint projects where agents have restricted willingness to pay (Tijs and Branzei (2004)).

The paper is organized as follows. In Section 2 we recall basic notions and results from the theory of cooperative interval games. In Section 3 we introduce two bankruptcy interval rules which are the extensions to the interval setting of the proportional rule and the rights-egalitarian rule for classical bankruptcy situations. It turns out that these bankruptcy interval rules generate allocations in the interval core of a related cooperative interval game. Final comments are given in Section 4.

# 2 Preliminaries

We start with some preliminaries from interval calculus (Moore (1979).
Let $I, J \in I(\mathbb{R})$ with $I = \left[\underline{I}, \overline{I}\right]$, $J = \left[\underline{J}, \overline{J}\right]$, $|I| = \overline{I} - \underline{I}$ and $\alpha \in \mathbb{R}_+$. Then, $I + J = \left[\underline{I} + \underline{J}, \overline{I} + \overline{J}\right]$; $\alpha I = \left[\alpha \underline{I}, \alpha \overline{I}\right]$. The partial substraction operator $I - J$ is defined (Alparslan Gök, Branzei and Tijs (2008b)), only if $|I| \geq |J|$, by $I - J = \left[\underline{I} - \underline{J}, \overline{I} - \overline{J}\right]$. Note that $\underline{I} - \underline{J} \leq \overline{I} - \overline{J}$. We recall that $I$ is

weakly better than $J$, which we denote by $I \succcurlyeq J$, if and only if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$. We also use the reverse notation $J \preccurlyeq I$, if and only if $\underline{J} \leq \underline{I}$ and $\overline{J} \leq \overline{I}$.

A cooperative interval game in coalitional form (Alparslan Gök, Miquel and Tijs (2008)) is an ordered pair $< N, w >$ where $N = \{1, 2, \ldots, n\}$ is the set of players, and $w : 2^N \to I(\mathbb{R})$ is the characteristic function such that $w(\emptyset) = [0, 0]$. For each $S \in 2^N$, the worth set (or worth interval) $w(S)$ of the coalition $S$ in the interval game $< N, w >$ is of the form $[\underline{w}(S), \overline{w}(S)]$, where $\underline{w}(S)$ is the lower bound and $\overline{w}(S)$ is the upper bound of $w(S)$. The family of all interval games with player set $N$ is denoted by $IG^N$. Some classical $TU$-games associated with an interval game $w \in IG^N$ play a key role, namely the border games $< N, \underline{w} >$, $< N, \overline{w} >$ and the length game $< N, |w| >$, where $|w|(S) = \overline{w}(S) - \underline{w}(S)$ for each $S \in 2^N$. Note that $\overline{w} = \underline{w} + |w|$.

Let $w_1, w_2 \in IG^N$. We say that $w_1 \preccurlyeq w_2$ if $w_1(S) \preccurlyeq w_2(S)$ for each $S \in 2^N$, and define $< N, w_1 + w_2 >$ by $(w_1 + w_2)(S) = w_1(S) + w_2(S)$ for each $S \in 2^N$. For $w_1, w_2 \in IG^N$ with $|w_1(S)| \geq |w_2(S)|$ for each $S \in 2^N$, $< N, w_1 - w_2 >$ is defined by $(w_1 - w_2)(S) = w_1(S) - w_2(S)$. Given $w \in IG^N$ and $\lambda \in \mathbb{R}_+$ we define $< N, \lambda w >$ by $(\lambda w)(S) = \lambda \cdot w(S)$ for each $S \in 2^N$.

Let $w \in IG^N$. The interval core $\mathcal{C}(w)$ is defined by

$$\mathcal{C}(w) = \left\{ (I_1, \ldots, I_n) \in I(\mathbb{R})^N | \sum_{i \in N} I_i = w(N), \sum_{i \in S} I_i \succcurlyeq w(S), \forall S \in 2^N \setminus \{\emptyset\} \right\}.$$

We say that $< N, w >$ is supermodular if

$$w(S) + w(T) \preccurlyeq w(S \cup T) + w(S \cap T) \text{ for all } S, T \in 2^N.$$

We call a game $w \in IG^N$ convex if $< N, w >$ is supermodular and its length game $< N, |w| >$ is convex. For details on cooperative interval games we refer the reader to Alparslan Gök, Branzei and Tijs (2008a,b).

