E C O N O M I C S   B U L L E T I N

Sunspots do matter: a simple disproof of Mas-Colell

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Abstract

Mas-Colell conjectured that, for economies admitting a unique deterministic equilibrium, sunspot equilibria that 'matter' would not exist. I prove an abundance of such equilibria using simple linear algebra.

I thank, with the usual caveats, Professors Eric Fisher and Bill Zame for helpful comments. This research note was supported by an Anderson Summer Program Doctoral Fellowship
Citation: Huesch, Marco, (2008) "Sunspots do matter: a simple disproof of Mas-Colell." *Economics Bulletin,* Vol. 5, No. 19 pp. 1-20
Submitted: December 18, 2006. **Accepted:** July 3, 2008.
URL: http://economicsbulletin.vanderbilt.edu/2008/volume5/EB-06E00010A.pdf

SUNSPOTS *DO* MATTER: A SIMPLE DISPROOF OF MAS-COLELL

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1 Introduction

In an economy with multiple spot market equilibria, it is well-known that perfectly foresighted rational agents may be able to use an exogenous random event (a sunspot) to correlate their expectations and achieve a unique equilibrium.

Now suppose the opposite. There is a deterministic economy admitting a unique spot market equilibrium, and there are sunspots with nothing to do with the fundamentals. There are sunspot-contingent assets with known payoffs.

Might there now be a different spot market equilibrium in each of the sunspot states? Would each agent's utilities be different in each of those states? A positive answer to both questions implies the existence of sunspot equilibria [SSE] that 'matter'.

Cass & Shell (1983, p 193) first asked the question, "Do Sunspots Matter?" Mas-Colell conjectured that there could be no SSE that matter if the underlying deterministic economy had a unique equilibrium (1992, cited in Barnett & Fisher, 2002 [hereafter BF], p1). Hens (2000) provided a 2 asset counter-example which BF showed was flawed.

Hens & Pilgrim (2004) provided a correct counter-example to Mas-Colell's conjecture in the ≥ 3 asset case (see Appendix 4). They also proved an un-intuitive general equivalence between SSE existence and the existence of the Leontief trade 'transfer paradox'.

In this note, I provide identical results using a simple linear algebraic method. I parameterize all economies with a certain number of assets and a unique underlying deterministic equilibrium and partition the resulting parameter space. I form a linear non-homogeneous system in the sunspot state probabilities. For SSE to matter, it suffices that the solutions to this system - the state probabilities - are all strictly bounded away from zero. In this system the matrix summarizes the agents' asset demand first-order conditions and the adding-up constraint of the state probabilities. The requirements that this matrix must be invertible, and that the solutions must be strictly positive, implies a certain set of inequalities must hold in every partition. I use these inequalities as 'test expressions'.

In the case of $n = 2$ asset economies, I proceed by *reductio ad absurdum* and find that in *no* partition of the parameter space is it possible to escape a contradiction if the assumptions of invertibility of this matrix and strictly positive state probabilities are maintained. This result hinges on my finding that the orderings of indirect utilities by state must be inverse across the two agents.

In the more general case of $n \geq 3$ asset economies, I assume invertibility of this matrix, by appealing to the density of invertible matrices and my freedom to arbitrarily perturb the

Date: June 24, 2008.

I thank, with the usual caveats, Sushil Bikhchandani for helpful comments. I am also grateful to Roger Farmer and Eric Fisher for useful suggestions on prior drafts. Finally I am indebted to Joe Ostroy and my advisor Bill Zame whose enjoyable GE courses sparked my interest in this material.

asset structure and agent risk aversion degree. I subsequently show that it is *not* possible to create a contradiction of the assumption of invertibility of this matrix. This result hinges on my finding that it is no longer possible, given orderings of indirect state utilities implied by the particular partition parameters of even $n-1$ agents, to determine the ordering of indirect state utilities of the final agent.

I conclude that as n increases beyond 3, any randomly chosen economy becomes arbitrarily close to some SSE that matters, implying that such economies essentially *generically* exhibit SSE that matter. My approach makes it clear that Mas-Colell's original conjecture was wrong: if the state space is large, then SSE that matter are in the appropriate sense very likely.

