E c o n o m i c s B u l l e t i n

A Note on Existence of Nash Networks in One-way Flow

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Abstract

In this note we provide conditions which ensure the existence of Nash networks in One-way flow models with cost heterogeneity.

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1 Introduction

The importance of networks is pervasive and now well documented. Among the growing litterature on the formation of networks, a few papers have explored the impact of heterogeneity, although this is a distinctive feature. Galeotti (2006, [3]) is one exception. The author introduces agents heterogeneity in one way flow models of network formation and characterize the architectures of Nash networks. However Nash networks do not alway exist. Heterogeneity in cost of forming links play a major role in this non existence.

In recent paper Billand, Bravard and Sarangi (BBS, 2008, [1]) show that Nash networks do not always exist under heterogeneity of costs by pairs. Then in Proposition 3 (pg. 505), they provide a sufficient condition for the existence of Nash networks. Subsequently, by means of a counterexample Derks and Tennekes (2008, [2]) showed that the condition given in Proposition 3 is not sufficient for the existence of Nash networks. In this comment we provide an additional condition which ensures the existence of Nash networks. We also show that although this is a fairly strong condition, it still allows us to have models with non-trivial cost heterogeneity.

2 Model Setup

Let $N = \{1, \ldots, n\}$ be the set of players. The network relations among these players are formally represented by directed graphs whose nodes are the players. A network $\boldsymbol{g} = (N, E)$ is a pair of sets: the set N of players and the edges set $E(\boldsymbol{g}) \subset N \times N$ of directed links. A link initiated by player i to player j is denoted by $i\ j$. Each player i chooses a strategy $\boldsymbol{g}_i = (\boldsymbol{g}_{i1}, \ldots, \boldsymbol{g}_{ii-1}, \boldsymbol{g}_{ii+1}, \ldots, \boldsymbol{g}_{in})$, $\boldsymbol{g}_{ij} \in \{0, 1\}$ for all $j \in N \setminus \{i\}$, which describes the decision of establishing links. More precisely, $\boldsymbol{g}_{ij} = 1$ if and only if $i\ j \in E(\boldsymbol{g})$. The interpretation of $\boldsymbol{g}_{ij} = 1$ is that player i forms a link with player $j \neq i$, and the interpretation of $\boldsymbol{g}_{ij} = 0$ is that i does not form a link with player j. We assume in the following that every player is always trivially connected to herself, so $\boldsymbol{g}_{ii} = 1$ for all $i \in N$ and do not include it in $\boldsymbol{g}_i$. We only use pure strategies. Note that $\boldsymbol{g}_{ij} = 1$ does not necessarily imply that $\boldsymbol{g}_{ji} = 1$. Indeed it is possible i is linked to j, but j is not linked to i. Let $\mathcal{G} = \times_{i=1}^{n} \mathcal{G}_i$ be the set of all possible networks where $\mathcal{G}_i$ is the set of all possible strategies of player $i \in N$.

We now provide some important graph theoretic definitions. For a directed graph, $\boldsymbol{g} \in \mathcal{G}$, a *path* $P(\boldsymbol{g})$ of length m in $\boldsymbol{g}$ from player j to i, $i \neq j$, is a finite sequence $i_0, i_1, \ldots, i_m$ of distinct players such that $i_0 = i$, $i_m = j$ and $\boldsymbol{g}_{i_k i_{k+1}} = 1$ for $k = 0, \ldots, m-1$. If $i_0 = i_m$, then the path is a cycle. We denote the set of cycles in the network $\boldsymbol{g}$ by $\mathcal{C}(\boldsymbol{g})$. Let $C(\boldsymbol{g})$ be a typical member of $\mathcal{C}(\boldsymbol{g})$. We assume that if $\ell \in N^{C(\boldsymbol{g})}$, the set of players who belong to the cycle $C(\boldsymbol{g})$, then $\ell + 1 \in N^{C(\boldsymbol{g})}$ and $\boldsymbol{g}_{\ell+1,\ell} = 1$. In the empty network, $\boldsymbol{g}^0$, there are no links between any agents.

