E C O N O M I C S B U L L E T I N

Risk, Ambiguity, and the Klibanoff Axioms

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Abstract

Machina (2007) formulates a number of experiments, and shows that they can be used to test the Choquet expected utility model. We show that one of them can also be used to test the class of maxmin expected utility preferences in Klibanoff (2001). Those preferences are not consistent with Choquet expected utility preferences in Machina's experiment.

I would like to thank Larry Epstein and a referee for useful comments.
Citation: Lo, Kin Chung, (2008) "Risk, Ambiguity, and the Klibanoff Axioms." *Economics Bulletin,* Vol. 4, No. 27 pp. 1-5
Submitted: March 31, 2008. **Accepted:** August 29, 2008.
URL: http://economicsbulletin.vanderbilt.edu/2008/volume4/EB-08D80018A.pdf

1 Introduction

Machina (2007) formulates a number of experiments à la Ellsberg (1961). One of them is as follows.[1] There is a coin and an urn in front of you. You are told that the coin is unbiased, and the urn contains one hundred balls; each ball in the urn can be either black or white, but the relative proportions are not specified (that is, there may be from zero to one hundred black/white balls). The coin will be flipped once, and simultaneously one ball will be drawn from the urn. Consider, for instance, f_1 in Figure 1. It is a typical act, with a payoff of \$0 if h (head) comes up in the coin flip, and a b (black) ball is drawn from the urn; \$100 if t (tail) comes up in the coin flip, and a w (white) ball is drawn from the urn, etc. Similar interpretations are given to f_2, f_3 and f_4.

	b	w
h	\$0	\$200
t	\$100	\$100

f_1

	b	w
h	\$0	\$100
t	\$200	\$100

f_2

	b	w
h	\$100	\$200
t	\$100	\$0

f_3

	b	w
h	\$100	\$100
t	\$200	\$0

f_4

Figure 1: Machina's experiment

In the above setting, the state space is $S_r \times S_a = \{h, t\} \times \{b, w\}$, where S_r involves risk, and S_a involves ambiguity. In general, an act $f\colon S_r \times S_a \to \mathbb{R}$ specifies the monetary payoff $f(s_r, s_a)$ you receive at every state $(s_r, s_a) \in S_r \times S_a$. Let $\succsim$ be your preference relation over acts, with $\succ$ and $\sim$ the induced strict preference and indifference relations, respectively. Since f_1 and f_4 (similarly, f_2 and f_3) are equivalent in substance, there is no question that any reasonable $\succsim$ must conform to one of the following three patterns:

$$f_1 \sim f_2 \sim f_3 \sim f_4$$
$$f_1 \sim f_4 \succ f_2 \sim f_3$$
$$f_2 \sim f_3 \succ f_1 \sim f_4$$

As Machina explains, only $f_1 \sim f_2 \sim f_3 \sim f_4$ is consistent with Choquet expected utility preferences (Schmeidler, 1989). So this experiment can be used to test the Choquet expected utility model. Machina offers various arguments that f_1 and f_2 are different in substance, and therefore $f_1 \sim f_2 \sim f_3 \sim f_4$ *might not* hold. However, he does not really say that this preference pattern *should not* hold.

The maxmin expected utility model (Gilboa and Schmeidler, 1989) is the classic alternative to Choquet expected utility. Confining to environments that are perfectly illustrated by the above experiment, Klibanoff (2001) provides axiomatic characterizations for a specific class of maxmin expected utility preferences. Loosely speaking, when you face two

[1]We are talking about Example A3 in the Appendix of Machina's manuscript (dated July 22, 2007), presented in a way as he describes in fn. 17. The latest version of his paper (forthcoming in *American Economic Review*) contains only experiments to which our analysis does not apply.

dimensions of uncertainty, you may have expected utility preference over acts defined on one dimension, and ambiguity-averse preference over acts on the other dimension; moreover, you may feel that the two dimensions are independent. The class of maxmin expected utility preferences with Klibanoff's structure reflects these features. As Klibanoff argues, that subset of maxmin expected utility has important implications, such as preference for randomization.