# 3 Bankruptcy situations with interval data and related rules

A *bankruptcy interval situation* with a fixed set of claimants $N = \{1, 2, \ldots, n\}$ is a pair $(E, d) \in I(\mathbb{R}) \times I(\mathbb{R})^N$, where $E = [\underline{E}, \overline{E}] \succcurlyeq [0, 0]$ is the estate to be divided and $d$ is the vector of interval claims with $i$-th coordinate $d_i = [\underline{d}_i, \overline{d}_i]$,

$i \in N$, such that $[0,0] \preccurlyeq d_1 \preccurlyeq d_2 \preccurlyeq \ldots \preccurlyeq d_n$ and $\overline{E} < \sum_{i=1}^{n} \underline{d}_i$. We note that all selections $(\tilde{E}, \tilde{d})$, where $\underline{E} < \tilde{E} < \overline{E}$ and $\underline{d}_i < \tilde{d}_i < \overline{d}_i$, for all $i \in N$, are traditional bankruptcy situations. We denote by $\underline{d}(N)$ the total lower claim and by $\overline{d}(N)$ the total upper claim. We also use the notations $\underline{d}(S) = \sum_{i\in S} \underline{d}_i$ and $\overline{d}(S) = \sum_{i\in S} \overline{d}_i$ for $S \subset N$. We denote by $BRI^N$ the family of bankruptcy interval situations with set of claimants $N$.

A *bankruptcy interval rule* for bankruptcy interval situations is a function $\mathcal{F} : BRI^N \rightarrow I(\mathbb{R})^N$ assigning to each bankruptcy interval situation $(E, d) \in BRI^N$ a vector $\mathcal{F}(E, d) = (\mathcal{F}_1(E, d), \ldots, \mathcal{F}_n(E, d)) \in I(\mathbb{R})^N$, such that

(i) $[0, 0] \preccurlyeq \mathcal{F}_i(E, d) \preccurlyeq d_i$ for each $i \in N$ (reasonability);

(ii) $\sum_{i=1}^{n} \mathcal{F}_i(E, d) = E$ (efficiency).

In this paper we look at the bankruptcy rules $PROP$ and $f^{RE}$ and extend them to the interval setting. We denote by $BRI_1^N$ the family of all bankruptcy situations $(E, d) \in BRI^N$ which satisfy the condition

$$\underline{E}/\underline{d}(N) \leq \overline{E}/\overline{d}(N), \quad (1)$$

and by $BRI_2^N$ the family of all bankruptcy situations $(E, d) \in BRI^N$ which satisfy the condition

$$|E| \geq |d(N)|\,. \quad (2)$$

Condition (1) can be read as: the available amount per-unit of lower-estate is weakly smaller than the available amount per-unit of upper estate. Condition (2) can be read as: the spread of uncertainty regarding the estate is weakly larger than the total spread of uncertainty regarding the claims. Note that conditions (1) and (2) are satisfied for all bankruptcy interval situations where all the claim intervals are degenerate, i.e. $\underline{d}_i = \overline{d}_i$ for all $i \in N$. Bankruptcy interval situations where the estate is a nondegenerate interval, i.e. $\underline{E} < \overline{E}$, and all the claims are uncertainty-free are studied in Branzei and Dall'Aglio (2008).

The inclusion $BRI_1^N \subset BRI^N$ might be strict as the following example illustrates.

