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The Maximal Equal Allocation of Nonseparable Costs on Multi-Choice Games

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Abstract

This note provides a generalization of the equal allocation of nonseparable costs (EANSC) on multi-choice games. Also, we extend to the multi-choice games case the complement reduced game and characterize the extended EANSC by means of related property of consistency.

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Introduction

Consistency is a crucial property of solutions. If a solution is not consistent, then a subgroup of agents might not respect the original compromise but revise the payoff distribution within the subgroup. The fundamental property of solutions has been investigated in various classes of problems by applying reduced games always. Various definitions of a reduced game have been proposed, depending upon exactly how the agents outside of the subgroup be paid off. Sobolev [9] and Peleg [7, 8] axiomatized the prenucleolus, the prekernel and the core, respectively, by means of consistency which respect to the reduced game due to Davis and Maschler [1]. Moulin [6] introduced an alternative version of a reduced game in the context of quasi-liner cost allocation problems. Hart and Mas-Colell [3] introduced a version of a reduced game to axiomatize the Shapley value, and so on.

In this note, we focus on the solution concept of the equal allocation of nonseparable costs (EANSC). The EANSC is a well-known solution concept in cooperative game theory. Moulin [6] introduced a notion of consistency and showed that the EANSC is in the sense a stable solution that satisfies the property of consistency.

A multi-choice TU game, introduced by Hsiao and Raghavan [4], is a generalization of a traditional TU game. In a traditional TU game, each player is either fully involved or not involved at all in participation with some other agents, while in a multi-choice game, each player is allowed to participate with finite many different activity levels. As we knew, solutions on multi-choice games could be applied in many fields such as economics, political sciences, accounting, and even military sciences.

In the framework of multi-choice games, we introduce a generalization of the EANSC and propose an extension of the complement reduced game introduced by Moulin [6]. Different from the axiomatization of Moulin [6], we offer an axiomatic justification which is an analogue of Hart and Mas-Colell's [3] characterization by applying related property of two-person standardness and consistency.

Preliminaries

Let U be the universe of players. Suppose each player i has $m_i \in \mathbb{N}$ levels at which he can actively participate. Let $m^U \in \mathbb{N}^U$ be the vector that describes the number of activity levels for each player, in which he can actively participate. Let $N \subseteq U$ be a set of players. Denote $m^U_N \in \mathbb{R}^N$ to be the restriction of m^U to N. For $i \in U$, we set $M_i = \{0, 1, \cdots, m_i\}$ as the action space of player i, where the action 0 means not participating, and $M_i^+ = M_i \setminus \{0\}$. For $N \subseteq U$, $N \neq \emptyset$, let $M^N = \prod_{i\in N} M_i$ be the product set of the action spaces for players N. Denote the zero vector in $\mathbb{R}^N$ by 0_N.

A **multi-choice game** is a triple (N, m, v), where N is a non-empty and finite set of players, m is the vector that describes the number of activity levels for each player, and $v : M^N \to \mathbb{R}$ is a characteristic function which assigns to each action vector $\alpha = (\alpha_i)_{i\in N} \in M^N$ the worth that the players can obtain when each player i plays at activity level $\alpha_i \in M_i$ with $v(0_N) = 0$. If no confusion can arise a game (N, m, v) will sometimes be denoted by its characteristic function v.

Denote the class of all multi-choice TU games by Γ. A **solution on** Γ is a function σ which associates with each $(N, m, v) \in \Gamma$ an element $\sigma(N, m, v)$ of $I\!R^N$. Let $(N, m, v) \in \Gamma$, $\alpha \in M^N$ and $S \subseteq N$, we define $S(\alpha) = \{i \in N \mid \alpha_i > 0\}$ and we denote $\alpha_S \in \mathbb{R}^S$ to be the restriction of α to S. Let $i \in N$, for convenience we introduce the substitution notation α_{-i} to stand for $\alpha_{N\setminus\{i\}}$ and let $\gamma = (\alpha_{-i}, t) \in \mathbb{R}^N$

be defined by $\gamma_{-i} = \alpha_{-i}$ and $\gamma_i = t$.