To sum up, a link from a player j to a player i ($\boldsymbol{g}_{ij} = 1$) allows player i to get resources from player j but since we are in a one-way flow model, this link does not allow player j to obtain resources from i. Moreover, a player i may receive information from other players through a sequence of indirect links. To be precise, information flows from player j to player i, if i and j are linked by a path in

$\boldsymbol{g}$ from j to i. Let

$$N_i(\boldsymbol{g}) = \{j \in N | \text{ there exists a path in } \boldsymbol{g} \text{ from } j \text{ to } i\},$$

be the set of players that player i can access in the network $\boldsymbol{g}$. By definition, we assume that $i \in N_i(\boldsymbol{g})$ for all $i \in N$ and for all $\boldsymbol{g} \in \mathcal{G}$. Information received from player j is worth V_{ij} to player i. Moreover, i incurs a cost c_{ij} when she initiates a direct link with j, i.e. when $\boldsymbol{g}_{ij} = 1$. We can now define the payoff function of player $i \in N$:

$$\pi_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g}) \setminus \{i\}} V_i - \sum_{j \in N \setminus \{i\}} \boldsymbol{g}_{ij} c_{ij}. \tag{1}$$

We assume that $c_{ij} > 0$ and $V_i > 0$ for all $i \in N$. Let $\bar{c} = \max_{(i,j) \in N \times N \setminus \{i\}} \{c_{i,j}\}$ and $\underline{c} = \min_{(i,j) \in N \times N \setminus \{i\}} \{c_{i,j}\}$. We now provide some useful definitions for studying the existence of Nash networks. Given a network $\boldsymbol{g} \in \mathcal{G}$, let $\boldsymbol{g}_{-i}$ denote the network obtained when all of player i's links are removed. The network $\boldsymbol{g}$ can be written as $\boldsymbol{g} = \boldsymbol{g}_{-i} \oplus \boldsymbol{g}_i$, where the operator $\oplus$ indicates that $\boldsymbol{g}$ is formed by the union of links in $\boldsymbol{g}_i$ and $\boldsymbol{g}_{-i}$. The strategy $\boldsymbol{g}_i$ is said to be a best response of player i to $\boldsymbol{g}_{-i}$ if:

$$\pi_i(\boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}) \geq \pi_i(\boldsymbol{g}'_i \oplus \boldsymbol{g}_{-i}), \text{for all } \boldsymbol{g}'_i \in \mathcal{G}_i.$$

The set of player i's best responses to $\boldsymbol{g}_{-i}$ is denoted by $\mathcal{BR}_i(\boldsymbol{g}_{-i})$. Furthermore, a network $\boldsymbol{g} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_i, \ldots, \boldsymbol{g}_n)$ is said to be a Nash network if $\boldsymbol{g}_i \in \mathcal{BR}_i(\boldsymbol{g}_{-i})$ for each $i \in N$.

3 Results

Proposition Let the payoff function be given by (1) for all $i \in N$. There always exists a Nash network if

- (P1) for all $i, j, j' \in N$, we have $| c_{i,j} - c_{i,j'} | \leq V_i$,
- (P2) for all $i \in N$, $j \in N$, $| V_j - V_i | \leq \frac{2\underline{c} - \bar{c}}{n-1}$.

Proof of Proposition. We state two lemmas that provide insights about the algorithmic process. The first explains an important property of payoff functions that satisfy (P1) and (P2), while the second explains a crucial feature of the process. The process we use is as follows.

Step 1. $\boldsymbol{g} = \emptyset$;
Step 2. While $\boldsymbol{g}$ is not Nash do
Step 3. Find $i \in N$ such that $\boldsymbol{g}_i \notin \mathcal{BR}_i(\boldsymbol{g}_{-i})$
Step 4. $\boldsymbol{g} := \boldsymbol{g}_{-i} \oplus \boldsymbol{g}'_i$, with $\boldsymbol{g}'_i \in \mathcal{BR}_i(\boldsymbol{g}_{-i})$;
Step 5. If $\sum_{j \in N \setminus \{i\}} g'_{i,j} > 1$, then compute a wheel and Stop
Else go to Step 6.
Step 6. Compute
$$N^1(\boldsymbol{g}) = \{(\ell, \ell') \in (N \setminus N^{C(\boldsymbol{g})}) \times N^{C(\boldsymbol{g})} \mid \boldsymbol{g}_{\ell,\ell'} = 1\}$$

