E C O N O M I C S B U L L E T I N

On the geometry of the consumer's surplus line integral

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Abstract

Consumer's surplus can be seen as a correct measure of the change in welfare under special conditions on the preferences of the consumer. The note addresses the question whether the intuitive appeal of the consumer's surplus concept in the one-price change case extends into cases where several prices of inter-related goods change. An intuitively justified attribution of the change in welfare is conjectured. Sufficient conditions for this attribution to be exactly consistent with the geometry of the consumer's surplus line integral are discussed.

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1. Introduction

Since Dupuit (1844) consumer's surplus has been proposed as a concept to measure the change in welfare in cases where the price of one good changes. In these cases the measure proposed, the consumer's surplus change, is geometrically the area under the demand curve between the price in the initial state and the price in the new state. The geometry of consumer's surplus gives rise to the interpretation of the welfare change as the variation, from the initial state to the new state, of the difference between the willingness to pay and the amount actually paid for the good.

The extension to the case of a set of inter-related goods was first tackled by Hotelling (1938) who proposed a line integral in the quantity space as generalisation of the integral representing total benefit, of which consumer's surplus is a part. In cases where the prices of more than one good change the proposal made by Hotelling leads as a measure of consumer's surplus change to a line integral over the price space with the demand functions for the different goods as integrands (what will be referred to hereafter as the consumer's surplus line integral, CSLI).

This measure is, however, not well defined as the line integral is generally path dependent. In fact, the demand function for one good shifts if the price of at least another good changes and the shift depends on the sequence of price pairs (in the case of a change in the price of two goods) followed from the price vector in the initial state to the price vector in the new state.

The relevance of the CSLI as a measure of the welfare change lies in its relation with the change in (indirect) utility. Under the assumption of a constant marginal utility of income the CSLI is directly proportional to the utility change and is path independent (a proof of this proposition is in Takayama, 1994).

The hypothesis of constancy of the marginal utility of income is subject to the qualifications first highlighted by Samuelson (1942), which translate into special assumptions on consumer's preferences. Chipman and Moore (1976) discussed two interpretations of the constancy of the marginal utility of income, the homotheticity of the demand functions, and the case of vertical Engel curves. Under such circumstances the CSLI is a correct measure of welfare change as it is in direct proportion to the utility change. Practically, the two cases of Chipman and Moore occur, respectively, when a constant proportion of the income is spent on each good, and when the expenditure on each of the goods subject to the price change is a small part of the whole consumer's expenditure.

This note deals with the geometry and interpretation of the CSLI. Implicitly it assumes that the above recalled conditions for the consumer's surplus to be a correct measure of the welfare change are satisfied. The conditions are ones where the CSLI is path independent. The CSLI is commonly solved on a piecewise linear path where the prices of the different goods are changed sequentially, each price being changed only once. This choice of the path is justified by computation reasons, as the CSLI reduces immediately to a sum of ordinary integrals. The note shows how a different choice of the integration path can be exploited to prove the correctness of an intuitively justified interpretation of the welfare change in the many-price change case.

First, the note deals with the exact geometry of the line integral evaluated over a linear (not piecewise) path. Second, it considers an intuitively justified interpretation of the welfare change based on a distinction between the contribution to the welfare change from the existing consumption and that from the new consumption (for the goods whose consumption is increased as a consequence of the change in the price vector; the contributions relate to the preserved consumption and to the lost consumption otherwise). Third, it discusses the consistency of this interpretation with the exact geometry of the line integral. Sufficient conditions for this interpretation to be consistent with the exact geometry are discussed. Finally, it shows how the change in the cost of living proposed by Bennet (1920)[1] is derived from the CSLI and discusses the approximation implicit in this measure compared with the exact CSLI.

2. Consumer's surplus line integral

We consider *n* goods with demand functions $x_i(\mathbf{p})$, i=1,..*n*, $\mathbf{p}=[p_1,\ldots p_n]$ being the price vector of the *n* goods.

Given the price vector in the initial state $\mathbf{p}^0$ and the price vector in the new state $\mathbf{p}^1$ we consider the consumer's surplus line integral (CSLI):

$$CSLI = -\int_{\mathbf{p}^0}^{\mathbf{p}^1} \sum_i x_i(\mathbf{p}) \cdot \mathrm{d}\, p_i \tag{1}$$

Following Takayama (1994) this quantity can be shown to be in simple relation to the change in utility under restrictive assumptions on the marginal utility of income which also ensure path independence for the CSLI. The conditions ensuring path independence are assumed to hold here.