Example 3.1. *Let $(E, d)$ be a three-person bankruptcy situation. We suppose that the claims of the players are closed intervals with $d_1 = [10, 20]$, $d_2 = [30, 50]$ and $d_3 = [30, 70]$, respectively and the estate is $E = [60, 100]$. Then, we obtain $\underline{E}/\underline{d}(N) = 6/7 > 5/7 = \overline{E}/\overline{d}(N)$.*

The inclusion $SBRI_2^N \subset SBRI^N$ might be also strict as we can see from Example 3.1, where $|E| = 40 < 70 = |d(N)|$. In the following we extend the proportional rule and the rights-egalitarian rule to the interval setting.
First, note that

$$PROP_i(\underline{E}, \underline{d}) = (\underline{d}_i/\underline{d}(N))\underline{E} \leq (\underline{d}_i/\overline{d}(N))\overline{E} \leq (\overline{d}_i/\overline{d}(N))\overline{E} = PROP_i(\overline{E}, \overline{d})$$

for each $i \in N$, where the first inequality follows from condition (1) and the second inequality follows from $[\underline{d}_i, \overline{d}_i] \in I(\mathbb{R})$.
We define the *proportional interval rule* $\mathcal{PROP} : BRI_1^N \to I(\mathbb{R})^N$ by

$$\mathcal{PROP}_i(E, d) = [PROP_i(\underline{E}, \underline{d}), PROP_i(\overline{E}, \overline{d})],$$

for each $(E, d) \in BRI_1^N$ and all $i \in N$. Second, note that

$$f_i^{RE}(\underline{E}, \underline{d}) = \underline{d}_i + \frac{1}{n}(\underline{E} - \underline{d}(N)) \leq \underline{d}_i + \frac{1}{n}(\overline{E} - \overline{d}(N)) \leq \overline{d}_i + \frac{1}{n}(\overline{E} - \overline{d}(N)) = f_i^{RE}(\overline{E}, \overline{d})$$

for each $i \in N$, where the first inequality follows from condition (2) and the second inequality follows from $[\underline{d}_i, \overline{d}_i] \in I(\mathbb{R})$.
We define the *rights-egalitarian interval rule* $\mathcal{F}^{RE} : BRI_2^N \to I(\mathbb{R})^N$ by

$$\mathcal{F}_i^{RE}(E, d) = [f_i^{RE}(\underline{E}, \underline{d}), f_i^{RE}(\overline{E}, \overline{d})],$$

for each $(E, d) \in BRI_2^N$ and all $i \in N$. The next proposition shows that $\mathcal{PROP}$ and $\mathcal{F}^{RE}$ are bankruptcy interval rules.

PROPOSITION 3.1. *Let $\mathcal{B} = \{\mathcal{PROP}, \mathcal{F}^{RE}\}$. Then, each interval rule $\mathcal{F} \in \mathcal{B}$ is efficient and reasonable.*

*Proof.* The efficiency of $\mathcal{F}$ follows from the efficiency of corresponding classical bankruptcy rule $f \in \{PROP, f^{RE}\}$, i.e. $\sum_{i \in N} f_i(\underline{E}, \underline{d}) = \underline{E}$ and $\sum_{i \in N} f_i(\overline{E}, \overline{d}) = \overline{E}$. Further, the reasonability of $\mathcal{F}$ follows from

$$0 \leq f_i(\underline{E}, \underline{d}) \leq \underline{d}_i \text{ and } 0 \leq f_i(\overline{E}, \overline{d}) \leq \overline{d}_i \text{ for each } i \in N.$$

□

In the following we define a subclass of $BRI^N$, denoted by $SBRI^N$, consisting of all bankruptcy interval situations such that

$$\text{for each } S \in 2^N \text{ with } \underline{d}(N \setminus S) \leq \underline{E} \text{ it holds } |d(N \setminus S)| \leq |E|. \quad (3)$$