While Machina does not argue against $f_1 \sim f_2 \sim f_3 \sim f_4$, Klibanoff in effect does. In this note, we show that Klibanoff's maxmin expected utility preferences must deliver $f_2 \sim f_3 \succ f_1 \sim f_4$. Thus, the experiment can also be used to test the validity of those preferences.[2]

2 Maxmin expected utility preferences

Since the unbiased coin and the ambiguous urn are two completely separate objects, you know that the probability law governing $S_r \times S_a$ is a product measure, assigning marginal probability 1/2 to each "row" (h or t); but the marginal probability of each "column" (b or w) is unknown. Under this circumstance, Klibanoff (p. 612, Theorem 1) would argue that $\succsim$ may be represented by the following utility function: For any act f, the utility of f

$$U(f) = \min_{p \in \Delta} \sum_{(s_r, s_a) \in S_r \times S_a} p(s_r, s_a) u(f(s_r, s_a)), \tag{1}$$

where $u \colon \mathbb{R} \to \mathbb{R}$ is a strictly increasing vNM index, and Δ is a closed and convex set of probability measures on $S_r \times S_a$, with the properties that

$$p(s_r, s_a) = \frac{p(S_r \times s_a)}{2} \quad \forall (s_r, s_a) \in S_r \times S_a \quad \forall p \in \Delta, \tag{2}$$

and

$$\max_{p \in \Delta} p(S_r \times s_a) = x > \frac{1}{2} \quad \forall s_a \in S_a. \tag{3}$$

Eq. (1) is a general maxmin expected utility function. The intuition is that Δ represents your beliefs, and you evaluate each act according to its minimum expected utility, where the minimum is taken over all the probability measures in Δ. The restrictions in Eqs. (2) and (3) reflect both ambiguity aversion and the information structure of the experiment. To elaborate, you think that the rows are stochastically independent of the columns (Δ contains only product measures), every row occurs with probability 1/2, and every column occurs with probability between $1 - x$ and x. You are ambiguity averse ($x > 1/2$), which is consistent with the typical preference pattern in Ellsberg's experiment.

It follows from Gilboa and Schmeidler (1993, p. 40, Propositions 2.1 and 2.2) and Klibanoff (p. 615, Theorem 2) that none of the preference relations satisfying Eqs. (1)–(3) is a Choquet expected utility preference. So the natural question arises: When restricted to

[2] Klibanoff also has an experiment (p. 609, Table 2). However, as he himself points out (p. 614, Remark 1; p. 617, Remark 5), that experiment cannot be used to distinguish his maxmin expected utility model from Choquet expected utility.

the experiment, are these maxmin expected utility preferences consistent with Choquet expected utility? Without loss of generality, let $u(\$0) = 0$, $u(\$100) = 1$, and $u(\$200) = y > 1$. According to (1)–(3),

$$\begin{aligned} U(f_1) &= \min_{p \in \Delta} \left[p(h,w)y + p(t,b) + p(t,w) \right] \\ &= \frac{(1-x)y}{2} + \frac{x}{2} + \frac{1-x}{2}, \end{aligned}$$

and

$$\begin{aligned} U(f_2) &= \min_{p \in \Delta} \left[p(h,w) + p(t,b)y + p(t,w) \right] \\ &= \begin{cases} \frac{1-x}{2} + \frac{xy}{2} + \frac{1-x}{2} & \text{if } y \le 2 \\ \frac{x}{2} + \frac{(1-x)y}{2} + \frac{x}{2} & \text{if } y > 2. \end{cases} \end{aligned}$$

Given $y > 1$, we have $U(f_2) > U(f_1)$ if and only if $x > 1/2$. One can also easily verify $U(f_1) = U(f_4)$ and $U(f_2) = U(f_3)$. So, all these maxmin expected utility preferences deliver the same preference pattern—which is ruled out by Choquet expected utility—in the experiment.

Proposition 1. *Suppose that* $\succsim$ *is represented by* (1)–(3). *Then* $f_2 \sim f_3 \succ f_1 \sim f_4$.

Note that if $x = 1$ and $y = 2$, then $U(f_1) = 1/2$ and $U(f_2) = 1$. This is consistent with Robert Nau's observation (mentioned in Machina, p. 12) that the expected value of f_2 must be \$100, but the expected value of f_1 could be as low as \$50, and as high as \$150. Proposition 1 is much more general than Nau's observation. If you are ambiguity averse, and find Klibanoff convincing, then no matter how much ambiguity aversion you have, and no matter what risk attitude you have, you strictly prefer f_2 over f_1.