Subsequently, we provide a generalization of the equal allocation of nonseparable costs (EANSC) on multi-choice games.

Definition 1 *The* **maximal equal allocation of nonseparable costs (MEANSC)** *on multi-choice games, $\overline{\beta}$, is the function on Γ which associates with $(N, m, v) \in \Gamma$ and each player $i \in N$ the value*[1]

$$\overline{\beta_i}(N, m, v) = \beta_i(N, m, v) + \frac{1}{|N|} \cdot \big[v(m) - \sum_{k \in N} \beta_k(N, m, v)\big],$$

where for all $i \in N$, $\beta_i(N, m, v) = \max_{j \in M_i^+} \{v(m_{-i}, j) - v(m_{-i}, 0)\}$. We say that the value $\beta_i(N, m, v)$ is the **maximal marginal contribution** *of player i.*

Axioms, Reduced Game and Characterization

In this section, we show thatthere exists a reduced game that can be used to characterize the MEANSC. We will make use of the following axioms.

Let σ be a solution on Γ. σ satisfies **efficiency (EFF)** if for all $(N, m, v) \in \Gamma$, $\sum_{i \in N} \sigma_i(N, m, v) = v(m)$. σ satisfies **standard for two-person games (STPG)** if for all $(N, m, v) \in \Gamma$ with $|N| = 2$, $\sigma(N, m, v) = \overline{\beta}(N, m, v)$.

Lemma 1 *The MEANSC satisfies EFF and STPG.*

Proof. It can easily be deduced from Definition 1 and the definitions of the EFF axiom and the STPG axiom. ∎

Next, consider the Moulin reduction. Given a payoff vector chosen by a solution for some game, and given a subgroup of players, Moulin [6] defined the reduced game as that in which each coalition in the subgroup could attain payoffs to its members only if they are compatible with the initial payoffs to "all" the members outside of the subgroup. A natural extension of Moulin reduction to the multi-choice games case is as follows.

Given $(N, m, v) \in \Gamma$, $S \subseteq N$ and a solution σ, the **reduced game** (S, m_S, v_S^σ) **with respect to** S **and** σ is defined by for all $\alpha \in M^S$,

$$v_S^\sigma(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0_S, \\ v\big(\alpha, m_{N \setminus S}\big) - \sum\limits_{i \in N \setminus S} \sigma_i(N, m, v) & \text{otherwise.} \end{cases}$$

The *Consistency*[2] requirement may be described informally as follows: Let σ be a solution on Γ. For any group of players in a game, one defines a " reduced game" among them by considering the amounts remaining after the rest of the players are given the payoffs prescribed by σ. Then σ is said to be *consistent* if, when it is applied to any reduced game, it always yields the same payoffs as in the original game.

Formally, a solution σ satisfies **consistency (CON)** if for all $(N, m, v) \in \Gamma$, for all $S \subseteq N$ with $S \neq \emptyset$ and for all $i \in N$, $\sigma_i(S, m_S, v_S^\sigma) = \sigma_i(N, m, v)$.

[1] Without loss of generality, we can assume that $S(m) = N$.

[2] The axiom was originally introduced by Harsanyi [2] under the name of bilateral equilibrium. For discussion of this axiom, please see Thomson [11].