Section 2 lays out the model, in which, purely to ensure comparability of model and results, I follow the notation of Hens (2000) and BF almost exactly. I collect most of the tedious algebra in the appendices.

2 Model

Let there be a two-period economy without production. There are two real goods, 2 sunspot-contingent assets and 3 sunspot states. Storage is assumed to be feasible and costless. Consumption is restricted to the second period. The only uncertainty lies in this second period in which any one of three sunspot-dependent states of nature, $s = 1, 2, 3$ may occur. Let it be known by both of the two agents, labeled $i = 1, 2$, that the probability of each sunspot-state occurring is $\pi(s) > 0$.

Spot markets for the two commodities, labeled $x = 1, 2$, exist only in period 2. I set good 2 as the numeraire of the economy. In period one only, there are spot markets for two sunspot contingent labeled $j = 1, 2$. Holdings by agent i of asset j are denoted y^i_j where both assets are in zero net supply. The two sunspot-contingent assets have different pay-offs denominated in units of the real numeraire across the three states as follows, with rows denoting states, and columns, assets:

$$V = \begin{pmatrix} 0 & 1 \\ b & 0 \\ c & 0 \end{pmatrix}$$

By my definition of markets, prices for the two assets exist in period 1, denoted by $(q, 1)$, while prices for the two goods exist in period 2 in each of the three states, denoted by $p(s) = (p_1(s), 1)$, where I normalize prices relative to the numeraire good.

Each agent has endowments ω^i, where $\omega^1 = (1, 0)'$ and $\omega^2 = (0, 1)'$. Each agent has Cobb-Douglas preferences $u^i(x^i) = (x^i_1)^{\alpha^i}(x^i_2)^{1-\alpha^i}$ over the two goods, where α^1 differs from α^2. His risk aversion is characterized by a strictly concave and strictly monotonic increasing transformation h^i composed with his preferences. As a von Neumann-Morgenstern expected utility maximizer, each agent thus attempts to maximize

$$U^i(x^i) = \sum_{s=1}^{3} \pi(s) h^i \circ u^i(x^i(s))$$

Supposing there were no sunspots, and given these assumptions on endowments and the nature of the objective function, it is clear that the underlying deterministic economy has a unique equilibrium. With sunspots however, each agent must evaluate risky consumption plans $x \in \mathrm{R}_+^{2\times 3}$, and the agents' decision problems are now to

$$\max_{x^i,\, y^i} (U^i(x^i)) \text{ subject to}$$

the self-financing portfolio budget restriction

$$qy^i_{{}_1} + 1y^i_{{}_2} \ \leq 0$$

and, where $V(s)$ is row s of V, the goods budget restriction

$$p_1(s)x^i_{{}_1}(s) + 1x^i_{{}_2}(s) \leq p_1(s)\omega^i_{{}_1}(s) + 1\omega^i_{{}_2}(s) + V(s)y^i$$

At equilibrium the self-financing portfolio restriction will hold with equality, since the second asset has a non-negative payoff vector and thus $qy^i_{{}_1}+1y^i_{{}_2} \ = 0$, while the asset market clearing condition implies that for asset one, $y_1^1 = -y_1^2$ and identically for asset two.

Feasible aggregate consumption plans will require that in each state, consumptions of both goods are less than or equal to the respective initial aggregate endowments (storage is assumed to be feasible and costless). Clearly this will hold with equality at equilibrium given preferences and goods.

An equilibrium in this sunspot economy is then, for each agent, a consumption plan across the two goods and three states in period 2, as well as a first period asset demand across the two assets. Equilibrium prices are an optimal asset price for the non-numeraire asset 1, as well as correctly foreseen spot prices for the non-numeraire good 2 in each of the three states' spot markets in period 2.