3 Stochastically independent preferences

From now on, an act may be denoted using a 2×2 matrix of numbers; for instance, the act f_1 in Figure 1 is $\left[\begin{smallmatrix} 0 & 200 \\ 100 & 100 \end{smallmatrix}\right]$. Say that $\succsim$ is *stochastically independent* if for all $a, b, c \in \mathbb{R}$,

$$\begin{bmatrix} a & a \\ b & b \end{bmatrix} \sim \begin{bmatrix} c & c \\ c & c \end{bmatrix} \implies \begin{bmatrix} a & c \\ b & c \end{bmatrix} \sim \begin{bmatrix} c & a \\ c & b \end{bmatrix} \sim \begin{bmatrix} c & c \\ c & c \end{bmatrix} \tag{4}$$

and

$$\begin{bmatrix} a & b \\ a & b \end{bmatrix} \sim \begin{bmatrix} c & c \\ c & c \end{bmatrix} \implies \begin{bmatrix} a & b \\ c & c \end{bmatrix} \sim \begin{bmatrix} c & c \\ a & b \end{bmatrix} \sim \begin{bmatrix} c & c \\ c & c \end{bmatrix}. \tag{5}$$

This definition (strictly speaking, Eq. (4) only) is adopted by Klibanoff (pp. 611–612). It is obvious that any $\succsim$ represented by (1)–(3) is stochastically independent. We are led to explore the question: How would stochastically independent (but not necessarily maxmin expected utility) preferences behave in Machina's experiment?

Say that $\succsim$ is *monotonic* (*strictly monotonic*, respectively) if for any two acts f and g with $f \neq g$,

$$f(s_r, s_a) \geq g(s_r, s_a) \quad \forall (s_r, s_a) \in S_r \times S_a \implies f \succsim g \quad (f \succ g, \text{respectively}).$$

Naturally, we restrict attention to (strictly) monotonic preferences.[3] Also, suppose $f_2 \sim f_3$, $f_1 \sim f_4$, and there exist $p_r \in \mathbb{R}$ and $p_a \in \mathbb{R}$ such that

$$\begin{bmatrix} 100-p_r & 100-p_r \\ 100-p_r & 100-p_r \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 200 & 200 \end{bmatrix} \sim \begin{bmatrix} 200 & 200 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 200+p_a \\ 0 & 200+p_a \end{bmatrix} \sim \begin{bmatrix} 200+p_a & 0 \\ 200+p_a & 0 \end{bmatrix}. \tag{6}$$

The number p_r can be interpreted as the risk premium of $\left[\begin{smallmatrix} 0 & 0 \\ 200 & 200 \end{smallmatrix}\right]$, and the number p_a the ambiguity premium of $\left[\begin{smallmatrix} 0 & 200+p_a \\ 0 & 200+p_a \end{smallmatrix}\right]$. Recall that $f_2 = \left[\begin{smallmatrix} 0 & 100 \\ 200 & 100 \end{smallmatrix}\right]$. If $\succsim$ is (weakly) risk averse in the sense that $p_r \geq 0$, and if $\succsim$ is monotonic, then

$$f_2 \succsim \begin{bmatrix} 0 & 100-p_r \\ 200 & 100-p_r \end{bmatrix}. \tag{7}$$

Eqs. (4), (5) and (6) imply

$$\begin{bmatrix} 0 & 100-p_r \\ 200 & 100-p_r \end{bmatrix} \sim \begin{bmatrix} 100-p_r & 100-p_r \\ 100-p_r & 100-p_r \end{bmatrix} \sim \begin{bmatrix} 0 & 200+p_a \\ 100-p_r & 100-p_r \end{bmatrix}. \tag{8}$$

Combining Eqs. (7) and (8), we have, for any risk-averse, monotonic, and stochastically independent $\succsim$,

$$f_2 \succsim \begin{bmatrix} 0 & 200+p_a \\ 100-p_r & 100-p_r \end{bmatrix}. \tag{9}$$

It can be established along the same line that

$$f_3 \succsim \begin{bmatrix} 100-p_r & 100-p_r \\ 200+p_a & 0 \end{bmatrix}. \tag{10}$$

Recall that $f_1 = \left[\begin{smallmatrix} 0 & 200 \\ 100 & 100 \end{smallmatrix}\right]$ and $f_4 = \left[\begin{smallmatrix} 100 & 100 \\ 200 & 0 \end{smallmatrix}\right]$. Eqs. (9) and (10) provide a partial answer to our question.

Proposition 2. *Suppose that $\succsim$ is risk neutral* (*in the sense that $p_r = 0$*), *ambiguity averse* (*in the sense that $p_a > 0$*), *strictly monotonic, and stochastically independent. Then $f_2 \sim f_3 \succ f_1 \sim f_4$.*

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[3]Any $\succsim$ represented by (1)–(3) is monotonic (strictly monotonic if $x < 1$).

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