Lemma 2 *The MEANSC $\overline{\beta}$ satisfies CON.*

Proof. Given $(N,m,v) \in \Gamma$ and $S \subseteq N$. Assume that $S = \{i\}$ for some $i \in N$. By EFF of $\overline{\beta}$ and the definition of $v_S^{\overline{\beta}}$,

$$\begin{aligned}
\beta_i(S, m_S, v_S^{\overline{\beta}}) &= v_S^{\overline{\beta}}(m_S) \\
&= v(m) - \sum_{k\in N\setminus\{i\}} \overline{\beta_k}(N,m,v) \\
&= \beta_i(N,m,v).
\end{aligned}$$

Assume that $|N| \geq 2$ and $2 \leq |S| \leq |N|$. By the definition of $\overline{\beta}$, for all $i \in S$,

$$\overline{\beta}_i(S, m_S, v_S^{\overline{\beta}}) = \beta_i(S, m_S, v_S^{\overline{\beta}}) + \frac{1}{|S|} \cdot \big[v_S^{\overline{\beta}}(m_S) - \sum_{k\in S} \beta_k(S, m_S, v_S^{\overline{\beta}})\big]. \tag{1}$$

By definitions of β and $v_S^{\overline{\beta}}$, for all $i \in S$,

$$\begin{aligned}
\beta_i(S, m_S, v_S^{\overline{\beta}}) &= \max_{j\in M_i^+} \{v_S^{\overline{\beta}}(m_{S\setminus\{i\}}, j) - v_S^{\overline{\beta}}(m_{S\setminus\{i\}}, 0)\} \\
&= \max_{j\in M_i^+} \{v(m_{-i}, j) - \sum_{k\in N\setminus S} \overline{\beta_k}(N,m,v) - v(m_{-i}, 0) + \sum_{k\in N\setminus S} \overline{\beta_k}(N,m,v)\} \\
&= \max_{j\in M_i^+} \{v(m_{-i}, j) - v(m_{-i}, 0)\} \\
&= \beta_i(N,m,v).
\end{aligned} \tag{2}$$

Hence, by equations (1), (2) and definitions of $v_S^{\overline{\beta}}$ and $\overline{\beta}$,

$$\begin{aligned}
\overline{\beta}_i(S, m_S, v_S^{\overline{\beta}}) &= \beta_i(N,m,v) + \tfrac{1}{|S|} \cdot \big[v_S^{\overline{\beta}}(m_S) - \sum_{k\in S} \beta_k(N,m,v)\big] \\
&= \beta_i(N,m,v) + \tfrac{1}{|S|} \cdot \big[v(m) - \sum_{k\in N\setminus S} \overline{\beta_k}(N,m,v) - \sum_{k\in S} \beta_k(N,m,v)\big] \\
&= \beta_i(N,m,v) + \tfrac{1}{|S|} \cdot \big[\sum_{k\in S} \overline{\beta_k}(N,m,v) - \sum_{k\in S} \beta_k(N,m,v)\big] \quad \text{(by EFF of } \overline{\beta}\text{)} \\
&= \beta_i(N,m,v) + \tfrac{1}{|S|} \cdot \Big[\tfrac{|S|}{|N|} \cdot \big[v(m) - \sum_{k\in N} \beta_k(N,m,v)\big]\Big] \quad \text{(by Definition 1)} \\
&= \beta_i(N,m,v) + \tfrac{1}{|N|} \cdot \big[v(m) - \sum_{k\in N} \beta_k(N,m,v)\big] \\
&= \overline{\beta}_i(N,m,v).
\end{aligned}$$

Hence, the MEANSC satisfies CON. ■

Subsequently, inspired by Hart and Mas-Colell [3], we characterize the MEANSC by means of the properties of two-person standardness and consistency.

Theorem 1 *A solution σ on Γ satisfies STPG and CON if and only if $\sigma = \overline{\beta}$.*

Proof. By Lemmas 1 and 2, $\overline{\beta}$ satisfies STPG and CON.