In the analysis that follows I find it more convenient to consider indirect utility in any state as follows, although clearly the agents are maximizing expected utility in their stochastic programs. Each agent's indirect utility v^i at prices $p(s)$ and given income w^i is denoted by the function $v^i(p(s), w^i(s))$. Each agent's ex-post income is given by $w^i(s) = p(s)\cdot\omega^i + V(s)y^i$ where y^i is the vector consisting of each agents equilibrium asset holdings. I follow BF in noting that each ex-post spot market has a unique equilibrium given the assumed preferences, and simply index the functions h, g (and, below, their derivatives h' and g' by s): $h(s) \equiv h^1 \circ v^1(p(s), w^1(s))$ and $g(s) \equiv h^2 \circ v^2(p(s), w^2(s))$. It is understood that these functions are evaluated at the appropriate arguments, not the state index s. I also write α^1 as α and α^2 as β.

I will need to identify certain equilibrium relationships between the various parameters, so taking first-order conditions of the objective function yields the following equalities for each of the two agents

$$\pi(1)qh'(1)/p_1(1)^\alpha - \pi(2)bh'(2)/p_1(2)^\alpha - \pi(3)ch'(3)/p_1(3)^\alpha = 0$$
$$\pi(1)qg'(1)/p_1(1)^\beta - \pi(2)bg'(2)/p_1(2)^\beta - \pi(3)cg'(3)/p_1(3)^\beta = 0$$

while the necessity for the three state probabilities to add up to 1 yields

$$\pi(1) + \pi(2) + \pi(3) = 1$$

Combining these, BF obtain the following system of non-homogeneous linear equations generated by the two asset demand equations and the vector of state probabilities summing to one in the form $A \cdot \pi = (0\,0\,1)'$:

$$\begin{pmatrix} qh'(1)/p_1(1)^\alpha & -bh'(2)/p_1(2)^\alpha & -ch'(3)/p_1(3)^\alpha \\ qg'(1)/p_1(1)^\beta & -bg'(2)/p_1(2)^\beta & -cg'(3)/p_1(3)^\beta \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi(1) \\ \pi(2) \\ \pi(3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Simplifying this matrix, impose that the a_{11} and a_{21} elements of the A matrix are 1, in view of the equivalence of expected utility up to an affine transformation yields

$$\begin{pmatrix} 1 & -bh'(2)/p_1(2)^{\alpha} & -ch'(3)/p_1(3)^{\alpha} \\ 1 & -bg'(2)/p_1(2)^{\beta} & -cg'(3)/p_1(3)^{\beta} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi(1) \\ \pi(2) \\ \pi(3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

BF recognized that the existence of SSE that matter was equivalent to $\pi >> 0$ which could only happen if all the 3 elements in the last column of A^{-1} were of the same sign.[1] BF were able to show that Hens' (2000) particular numerical example was flawed by showing that it could not produce this required equality in sign. However they explicitly left open the possibility that some other constructive example could be evinced.

3 Mas-Colell is affirmed for 2 Asset Economies

I adapt and extend the linear algebraic technique used by BF, and apply a similar proof methodology over the universe of two-asset economies parameterized such that there is a unique fundamental equilibrium. I obtain six tests, derived from the non-singularity of A and the requirement that $\pi >> 0$, with three for positive determinant:

(1a) $$bc[(h'(2)/g'(2))(g'(3)/h'(3))] > bc[(p(3)/p(2))^{\beta-\alpha}]$$

(1b) $$c[(h'(3)/g'(3))(g'(1)/h'(1))] < c[(p(3)/p(1))^{\alpha-\beta}]$$

(1c) $$b[(h'(2)/g'(2))(g'(1)/h'(1))] > b[(p(2)/p(1))^{\alpha-\beta}]$$

with the inequalities in the other direction for the case of negative determinant. If an economy fails at least one test in both groups, it cannot admit SSE.

I partition the universe of economies in the two-asset context into sixteen generic cases. In the first eight cases, both sunspot-contingent payoffs b and c are positive, and I distinguish only between whether α is less than or greater than β, whether for the sunspot-contingent payoffs of the first asset b is less than or greater than c and whether y_1 is positive or negative.[2] It cannot of course be zero, since no trade would imply the unique deterministic equilibrium.

In the remaining eight cases, I have assumed without loss of generality that b is negative and c is positive (since the state numbering is arbitrary, and since they cannot both be negative or no rational agent would purchase that asset), and I distinguish again between whether α is less than or greater than β, but now whether $b+q$ is less than or greater than 0 and again whether y_1 is positive or negative.