We consider the linear path *l* between the points $\mathbf{p}^0$ and $\mathbf{p}^1$, i.e. the segment [$\mathbf{p}^0$, $\mathbf{p}^1$]:

$$\mathbf{p} = \mathbf{p}^0 + (1-\alpha)\cdot \mathbf{p}^1 \quad 0 \le \alpha \le 1 \tag{2}$$

Over the path *l* the CSLI is equivalent to the sum of ordinary integrals:

$$-\int_{\mathbf{p}^0,l}^{\mathbf{p}^1} \sum_i x_i(\mathbf{p}) \cdot \mathrm{d}\, p_i = -\sum_i \int_{p_i^0}^{p_i^1} x_i'(p_i) \cdot \mathrm{d}\, p_i \tag{3}$$

where x_i' is the demand function for good *i* evaluated over the linear path *l* and reduced to a function of only p_i by substitution using the parametric equations of *l*:

$$\begin{aligned} p_k &= p_k(p_i) = p_k^0 - \frac{\Delta p_k}{\Delta p_i} \cdot p_i^0 + \frac{\Delta p_k}{\Delta p_i} \cdot p_i \qquad k \ne i \\ \Delta p_h &= p_h^1 - p_h^0, \qquad h = k, i \end{aligned} \tag{4}$$

[1] This part of the note follows the line of reasoning in Williams (1976) who used a linear path for the evaluation of the CSLI to derive a measure of users' benefits widely used in transport planning, known in the transport jargon as rule-of-a-half; the rule-of-a-half is formally identical to the Bennet change in the cost of living.

A proof of eqn (3) is in the Appendix. We call pseudo-demand function the demand function $x_i'(p_i)$ for good i evaluated over the linear path l. Each ordinary integral in the right-hand side of eqn (3) is formally a consumer's surplus in the one-price change case with the pseudo-demand function as integrand.

On the basis of eqn (3) the usual geometry of the consumer's surplus in the one-price change case is retrieved: the CSLI is equivalent to a sum of areas under well defined demand functions depending only on the price of the corresponding good and the end-points are the price for the good in the initial and in the new state. The caveat is that, as we have the pseudo-demand functions as integrands, the prices of the other goods are adjusted as we move along each curve.

We now introduce an intuitively justified attribution of the welfare change for a change in the price vector.

Preliminarily we note that as a consequence of the change of the price vector from the initial to the new state the demand for each good changes. As several prices of inter-related goods change simultaneously, a decrease in the price of one good does not necessarily mean that the consumption for that good increases. We have two cases. If the demand for the good increases there is an existing consumption and a new consumption. If the demand for the good decreases there is a preserved consumption and a lost consumption.

The welfare change for the existing or preserved consumption is for each good simply the change in price multiplied by the existing or preserved demand. The welfare change has a positive sign for a price decrease.

We consider then the welfare change for the new consumption for the goods whose demand increases, and for the lost consumption for the goods whose demand decreases. The total welfare change for the new and lost consumption is simply the difference between the CSLI and the sum over the goods of the welfare changes for the existing or preserved consumption.

This attribution is expressed mathematically as follows. The welfare change ΔS_{Ai} for the existing or preserved consumption $\bar{x}_i$ is for each good i:

$$\begin{aligned} &\Delta S_{Ai} = -\bar{x}_i \cdot \left(p_i^1 - p_i^0\right) \\ &\bar{x}_i = \min\left\{x_i^0, x_i^1\right\} \\ &i = 1, \ldots n \end{aligned} \tag{5}$$

The welfare change ΔS_B for the new and lost consumption for all goods is:

$$\Delta S_B = CSLI - \sum_i \Delta S_{Ai} \tag{6}$$

which yields for the CSLI:

$$CSLI = \sum_i \Delta S_{Ai} + \Delta S_B \tag{7}$$

The geometry of the CSLI based on the equivalence of eqn (3) is consistent with this attribution of the welfare change. In fact the right-hand side of eqn (3) provides a sum of ordinary integrals which are geometrically areas under pseudo-demand curves. Each of these areas can be decomposed into a rectangle corresponding to the welfare change ΔS_{Ai} for the existing or preserved demand plus a curvilinear triangle. The sum over the goods of the areas of the curvilinear triangles provides the welfare change ΔS_B for the new and lost consumption.

However, for this decomposition to be correct it is necessary that each pseudo-demand curve $x_i'(p_i)$, between the price in the initial state and the price in the new state, lies above the rectangle which has as height the existing or preserved demand $\bar{x}_i$. Mathematically the necessary condition[2] is:

$$\begin{aligned} & x_i'(p_i) \geq \bar{x}_i \\ & p_i^0 \leq p_i \leq p_i^1 \ \text{ if } \ p_i^0 \leq p_i^1 \quad \text{or} \quad p_i^1 \leq p_i \leq p_i^0 \ \text{ if } \ p_i^1 \leq p_i^0 \\ & i = 1, \ldots n \end{aligned} \tag{8}$$

For this condition to hold it is sufficient (but not necessary) that each pseudo-demand curve $x_i'(p_i)$ is monotone between the price in the initial state and the price in the new state.

Sufficient but more restrictive conditions are that each demand function $x_i(\mathbf{p})$ is quasi-monotone. By definition (Martos, 1975) a scalar function of many variables $x_i(\mathbf{p})$ is quasi-monotone in a convex set $C \subset E^n$ if it is increasing or decreasing along any segment $[\mathbf{p}^0, \mathbf{p}^1] \subset C$. A theorem (Bazaraa et al., 1993) states that a function $x_i(\mathbf{p})$ is quasi-monotone in a convex set $C \subset E^n$ if and only if the level surface $\{\mathbf{p} \in C : x_i(\mathbf{p}) = k\}$ is convex for all $k \in E^1$.