Step 7. While $N^1(\boldsymbol{g}) \neq \emptyset$ do

Choose $(\ell, \ell') \in N^1(\boldsymbol{g})$

$\boldsymbol{g} := \boldsymbol{g} \oplus \ell' + 1 \quad \ell \ominus \ell' + 1 \quad \ell'$

Compute $N^1(\boldsymbol{g})$

end

end

Lemma 1 *Suppose the payoff function satisfies equation (1), (P1) and (P2). If there exists a network* $\boldsymbol{g}$ *and a player* $j \in N$ *such that* $\boldsymbol{g}_j \in \mathcal{BR}(\boldsymbol{g}_{-j})$ *and* $\sum_{i \in N} \boldsymbol{g}_{j,i} \geq 2$*, then wheels are Nash networks.*

Proof Note that in a wheel each player i obtains net benefits which are at least $(n-1)V_i - \bar{c}$. Let player j be such that there is a network $\boldsymbol{g}$ with $\boldsymbol{g}'_j \in \mathcal{BR}_j(\boldsymbol{g}_{-j})$ and $\sum_{i \in N} \boldsymbol{g}'_{j,i} \geq 2$. Then player j incurs a cost in $\boldsymbol{g}' = \boldsymbol{g}_{-j} \oplus \boldsymbol{g}'_j$ greater or equal to $2\underline{c}$, and obtains at most $(n-1)V_j$ values. Since j plays a best response, we have: $(n-1)V_j - 2\underline{c} \geq 0$. First, consider players j' such that $V_{j'} \geq V_j$. We have $V_{j'} - V_j \geq 0 \geq \bar{c} - 2\underline{c}$, since $2\underline{c} - \bar{c} \geq \mid V_j - V_i \mid \geq 0$. Therefore $V_{j'} - \bar{c} > V_j - 2\underline{c}$ and $(n-1)V_{j'} - \bar{c} > (n-1)V_j - 2\underline{c}$. It follows that player j' has an incentive to be in a wheel instead being isolated. Moreover, by $(P1)$, in a wheel player j' has no incentive to replace her link. Second, consider players j' such that $V_{j'} < V_j$. We have $(n-1)(V_j - V_{j'}) < 2\underline{c} - \bar{c}$, that is $0 \leq (n-1)V_j - 2\underline{c} < (n-1)V_{j'} - \bar{c}$. By using the same argument as above, it follows that player j' plays a best response in a wheel. □

Lemma 2 *Let* $\boldsymbol{g}^0$, $\boldsymbol{g}^1$, *...,* $\boldsymbol{g}^{t-1}$, $\boldsymbol{g}^t$, *... be the sequence of networks obtained in the process. Then* $N_i(\boldsymbol{g}^{t-1}) \subseteq N_i(\boldsymbol{g}^t)$ *for all* $i \in N$.

Proof 1. By (P1) there is no network $\boldsymbol{g}$ in the process which contains a player i who does not belong to a cycle such that $\boldsymbol{g}_{\ell,i} = \boldsymbol{g}_{\ell',i} = 1, \ell \neq \ell'$.