[1] I am grateful to Professor Zame for pointing out that in this simple two agent case, it is clear that A must be invertible. To see the economic reasons for this, assume there is collinearity in rows and/or columns. Assume firstly collinearity in the columns, implying equality between them, and thus equal marginal utilities for income by agent across states. But then SSE don't 'matter ' and we're done. On the other hand, assume collinearity in rows, implying equality between them, and thus equal marginal rates of substitution for income across agents. But then trade won't occur, and neither will SSE, since differing consumptions by state require asset trade.

[2] I follow both papers by expressing *all* equilibrium holdings of assets in terms of the single scalar $y_1 \equiv y_1^1$, that is the amount of asset 1 demanded by agent 1. I can do this because the budget identity $qy_1^i + y_2^i = 0$ holds $\forall i$, and for the vectors of asset demands, y^i, the clearing identity $y^1 = -y^2$ holds by assumption.

I shall compute a series of relationships, which hold generically for any parameter values in a particular partition of the parameter space, between ratios of the derivatives of the h and g functions. I perform this in a series of steps, shown here for Case 2 of 16 in which $\alpha < \beta$, $b > c$, and $y_1 < 0$. Appendix 2 contains detailed calculations for this and the other fifteen cases.

I firstly derive the following orderings between good 1 state spot market prices $p_1(1) \leq p_1(3) \leq p_1(2)$ and the indirect utility to agent 1 in each of those states $v^1(2) \leq v^1(3) \leq v^1(1)$.

Unfortunately, algebraic manipulation of the particular parameters in this partition will not produce an unequivocal ordering of indirect utility to agent 2. I overcome this by observing that under SSE we have a multiplicity of competitive spot market equilibria, in which Pareto-efficiency implies Lemma 1. The proof is relegated to Appendix 1.

Lemma 1. *In the two agent, two good, three state economy, the orderings of indirect utilities by state must be inverse across the two agents.*

Thus the indirect utility to agent 2 are ordered $v^2(2) \geq v^2(3) \geq v^2(1)$ across the states. Recall that the functions h and g are strictly increasing concave risk aversion transformations of the underlying utility functions, so the relationships between indirect utilities can be 'inverted' to arrive at an ordering of first derivatives of h by state: $h'(2) \geq h'(3) \geq h'(1)$ and for g similarly $g'(1) \geq g'(3) \geq g'(2)$.

Now I test the orderings computed for Case 2 with the inequalities in one of the tests, e.g. Test 1b: $c[(h'(3)/g'(3))(g'(1)/h'(1))] < c[(p(3)/p(1))^{\alpha-\beta}]$. I have $(h'(3)/g'(3))(g'(1)/h'(1)) = (h'(3)/h'(1))(g'(1)/g'(3)) \geq 1$, but since the relevant price ratio is greater than one, $(p_1(3)/p_1(1))^{\alpha-\beta} \leq 1$. The contradiction is clear, and it turns out in Case 2 we can exclude SSE that matter. Similar results can be shown for the other fifteen cases (see Appendix 2 for the simple but tedious algebra).

In the two-asset case I've started with the assumption that SSE that matter do exist, and then used economic reasoning to justify the non-singularity of the A matrix characterizing the first-order conditions of the agents' asset demands. The implications of non-singularity were then shown to lead to a series of contradictions, establishing the following:

Proposition 1. *Given a 2 asset economy with a unique deterministic economy, no partition of the parameter space contains SSE that matter.*

In the next section I show that this proof will not go through in the case of $n \geq 3$ assets: the critical point is that Lemma 1 fails to hold.

4 Mas-Colell is rejected for ≥ 3 Asset Economies

Hens, Mayer & Pilgrim (2004, p27-29) were the first to provide a simple constructive (computer generated) example of a fully specified 3 asset economy, with a unique deterministic equilibrium, nonetheless exhibiting sunspot equilibria that matter. This definitively falsified Mas-Colell's conjecture and is included in Appendix 4 for information.