To prove uniqueness, suppose σ satisfies STPG and CON. By STPG and CON of σ, it is easy to derive that σ also satisfies EFF, hence we omit it. Let $(N,m,v) \in \Gamma$. If $|N| = 2$, then by STPG of σ, $\sigma(N,m,v) = \overline{\beta}(N,m,v)$. If $|N| = 1$, the proof is similar to the coalition game by adding a "dummy" player to one-person geam; hence, we omit it. The case $|N| > 2$: Let $i \in N$ and $S = \{i, k\}$ for some $k \in N \setminus \{i\}$, then

$$\begin{array}{rl} & \sigma_i(N,m,v) - \sigma_k(N,m,v) \\ = & \sigma_i(S,m_S,v_S^\sigma) - \sigma_k(S,m_S,v_S^\sigma) \ \ \text{(by CON of } \sigma) \\ = & \overline{\beta_i}(S,m_S,v_S^\sigma) - \overline{\beta_k}(S,m_S,v_S^\sigma) \ \ \text{(by STPG of } \sigma) \\ = & \beta_i(S,m_S,v_S^\sigma) - \beta_k(S,m_S,v_S^\sigma) \ \ \text{(by Equation (1))} \\ = & \max\limits_{j \in M_i^+} \{v_S^{\overline{\beta}}(m_{S\setminus\{i\}}, j) - v_S^{\overline{\beta}}(m_{S\setminus\{i\}}, 0)\} - \max\limits_{j \in M_k^+} \{v_S^{\overline{\beta}}(m_{S\setminus\{k\}}, j) - v_S^{\overline{\beta}}(m_{S\setminus\{k\}}, 0)\} \\ = & \max\limits_{j \in M_i^+} \{v(m_{-i}, j) - \sum\limits_{t \in N\setminus S} \overline{\beta_t}(N,m,v) - v(m_{-i},0) + \sum\limits_{t \in N\setminus S} \overline{\beta_t}(N,m,v)\} \\ & - \max\limits_{j \in M_k^+} \{v(m_{-k}, j) - \sum\limits_{t \in N\setminus S} \overline{\beta_t}(N,m,v) - v(m_{-k},0) + \sum\limits_{t \in N\setminus S} \overline{\beta_t}(N,m,v)\} \\ = & \max\limits_{j \in M_i^+} \{v(m_{-i}, j) - v(m_{-i},0)\} - \max\limits_{j \in M_k^+} \{v(m_{-k}, j) - v(m_{-k},0)\} \end{array}$$

Similarly,

$$\overline{\beta_i}(N,m,v) - \overline{\beta_k}(N,m,v) = \max_{j \in M_i^+} \{v(m_{-i}, j) - v(m_{-i},0)\} - \max_{j \in M_k^+} \{v(m_{-k}, j) - v(m_{-k},0)\}.$$

Hence,

$$\sigma_i(N,m,v) - \sigma_k(N,m,v) = \overline{\beta_i}(N,m,v) - \overline{\beta_k}(N,m,v).$$

By EFF of σ and $\overline{\beta}$,

$$\begin{array}{rl} |N| \cdot \sigma_i(N,m,v) - v(m) & = \sum\limits_{k \in N} [\sigma_i(N,m,v) - \sigma_k(N,m,v)] \\ & = \sum\limits_{k \in N} [\overline{\beta_i}(N,m,v) - \overline{\beta_k}(N,m,v)] \\ & = |N| \cdot \overline{\beta_i}(N,m,v) - v(m). \end{array}$$

Hence, for all $i \in N$, $\sigma_i(N,m,v) = \overline{\beta_i}(N,m,v)$. ■

The following examples are to show that each of the axioms used in Theorem 1 is logically independent of the remaining axioms.

Example 1 *Define a solution σ on Γ by for all $(N,m,v) \in \Gamma$ and for all $i \in N$, $\sigma_i(N,m,v) = 0$. It's easy to verify that σ satisfies CON, but it violates STPG.*

Example 2 *Given $\varepsilon \in \mathbb{R} \setminus \{0\}$. Define a solution σ on Γ by for all $(N,m,v) \in \Gamma$ and for all $i \in N$,*

$$\sigma_i(N,m,v) = \begin{cases} \overline{\beta_i}(N,m,v) & \text{if } |N| \le 2 \\ \overline{\beta_i}(N,m,v) - \varepsilon & \text{otherwise} \end{cases}.$$

It's easy to verify that σ satisfies STPG, but it violates CON.