2. By definition $N_i(\boldsymbol{g}^0) \subseteq N_i(\boldsymbol{g}^1)$, for all $i \in N$. We now show that $N_i(\boldsymbol{g}^{t-1}) \subseteq N_i(\boldsymbol{g}^t)$ for all i and t. To introduce a contradiction, let $\boldsymbol{g}^t$ be the first network, in the process where there is at least a player j such that $N_j(\boldsymbol{g}^t) \not\supseteq N_j(\boldsymbol{g}^{t-1})$. Then by construction of the process, the player who plays a best response between $\boldsymbol{g}^{t-1}$ and $\boldsymbol{g}^t$, say player i, must be such that $N_i(\boldsymbol{g}^t) \not\supseteq N_i(\boldsymbol{g}^{t-1})$. Let $\boldsymbol{g}^{t'}$ be the first network where player i has formed a link with player $i' \in N_i(\boldsymbol{g}^{t-1}) \setminus N_i(\boldsymbol{g}^t)$. By definition, all players $j \in N \setminus \{i\}$ are such that $N_j(\boldsymbol{g}^{t''-1}) \subseteq N_j(\boldsymbol{g}^{t''})$, with $t'' \leq t-1$. We show that player i does not play a best response in $\boldsymbol{g}^t$ if $N_i(\boldsymbol{g}^t) \not\supseteq N_i(\boldsymbol{g}^{t-1})$.
Suppose player i belongs to a component which is not a cycle in $\boldsymbol{g}^t$. Then $N_i(\boldsymbol{g}^t) \supseteq N_i(\boldsymbol{g}^{t-1})$. Indeed, since player i has formed a link with i' in $\boldsymbol{g}^{t'} \in \{\boldsymbol{g}^0, \ldots, \boldsymbol{g}^{t-1}\}$, we have $\pi_i(\boldsymbol{g}^{t'}) - \pi_i(\boldsymbol{g}^{t'} \ominus i \quad i') \geq 0$. Moreover, since $N_{i'}(\boldsymbol{g}^{t'}) \subseteq N_{i'}(\boldsymbol{g}^{t-1})$, by construction of the process we have $\pi_i(\boldsymbol{g}^t \oplus i \quad i') - \pi_i(\boldsymbol{g}^t) \geq \pi_i(\boldsymbol{g}^{t'}) - \pi_i(\boldsymbol{g}^{t'} \ominus i \quad i') \geq 0$. Therefore player i has no incentive to delete her link with i'. Finally, there is no player who has formed a link with player $k \in N_{i'}(\boldsymbol{g}^t)$ between $\boldsymbol{g}^{t'}$ and $\boldsymbol{g}^t$ by point 1. It follows that player i has no opportunity to replace her link to obtain a subset of $N_{i'}(\boldsymbol{g}^t)$.
Suppose player i belongs to a cycle in $\boldsymbol{g}^t$. If player i belongs to a cycle in $\boldsymbol{g}^{t-1}$, then she has no

incentive to remove her link (otherwise she has not played a best response during the process) and by (P1) she has no incentive to replace her link by a link with a player k who belongs to the same cycle. It follows that $N_i(\boldsymbol{g}^{t-1}) \subseteq N_i(\boldsymbol{g}^t)$.
If player i does not belong to a cycle in $\boldsymbol{g}^{t-1}$, then there are two cases. Either player i forms a cycle by using her best response in Step 4, or she forms a link with a cycle. In both cases, either player i does not form her first link and in that case a wheel is formed and the lemma is true, or player i forms her first link and we have $\{i\} = N_i(\boldsymbol{g}^{t-1}) \subset N_i(\boldsymbol{g}^t)$ since she cannot replace one of her links by point 1 and the construction of the process. □

If player i plays a best response in the process, then she must obtain strictly more payoffs and by Lemma 2 $N_i(\boldsymbol{g}^{t-1}) \subseteq N_i(\boldsymbol{g}^t)$. It follows that $\eta(\boldsymbol{g}^t) = \sum_{i \in N} |N_i(\boldsymbol{g}^t)|$ is stricly increasing in t. Since $\eta(\boldsymbol{g}^t)$ is finite this concludes our proof.

□

In the following example, we show that condition (P2) which is quite strong is difficult to improve upon. Indeed, even if all players obtain the same value V from others, the condition on costs heterogeneity remains strong.

Example 1. We show that even with homogeneous values, if a slightly weaker condition than (P2) is not satisfied, then Nash networks do not always exist. Indeed, if we have $\mid V - V \mid = 0 > (3\underline{c} - \bar{c})/(n-1)$, then there exist parameters such that a Nash network does not exist. Let $N = \{1, \ldots, 4\}$, $V_i = V$ for all $i \in N$, $\bar{c} = 3V + \varepsilon$ with $\varepsilon > 0$. We assume that $c_{1,2} = V - \varepsilon$, $c_{2,3} = c_{3,1} = 2V - \varepsilon$, $c_{2,4} = 2(V - \varepsilon)$, $c_{4,1} = 3V - \varepsilon$, and all other links have a cost of $\bar{c}$. Under these parameters, a Nash network does not exist for all $\varepsilon > 0$.

In the following example we show that our conditions allow for some parameter heterogeneity.

Example 2. Let $N = \{1, 2, 3\}$, $c_{1,2} = 10$, $c_{1,3} = 11$ $c_{2,1} = 12$ $c_{2,3} = 13$ $c_{3,1} = 14$ $c_{3,2} = 15$ and $V_1 = 11$, $V_2 = 12$, $V_3 = 13$. It is easy to check that (P1) and (P2) are satisfied and therefore a Nash network will exist.

References

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