E C O N O M I C S   B U L L E T I N

An allais paradox for generalized expected utility theories?

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Abstract

This article reports the results of an experiment which aims at providing a test of ordinal independence, a necessary property of Generalized Expected Utility theories such as Rank-Dependent Expected Utility theory (RDEU). Our experiment is based on a modified version of the Allais paradox proposed by Machina, which allows testing ordinal independence restricted to simple lotteries, i.e. the tail-separability property. The results tend to support RDEU models since tail-separability is not violated by 71% of subjects while 73% violate the independence condition of classic Allais paradox. This confirms the relative theoritical soundness of RDEU models over Expected Utility model for the particular context of risk.

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1. Introduction

In 1953, Allais severly questionned classical Expected Utility (vNM: von Neumann and Morgenstern, 1947) suggesting that choice behavior could not be consistent with a necessary condition of the theory, the independence axiom. With its empirical confirmations (MacCrimmon and Larson 1979, Kahneman and Tversky 1979), the so-called Allais paradox belongs to a broad series of systematic violations of Expected Utility (see Starmer, 2000 for a survey) which compromised Expected Utility as a descriptively valid theory of choice under risk.

Following Allais'intuition, alternative models were developped in order to allow non linear treatments of probabilities. Among these models, Quiggin's (1982) Rank Dependent Expected Utility (RDEU) model was the first successful, since it avoided violations of stochastic dominance through the introduction of decision weights that incorporate the relative ranking of the outcomes instead of a direct transformation of probability (for other rank-dependent generalizations, see Yaari 1987, Segal 1987). All these expected utility generalizations involving rank-dependence are based on a weaker form of the vNM's original independence axiom, namely comonotonic independence. The comonotonicity requirement was further weakened in the ordinal independence property (Green and Jullien 1988, Quiggin 1993, Wakker and Zank 2002), also called tail-separability when restricted to simple lotteries.

Machina (2007) proposes thought experiments in the spirit of Allais and Ellsberg that points out the possible vulnerability of RDEU models through the tail-separability property. Moreover, recent results show that the tail-separability property defined over events challenges the descriptive validity of the counterpart of RDEU under uncertainty, i.e. Choquet Expected Utility (L'Haridon and Placido, 2008). In the present paper, we provide an empirical test based on Machina's examples in order to empirically confront RDEU in the particular context of risk.

The rest of the paper is organized as follows: In Section 2 we present the general framework of the experiment. We report the findings of the experiment in Section 3. Section 4 concludes.

2. Framework of the experiment

2.1. Rank-Dependent Expected Utility

We consider an individual who has to make a choice between three-outcome risky lotteries. We restrict the formulation of RDEU to such lotteries. Let $L = (x_1, p_1; x_2, p_2; x_3, p_3)$ denotes the risky lottery which yields the monetary payoff x_i with probability p_i, $i = 1, ..., 3$. Monetary payoffs are rank-ordered: $x_1 \geq x_2 \geq x_3$. The RDEU of lottery L is given by $V_{RDEU}(L)$:

$$V_{RDEU}(L) = w(p_1)u(x_1) + [w(p_1 + p_2) - w(p_1)]u(x_2) + [1 - w(p_1 + p_2)]u(x_3) \quad (1)$$

$u(.)$ is a strictly increasing utility function over payoffs and $w(.)$ a strictly increasing probability weighting function from $[0, 1]$ to $[0, 1]$ with $w(0) = 0$ and $w(1) = 1$.

A decision maker who has RDEU preferences satisfies all EU axioms except the independence axiom, which is replaced by a similar condition on rank-dependence, the ordinal independence axiom (see Quiggin, 1993 and Marley and Luce, 2005 for a review). Ordinal independence requires that, if two lotteries agree on a given segment of the cumulative distribution function, the value they take on that segment should not affect their ranking (Quiggin, 1993). In our experiment, we focus on tail-separability, a special case of ordinal independence where the common segment of the cumulative distribution function is one of both tails. Tail-separability implies that if two lotteries share a common tail then the substitution of another common tail maintains the preference order between the lotteries.