It is worth noting that we don't attribute the welfare change for the new and lost consumption to the individual goods but consider the total ΔS_B. The reason is that such attribution would be ambiguous. In fact, while the welfare change for the existing or preserved consumption can be unambiguously attributed to each good based on the corresponding price change, the welfare change for the new and lost consumption is provided for each good by the area of a curvilinear triangle which changes if a different path is chosen.

It is straightforward to derive from the CSLI the change in the cost of living proposed by Bennet (1920):

$$\Delta I = \sum_i \frac{1}{2}(x_i^0 + x_i^1) \cdot \left(p_i^1 - p_i^0\right) \tag{9}$$

Each term of (9) is the area under a linearised demand function for one good between its price in the initial state and that in the new state. Once this is recognised the quantity in eqn (9) is immediately obtained from the right hand side of eqn (3) by linearization of the pseudo-demand functions. The geometry of the change in the cost of living implies the attribution of the welfare change expressed by eqns (5) and (6).

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[2] As the linear path *l* considered is not the only evaluation path for the CSLI, the conditions (8) are actually sufficient conditions for the attribution expressed by eqns (5) and (6) to be exact.

As it neglects the curvature of the demand functions the change in the cost of living is only an approximation of the exact CSLI. For the same reason to have the consistency of the attribution of the welfare change implicit in eqn (9) with the exact geometry of the CSLI evaluated over the linear path it is necessary to check whether conditions (8) are satisfied.

3. Conclusions

The note has shown how an intuitively justified attribution of the welfare change consequent to a change in the price vector can be retrieved from the exact geometry of the consumer's surplus line integral (CSLI). In that it has been implicitly assumed that the conditions for the consumer's surplus to be a correct measure of the welfare change are satisfied. The attribution considers the contribution to the welfare change from the existing and preserved consumption, simply equal for each good to the price variation multiplied by the quantity demanded, and that from the new and lost consumption.

An analysis of the geometry of the CSLI evaluated over a linear path between the price vector in the initial state and that in the new state has provided sufficient conditions on the demand functions for the attribution to be exact. These conditions are easily checked as they only require the estimation of the demand functions along the linear path, which is obtained using the parametric equations of the path. More restrictive conditions requiring the monotonicity of the demand functions along the path have been discussed. The analysis of the geometry of the CSLI has also provided a clarification of the assumptions needed to derive from the consumer's surplus the Bennet change in the cost of living.

The investigation here shows that the intuitive appeal of the consumer's surplus measure of the change in welfare in the one-price change case extends only partially into the many-price change case. The attribution conjectured in this note does not provide an interpretation based on the willingness to pay. Nonetheless it has an intuitive justification. Mild conditions on the demand functions are sufficient for the attribution to be exact.

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Appendix. Proof of eqn (3)

The CSLI (1) can be rewritten, by definition of line integral (Kaplan, 1984):

$$-\int_{\mathbf{p}^0}^{\mathbf{p}^1} \sum_i q_i(\mathbf{p}) \cdot \mathrm{d}\, p_i = -\sum_i \int_{\mathbf{p}^0}^{\mathbf{p}^1} q_i(\mathbf{p}) \cdot \mathrm{d}\, p_i \qquad \text{(A.1)}$$

For the theorem on the existence of line integrals (Kaplan, 1984), if each demand function $x_i(\mathbf{p})$ is continuous on the linear path l defined by eqn (2) we have that each term in the right-hand side of eqn (A.1) when evaluated on l is equivalent to the ordinary integral:

$$\int_{\mathbf{p}^0,l}^{\mathbf{p}^1} q_i(\mathbf{p}) \cdot \mathrm{d}\, p_i = \int_{p_i^0}^{p_i^1} q_i\left[p_1(p_i), \ldots p_i, \ldots p_n(p_i)\right] \cdot \mathrm{d}\, p_i \qquad \text{(A.2)}$$
$$i = 1, \ldots n$$

where:

$$p_k(p_i) = p_k^0 - \frac{\Delta p_k}{\Delta p_i} \cdot p_i^0 + \frac{\Delta p_k}{\Delta p_i} \cdot p_i \qquad k \neq i \qquad \text{(A.3)}$$
$$\Delta p_h = p_h^1 - p_h^0, \qquad h = k, i$$

provide the parametric equations of line l with p_i as parameter. Eqns (A.3) are immediately obtained from the symmetric equations of the line l in the n-dimensional space:

$$\frac{p_1 - p_1^0}{p_1^1 - p_1^0} = \ldots = \frac{p_i - p_i^0}{p_i^1 - p_i^0} = \ldots = \frac{p_n - p_n^0}{p_n^1 - p_n^0} \qquad \text{(A.4)}$$

Substituting eqns (A.2) in eqn (A.1) yields eqn (3).