Modified Moulin Reduction

In the axiomatic formulation of cooperative games, based on the Moulin reduction, Tadenuma [10] and Moulin [6] offered characterizations of the core and of the EANSC, respectively. In this section, we propose an alternative extension of Moulin reduction and show that the duplicate core introduced by Hwang and Liao [5] satisfies related property of consistency.

Given $(N,m,v) \in \Gamma$. A **payoff vector** of (N,m,v) is a vector $(x_i)_{i\in N} \in \mathbb{R}^N$, where the number x_i represents the payoff that player i receives for all $i \in N$. Given $(N,m,v) \in \Gamma$, a payoff vector x, $\alpha \in M^N$ and $S \subseteq N$, we denote $\alpha_S \in \mathbb{R}^S$ to be the restriction of α to S, and $x(\alpha) = \sum_{i\in N} \alpha_i \cdot x_i$. A payoff vector x of $(N,m,v) \in \Gamma$ is **efficient (EFF)** if $x(m) = v(m)$. A payoff vector x of $(N,m,v) \in \Gamma$ is **individual rational (IR)** if for all $i \in N$ and for all $j \in M_i^+$, $x(m_{-i}, j) \geq v(m_{-i}, j)$. A payoff vector x of $(N,m,v) \in \Gamma$ is **coalitional rational (CR)** if for all $\alpha \in M^N$, $x(\alpha) \geq v(\alpha)$.

The set of **feasible payoff vectors** of (N,m,v) is denoted by

$$X^*(N,m,v) = \{\ x \in \mathbb{R}^N \mid x(m) \leq v(m)\},$$

whereas

$$X(N,m,v) = \{\ x \in \mathbb{R}^N \mid x \text{ is EFF }\}$$

is the set of **preimputations** of (N,m,v) and the set of imputations of (N,m,v) is denoted by

$$I(N,m,v) = \{\ x \in \mathbb{R}^N \mid x \text{ is EFF and IR in } (N,m,v)\ \}.$$

A **non-single value solution** on Γ is a function σ which associates with each $(N,m,v) \in \Gamma$ a subset $\sigma(N,m,v)$ of $X^*(N,m,v)$. A solution σ on Γ satisfies **non-emptiness (NE)** if for all $(N,m,v) \in \Gamma$, $\sigma(N,m,v) \neq \emptyset$. A solution σ on Γ satisfies **efficiency (EFF)** if for all $(N,m,v) \in \Gamma$, $\sigma(N,m,v) \subseteq X(N,m,v)$. A solution σ on Γ satisfies **individual rational (IR)** if for all $(N,m,v) \in \Gamma$, $\sigma(N,m,v) \subseteq I(N,m,v)$.

The duplicate core of a multi-choice game (N,m,v) (Hwang and Liao [5]) is as follows.

Definition 2 *Let $(N,m,v) \in \Gamma$. The* **duplicate core** *$C(N,m,v)$ of $(N,m,v) \in \Gamma$ consists of all $x \in X(N,m,v)$ which satisfy CR.*

Given $(N,m,v) \in \Gamma$, $S \subseteq N$ and a payoff vector x, the **modified reduced game** $(S, m_S, v_{S,x})$ **with respect to** S **and** x is defined by for all $\alpha \in M^S$,

$$v_{S,x}(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0_S, \\ v\big(\alpha, m_{N\setminus S}\big) - \sum\limits_{i\in N\setminus S} m_i \cdot x_i & \text{otherwise.} \end{cases}$$

Modified consistency requires that if x is prescribed by a non-single value solution σ for a game (N,m,v), then the projection of x to S should be prescribed by σ for the reduced game with respect to S and x for all S. Thus, the projection of x to S should be consistent with the expectations of the members of S as reflected by their reduced game.