When the common tail concerns the higher payoffs, we more precisely refer to upper tail-separability. As an illustration, we consider the four following lotteries:

$$L_1 = (x_1, p_1; x_2; p_2; x_3, p_3) \text{ vs. } L_2 = (x_1, p_1; x_2; q_2; x_3, q_3)$$

$$L_3 = (x_1, p_1'; x_2; p_2; x_3, p_3) \text{ vs. } L_4 = (x_1, p_1'; x_2; q_2; x_3, q_3)$$

Lotteries L_1 and L_2 share the common upper tail (x_1, p_1) and lotteries L_3 and L_4 share the common upper tail (x_1, p_1'). Under RDEU, the choice between L_1 and L_2, (L_3 and L_4) depends on the sign of the difference $V_{RDEU}(L_1) - V_{RDEU}(L_2)$ ($V_{RDEU}(L_3) - V_{RDEU}(L_4)$) and the intensity of preference is given by the absolute amount of this difference (up to a positive affine transformation). Applying (1) one gets:

$$V_{RDEU}(L_1) - V_{RDEU}(L_2) = [w(p_1 + p_2) - w(p_1 + q_2)][u(x_2) - u(x_3)] \tag{2}$$

$$V_{RDEU}(L_3) - V_{RDEU}(L_4) = [w(p_1' + p_2) - w(p_1' + q_2)][u(x_2) - u(x_3)] \tag{3}$$

Using $p_1 + p_2 + p_3 = p_1 + q_2 + q_3 = 1$ and $p_1' + p_2 + p_3 = p_1' + q_2 + q_3 = 1$, (2) and (3) become

$$V_{RDEU}(L_1) - V_{RDEU}(L_2) = [w(1 - p_3) - w(1 - q_3)][u(x_2) - u(x_3)] \tag{4}$$

$$V_{RDEU}(L_3) - V_{RDEU}(L_4) = [w(1 - p_3) - w(1 - q_3)][u(x_2) - u(x_3)] \tag{5}$$

As a consequence, $V_{RDEU}(L_1) - V_{RDEU}(L_2)$=$V_{RDEU}(L_3) - V_{RDEU}(L_4)$. Any shift of the common upper tail (x_1, p_1) (x_1, p_1') between lotteries L_1 and L_2 (L_3 and L_4) does not change the preference order neither the intensity of preference.

Similarly, when the common tail concerns the lower payoffs, we more precisely defined lower tail-separability. As an illustration, we consider the four following lotteries:

$$L_1 = (x_1, p_1; x_2; p_2; x_3, p_3) \text{ vs. } L_2 = (x_1, q_1; x_2; q_2; x_3, p_3)$$

$$L_3 = (x_1, p_1; x_2; p_2; x_3, p_3') \text{ vs. } L_4 = (x_1, q_1; x_2; q_2; x_3, p_3')$$

Lotteries L_1 and L_2 share the common lower tail (x_3, p_3) and lotteries L_3 and L_4 share the common lower tail (x_3, p_3') Applying (1) to binary choices between L_1 and L_2 and L_3 and L_4 gives:

$$V_{RDEU}(L_1) - V_{RDEU}(L_2) = [w(p_1) - w(q_1)][u(x_1) - u(x_2)] = V_{RDEU}(L_3) - V_{RDEU}(L_4) \tag{6}$$

Lower tail-separability applies since any shift of the common lower tail (x_3, p_3) does not change the preference order and the intensity of preference.

2.2. Allais-like choices

Machina (2007) proposes choices built on Allais classic paradox that may question RDEU ordinal independence axiom in the same way that the Allais paradox questionned the EU independence axiom. The choices proposed are as follows:

$$L_1 = (75, 0.05; 45, 0.90; 15, 0.05) \text{ vs. } L_2 = (75, 0.05; 60, 0.45; 15, 0.50)$$

$$L_3 = (60, 0.05; 45, 0.90; 0, 0.05) \text{ vs. } L_4 = (60, 0.50; 15, 0.45; 0, 0.05)$$

These modified Allais lotteries are used in our experiment to test ordinal independence through tail-separability. Each pair of lotteries shares both a common upper tail (a 5% chance to get the best consequence) and a common lower tail (a 5% chance to get the worst consequence). The remaining 90% are split among monetary payoffs: concentrated on the intermediary payoff 45 in L_1 and L_3, split between the intermediary 60 and the worst payoff 15 in L_2 and between the intermediary 15 and the best payoff 60 in L_4. Common tails become apparent if one writes:

$$L_1 = (75, 0.05; 45, 0.90; 15, 0.05) \text{ vs. } L_2 = (75, 0.05; 60, 0.45; 15, 0.45; 15, 0.05)$$

$$L_3 = (60, 0.05; 45, 0.90; 0, 0.05) \text{ vs. } L_4 = (60, 0.05; 60, 0.45; 15, 0.45; 0, 0.05)$$