We call a bankruptcy interval situation in $SBRI^N$ a *strong bankruptcy interval situation.* With each $(E,d) \in SBRI^N$ we associate a cooperative interval game $< N, w_{E,d} >$ defined by $w_{E,d}(S) = [v_{\underline{E},\underline{d}}(S), v_{\overline{E},\overline{d}}(S)]$ for each $S \subset N$. Note that (3) implies $v_{\underline{E},\underline{d}}(S) \leq v_{\overline{E},\overline{d}}(S)$ for each $S \in 2^N$. We denote by $SBRIG^N$ the family of all bankruptcy interval games $w_{E,d}$ with $(E,d) \in SBRI^N$. We notice that $w_{E,d} \in SBRIG^N$ is supermodular because $v_{\underline{E},\underline{d}}$ and $v_{\overline{E},\overline{d}} \in G^N$ are convex (see Proposition 3.2 in Alparslan Gök, Branzei and Tijs (2008b)). The following example illustrates that $w_{E,d} \in SBRIG^N$is supermodular but not necessarily convex.

Example 3.2. *Let $(E,d)$ be a two-person bankruptcy situation. We suppose that the claims of the players are closed intervals $d_1 = [70,70]$ and $d_2 = [80,80]$, respectively and the estate is $E = [100,140]$. Then, for each $i = 1,2$ the corresponding game $< N, w_{E,d} >$ is given by $w_{E,d}(\emptyset) = [0,0]$, $w_{E,d}(1) = [20,60]$, $w_{E,d}(2) = [30,70]$ and $w_{E,d}(1,2) = [100,140]$. This game is supermodular, but is not convex because $|w_{E,d}| \in G^N$ is not convex.*

In the following we consider the restriction of the interval rule $\mathcal{PROP}$ to $SBRI_1^N = BRI_1^N \cap SBRI^N$, and the restriction of the interval rule $\mathcal{F}^{RE}$ to $SBRI_2^N = BRI_2^N \cap SBRI^N$. In the next proposition we consider $(E,d) \in SBRI_1^N$ if $\mathcal{F}$ is $\mathcal{PROP}$, and $(E,d) \in SBRI_2^N$ if $\mathcal{F}$ is $\mathcal{F}^{RE}$.

Proposition 3.2. *Let $\mathcal{F} \in \mathcal{B}$. Then, $\mathcal{F}(E,d) \in \mathcal{C}(w_{E,d})$ for each $w_{E,d} \in SBRIG^N$.*

*Proof.* First, we have

$$\sum_{i=1}^{n} \mathcal{F}_i(E,d) = E = E - \sum_{i \in \emptyset} d_i = w_{E,d}(N),$$

where the first equality follows from efficiency of the bankruptcy interval rules.
Second, take $S \subset N$. Then,

$$\sum_{i \in S} \mathcal{F}_i(E,d) = w_{E,d}(N) - \sum_{i \in N \setminus S} \mathcal{F}_i(E,d) \succcurlyeq E - \sum_{i \in N \setminus S} d_i,$$

where the equality follows from efficiency and the inequality follows from reasonability of the bankruptcy interval rules. Also, $\sum_{i \in S} \mathcal{F}_i(E,d) \succcurlyeq [0,0]$ by reasonability. So, $\sum_{i \in S} \mathcal{F}_i(E,d) \succcurlyeq w_{E,d}(S)$. Hence, $\mathcal{F}(E,d) \in \mathcal{C}(w_{E,d})$. $\square$

# 4 Final comments

In this paper we define two bankruptcy interval rules by extending the proportional rule and the rights-egalitarian rule to bankruptcy interval situations. An interesting topic for further research is to extend to the interval setting the axiomatic characterizations of $PROP$ and $f^{RE}$ and compare them in the spirit of Herrero, Maschler and Villar (1999). Note that to compare $\mathcal{PROP}$ with $\mathcal{F}^{RE}$ we need to consider the restricted class $BRI_1^N \cap BRI_2^N$.

The use of the allocations generated by the rules $\mathcal{PROP}$ and $\mathcal{F}^{RE}$ in practical bankruptcy-like situations with interval uncertainty is two-fold. Firstly, these interval allocations are used to inform claimants about what they can expect, between two boundaries, from the division problem at stake. Secondly, when the realization of the estate occurs, they are used to obtain standard allocations. We refer the reader to Branzei, Tijs and Alparslan Gök (2008) for ways to transform vectors of intervals in vectors of real numbers.

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