Formally, a non-single value solution σ satisfies **modfied consistency (MCON)** if for all $(N,m,v) \in \Gamma$, for all $S \subseteq N$ with $S \neq \emptyset$ and for all $x \in \sigma(N,m,v)$, $(S, m_S, v_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, m_S, v_{S,x})$. Next, we show that the duplicate core satisfies modfied consistency. We keep the terminology used in cooperative game theory, and say that the multi-choice game (N,m,v) is *balanced* [3] if $C(N,m,v) \neq \emptyset$. Let Γ_c denote the set of all balanced multi-choice games.

Lemma 3 *Let $(N,m,v) \in \Gamma$ and x be a payoff vector of (N,m,v). Then for all $S \subseteq N$, $S \neq \emptyset$,*

[3] A characterization of balanced multi-choice games was given by Hwang and Liao (2007).

1. *if x is EFF in (N,m,v), then x_S is EFF in $(S,m_S,v_{S,x})$.*

2. *if x is CR in (N,m,v), then x_S is CR in $(S,m_S,v_{S,x})$.*

Proof. Let $(N,m,v)\in\Gamma$, $S\subseteq N$, $S\neq\emptyset$ and x be a payoff vector of (N,m,v). To verify (1), let x be EFF in (N,m,v). By the definition of $v_{S,x}$ and EFF of x in (N,m,v),

$$\begin{aligned} v_{S,x}(m_S) &= v(m_S,m_{N\setminus S})-\sum_{i\in N\setminus S}x_i \\ &= v(m)-\sum_{i\in N\setminus S}m_i\cdot x_i \\ &= \sum_{i\in S}m_i\cdot x_i. \end{aligned}$$

That is, x_S is EFF in $(S,m_S,v_{S,x})$.

To verify (2), let x be CR in (N,m,v). For all $t\in M^S$,

$$\begin{aligned} & v_{S,x}(t)-\sum_{i\in S}t_i\cdot x_i \\ = \; & v(t,m_{N\setminus S})-\sum_{i\in N\setminus S}m_i\cdot x_i-\sum_{i\in S}t_i\cdot x_i. \end{aligned}$$

Since x is CR in (N,m,v), $v(t,m_{N\setminus S})-\sum_{i\in N\setminus S}m_i\cdot x_i-\sum_{i\in S}t_i\cdot x_i\leq 0$. That is, for all $t\in M^S$, $x_S(t)\geq v^M_{S,x}(t)$. Hence x_S is CR in $(S,m_S,v_{S,x})$. ■

Lemma 4 *On both Γ and Γ_c, the core satisfies MCON.*

Proof. It can easily be deduced from the proof of Lemma 3. ■

Remark 1 *As we have known, in the framework of TU games, Tadenuma [10] proved that on the domain of balanced games, the core is the only solution satisfying non-emptiness, individual rationality and consistency (w.r.t. Moulin reduction). However, in the framework of multi-choice games, the duplicate core is "not" the only solution satisfying non-emptiness, individual rationality and modified consistency on the domain of balanced multi-choice games. The reason is by way of illustration: we define a solution on the domain of balanced multi-choice games by for each balanced multi-choice game (N,m,v), $\sigma(N,m,v)$ is the set of all $x\in I(N,m,v)$ that satisfy for all $\alpha\in M^N$ with $\alpha_i=m_i$ for some $i\in N$, $x(\alpha)\geq v(\alpha)$. Clearly, $C(N,m,v)\subseteq\sigma(N,m,v)$ and it is not difficult to derive that σ satisfies non-emptiness, individual rationality and modified consistency on the domain of balanced multi-choice games . This points out that the duplicate core can "not" be characterized by non-emptiness, individual rationality and modfied consistency on the domain of balanced multi-choice games. Hence, based on the Moulin reduction, the related issue is still an open question.*

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