My contribution is instead to demonstrate that sunspot equilibria that matter will generically exist in the ≥ 3 asset economy with a unique deterministic equilibrium. To prove that SSE that matter *do* exist in the ≥ 3 asset case, I use a different but related device.

Firstly, I appeal to the continuity of the function mapping the A matrix entries to the determinant, i.e. I appeal to the density of invertible matrices in the space of all matrices of the appropriate size. Since I am allowed to choose the asset structure and the degree of risk aversion of the agents, I can always arbitrarily perturb *any* given economy to reach a close neighboring one which does exhibit an invertible A. Thus the following proof applies to *almost all* economies parameterized in a particular partition of the parameter space.

In what follows I sketch a proof of the generic existence of such SSE for the case $n = 3$. Proceeding exactly as before, I will create a non-homogeneous system of linear equations where the first three represent solutions to the first-order conditions of optimal asset demand, and the fourth one is an adding up constraint on the strictly positive state probabilities to 1. I will obtain a total of four inequalities in the case where the determinant is positive, and another four inequalities running in the opposite direction, in the case where the determinant is negative. Analogously to the $n = 2$ case, simple algebra produces a series of test expressions to be validated in the appropriate parameter-space partition.

I now consider an arbitrary parameter-space partition and an arbitrary test expression as an example, where $f'(2)/f'(1) > 1$ and $g'(1)/g'(2) > 1$ by virtue of the particular parameter space partition, and the product $(f'(2)/f'(1))(g'(1)/g'(2))(h'(2)/h'(1))$ ought to be greater than 1.

Then, for a particular economy in which the final ratio $h'(2)/h'(1)$ is greater than 1, no contradiction has been found to the requirements for strictly positive state probabilities. However if this ratio is less than 1, than such a contradiction has been found, and in this particular parameter space partition an SSE that matters exists.

Given the risk aversion transformation's strict concavity and monotonicity, this is equivalent to showing that $v_3(1)/v_3(2)$ is indeterminate when compared to 1. That is, the third agent's ordering of indirect utilities by state is not uniquely determined. I claim that this is indeed so: the only ordering that is explicitly ruled out by Pareto-efficiency in the state spot market is an ordering identical to that of the first two agents. Contrary to the two agent economy, in the three agent economy, knowledge of the orderings of indirect utility for two of the three agents is insufficient to infer the ordering for the third economy as per the following Lemma 2. While intuitive, I prove this lemma by constructive counter-example in Appendix 3.

Lemma 2. *In the three agent, three asset, three good, four state economy, the orderings of indirect utilities by state for two of the agents do not necessarily imply a particular ordering for the final agent.*

By Lemma 2, it follows immediately that the ratio $v_3(1)/v_3(2)$ could be greater or less than 1. Thus the arbitrary test expression considered in the example above is not able to be contradicted in that arbitrary parameter-space partition.

Recall that these test expressions are the only opportunities for an economy to fail to exhibit SSE, since the other requirements for goods price positivity, asset price positivity, and ex-post income non-negativity (and thus ex-post consumption non-negativity) are able to be satisfied for a continuum of economies in a given partition of the parameter space.[3]

Accordingly, an SSE that matters can be found in that particular arbitrary partition of the parameter space. Generalizing this example to any partition of the parameter space of a 3 asset economy, and then generalizing the result to n-assets leads to the following proposition:

Proposition 2. *Given a $n \geq 3$ asset economy with a unique deterministic equilibrium, an SSE that matters can be found in every partition of the parameter space.*

Hence as n becomes larger than 3, any randomly chosen economy becomes arbitrarily close to some SSE that matters, implying that such economies essentially *generically* exhibit SSE that matter. These conclusions stands in contrast to Mas-Colell's original intuition.

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[3]To see this, recall that all my economies are Cobb-Douglas whose unique deterministic equilibrium has positive goods prices. Assume a given economy is located in the parameter space such that no trade in assets occurs, and the deterministic equilibrium supervenes. Then negligibly perturb this economy's asset structure and its agents' degree of risk aversion such that asset prices rise above zero, asset trade occurs, and ex-post income is thus non-negative.