Then, choice between L_1 and L_2, and L_3 and L_4 are given by the sign of the following differences:

$$\begin{aligned} V_{RDEU}(L_1) - V_{RDEU}(L_2) &= -[w(0.50) - w(0.05)]u(60) + [w(0.95) - w(0.05)]u(45) \\ &\quad -[w(0.95) - w(0.50)]u(15) \end{aligned}$$

$$\begin{aligned} V_{RDEU}(L_3) - V_{RDEU}(L_4) &= -[w(0.50) - w(0.05)]u(60) + [w(0.95) - w(0.05)]u(45) \\ &\quad -[w(0.95) - w(0.50)]u(15) \end{aligned}$$

As a consequence: $V_{RDEU}(L_1) - V_{RDEU}(L_2) = V_{RDEU}(L_3) - V_{RDEU}(L_4)$. A decision maker who exhibits preference for L_3 over L_4 should also exhibit a preference for L_1 over L_2. An individual may prefer L_1 over L_2 because the latter offer a slightly higher chance to get the worst outcome while the chance to obtain the best outcome stays unchanged. However, people may also prefer L_4 over L_3 because the latter offers a higher chance to get the best outcome while the chance to get the worst outcome is unchanged. An individual who exhibit theses preferences violates tail-separability and consequently RDEU.

2.3. Experiment

Ninety-four students (39 females and 55 males) took part in the experiment. Students were enrolled in economics courses at IUFM and Ecole Centrale Paris. Most of the students were acquainted with probability theory but they had never heard of decision theory. The experiment consisted of a paper-pencil questionnaire where subjects were confronted with the two pairs of binary choices presented above. Subjects were told there were neither right nor wrong answers, and they had to choose the situation they prefered, without any time constraint. We run three sessions and within each session, subjects were informed that one of them would be randomly selected to have her choice played out for real. In order to control for order effects, we permuted situations on the questionnaire. As an introduction, subjects faced a version of the Allais paradox including monetary payoffs similar to the one used in lotteries L_1 to L_4. Classic Allais choices were the following:

$$L_{A1}=(\text{€}15,1) \text{ vs. } L_{A2}=(\text{€}75,0.10;\text{€}15;0.89;\text{€}0,0.01)$$

$$L_{A3}=(\text{€}75,0.1;\text{€}0,0.90) \text{ vs. } L_{A4}=(\text{€}15,0.11;\text{€}0,0.89)$$

3. Results

Table 1 summarizes subjects' choices for the two choice situations designed to test tail-separability. For each pair of lotteries, the following table gives the number of subjects that chose each of the four possible patterns of choice. Overall 68% of subjects revealed choices consistent with RDEU and 32% of subjects exhibit a preference reversal under RDEU. Moreover results from the first part of the experiment on Allais paradox also plead in favor of RDEU: 73% of the subjects satisfied the Allais paradox and thus exhibit preference reversals under EU.

Choice	L_1L_3	L_2L_4	L_1L_4	L_2L_3
n	45	19	19	11

Table 1: Subjects' choices

At the individual level, over the 25 subjects whose answers where EU compatible in the classic Allais paradox part of the experiment, 15 also gave answers RDEU compatible (60%). Among the 69 subjects whose answers were incompatible with EU in Allais, 49 gave answers compatible with RDEU (71%) in modified Allais choices. This left 20 subjects who gave answers incompatible with both EU and RDEU. The most common pattern of choice was therefore incompatible with EU preferences and the independence axiom but compatible with ordinal independence necessary to RDEU preferences
One should also notice that we found no significant effects from order, age, gender, and session (p-values of correlations between each variable and pattern of choices were all greater than 0.12).

4. Conclusion

Our results show that the majority of subjects exhibit a behavior that violate the independence axiom (and hence EU) but that is compatible with the ordinal independence axiom (i.e RDEU). This suggests that RDEU models are less vulnerable to independence-type violations in comparaison with EU. Thus, we found no "Allais paradox" for generalized expected utility theory. Our results are consistent with existing litterature. Weber and Kirsner (1997) show that the number of violation of comonotonic independence are significantly less that thoses for non comonotonic independence. Wu (1994) tests violations of tail-separability under risk and finds similar results in a different setting. Wu reports a 38% of within-subjects violation of upper tail-separability whereas we found only 32%. Wakker, Erev and Weber (1994) test comonotonic independence. They show that this axiom is well suited for Allais-type choices but loses in performance in more general choice contexts (in particularly when the certainty effect does not apply). Our experiment reinforces such evidence.

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