Appendix 1: Proof of Lemma 1

Let the following ordering of state utility be given for agent 1, without loss of generality, $v^1(1) > v^1(2)$ for two arbitrary states, $s = 1\,2$. By the assumed monotonicity of the utility functions, it must be that for at least one of the goods, call it good 1 without loss of generality, that the equilibrium allocation of that good 1 to agent 1 obeys the inequality $x_1^1(1) \geq x_1^1(2)$. Consumption existing only in this second period and the fixed total endowment implies that for agent 2, $x_1^2(1) \leq x_1^2(2)$ must hold.

Two cases now pertain: either this inequality is in the same direction as the ranking of indirect utilities for agent 2 namely $v^2(1) < v^2(2)$ and I'm done, or in fact $v^2(1) > v^2(2)$.

Suppose, to obtain a contradiction, that $v^2(1) > v^2(2)$ were true. This implies $x_2^2(1) > x_2^2(2)$. In turn this implies $x_2^1(1) < x_2^1(2)$.

I now appeal to the equalization of marginal rates of substitution [MRS] within each spot market by state. Let the marginal utilities for good 1 be in the numerator. Let the MRS for agent 1 be equated to that of agent 2, without loss of generality, in state 1.

I claim that the computed state-by-state allocations would now prevent the MRS being equalized in state 2. Note that in state 2 the allocation of good 1 to agent 1 fell compared to state 1. This increases the numerator of his MRS. Similarly the denominator must decrease, since the allocation of good 2 to agent 1 increased. Thus the MRS increases. However it is easy to verify that the opposite effects occur for agent 2, thus decreasing his MRS. This contradiction establishes the claim and the lemma.

Appendix 2: Derivation of (Two-Asset Economy) Utility Orderings

Setting $y_1 = y_1^1$ [$\equiv \theta$ in Hens (2000)] as the unique portfolio scalar corresponding to Agent 1s holding of Asset 1 the following prices obtain (Hens 2000): $p_1(1) = [(\beta - \alpha)q_1 y_1 + \beta]/(1-\alpha)$, $p_1(2) = [(\alpha - \beta)by_1 + \beta]/(1-\alpha)$, and $p_1(3) = [(\alpha - \beta)cy_1 + \beta]/(1-\alpha)$. Clearly $p_1(2) > p_1(3)$ when $p_1(2) - p_1(3) > 0$ or when $(b-c)(\alpha - \beta)y_1 > 0$. Similarly $p_1(2) > p_1(1)$ when $p_1(2) - p_1(1) > 0$ or when $(b+q)(\alpha - \beta)y_1 > 0$. Finally $p_1(3) > p_1(1)$ when $p_1(3) - p_1(1) > 0$ or when $(c+q)(\alpha - \beta)y_1 > 0$. The left hand side [LHS] of all of these inequalities is positive if and only if an odd number of the LHS terms are positive.

Table A1: Price Inequalities by Parameter Region

Case	$p_1(2) > p_1(3)$	$p_1(2) > p_1(1)$	$p_1(3) > p_1(1)$
#1	*no*	*no*	*no*
#2	*yes*	*yes*	*yes*
#3	*yes*	*no*	*no*
#4	*no*	*yes*	*yes*
#5	*no*	*yes*	*yes*
#6	*no*	*no*	*no*
#7	*no*	*yes*	*yes*
#8	*yes*	*no*	*no*
#9	*yes*	*no*	*no*
#10	*no*	*yes*	*yes*
#11	*yes*	*yes*	*no*
#12	*no*	*no*	*yes*
#13	*no*	*yes*	*yes*
#14	*yes*	*no*	*no*
#15	*no*	*no*	*yes*
#16	*yes*	*yes*	*no*

Combining these results yields the final table of price relationships across cases:

Table A2: Summary Table of Prices Relationships By Case

Case	$p_1(3) \sim p_1(1) \sim p_1(2)$
#1	$p_1(1) > p_1(3) > p_1(2)$
#2	$p_1(2) > p_1(3) > p_1(1)$
#3	$p_1(1) > p_1(2) > p_1(3)$
#4	$p_1(3) > p_1(2) > p_1(1)$
#5	$p_1(2) > p_1(3) > p_1(1)$
#6	$p_1(1) > p_1(3) > p_1(2)$
#7	$p_1(3) > p_1(2) > p_1(1)$
#8	$p_1(1) > p_1(2) > p_1(3)$
#9	$p_1(1) > p_1(2) > p_1(3)$
#10	$p_1(3) > p_1(2) > p_1(1)$
#11	$p_1(2) > p_1(1) > p_1(3)$
#12	$p_1(3) > p_1(1) > p_1(2)$
#13	$p_1(3) > p_1(2) > p_1(1)$
#14	$p_1(1) > p_1(2) > p_1(3)$
#15	$p_1(3) > p_1(1) > p_1(2)$
#16	$p_1(2) > p_1(1) > p_1(3)$

I will now derive the orderings of ex-post income in Table A3. Recall that the ex-post income to agent i in state s was $w^i(s) = p(s) \cdot \omega^i + V(s) \cdot y^i$, where $p(s) = [p_1(s),], 1]\ \forall s$. Recall that $y^i = [y_1^i,\ y_2^i]$ can be re-written for Agent 1 as $y^1 = [y_1^1,\ y_2^1] = [y_1^1,\ -qy_1^1]$, since the budget identity $qy_1^i + y_2^i = 0$ holds $\forall i$, and as $y^2 = [y_1^2,\ y_2^2] = [-y_1^1,\ qy_1^1]$ since the additional budget identity $y^1 = -y_2$ holds as well. I write y_1 for y_1^1.

Table A3: Ex-Post Incomes by Agent and by State

	$p(s)$	ω^i	$V(s)$	y^i	$Income$
$w^1(1)$	$[\mathrm{p_1(1),\ 1}]$	$[1,\ 0]^{\mathrm{T}}$	$[0,\ 1]$	$[\mathrm{y_1,\ -qy_1}]^{\mathrm{T}}$	$\mathrm{p_1(1) - qy_1}$
$w^1(2)$	$[\mathrm{p_1(2),\ 1}]$	$[1,\ 0]^{\mathrm{T}}$	$[\mathrm{b},\ 0]$	$[\mathrm{y_1,\ -qy_1}]^{\mathrm{T}}$	$\mathrm{p_1(2) + by_1}$
$w^1(3)$	$[\mathrm{p_1(3),\ 1}]$	$[1,\ 0]^{\mathrm{T}}$	$[\mathrm{c},\ 0]$	$[[\mathrm{y_1,\ -qy_1}]^{\mathrm{T}}$	$\mathrm{p_1(3) + cy_1}$
$w^2(1)$	$[\mathrm{p_1(1),\ 1}]$	$[0,\ 1]^{\mathrm{T}}$	$[0,\ 1]$	$[\mathrm{-y_1,\ qy_1}]^{\mathrm{T}}$	$\mathrm{1 + qy_1}$
$w^2(2)$	$[\mathrm{p_1(2),\ 1}]$	$[0,\ 1]^{\mathrm{T}}$	$[\mathrm{b},\ 0]$	$[\mathrm{-y_1,\ qy_1}]^{\mathrm{T}}$	$\mathrm{1 - by_1}$
$w^2(3)$	$[\mathrm{p_1(3),\ 1}]$	$[0,\ 1]^{\mathrm{T}}$	$[\mathrm{c},\ 0]$	$[\mathrm{-y_1,\ qy_1}]^{\mathrm{T}}$	$\mathrm{1 - cy_1}$

Turning to the derivation of indirect utility ratio relationships, I follow BF, who write the indirect utilities $v^i(p(s),\ w^i(s))$ are $v^1(p(s),\ w^1(s)) = \alpha^\alpha(1-\alpha)^{1-\alpha} w^1(s)/p_1(s)^\alpha$ and a similar expression for $v^2(p(s), w^2(s)) = \beta^\beta(1-\beta)^{1-\beta} w^2(s)/p_1(s)^\beta$. For my purposes, purely the *ratios* of state-dependent indirect utilities for the same Agent are of concern, so I will dispense with calculating the Cobb Douglas parameter expressions $\alpha^\alpha(1-\alpha)^{1-\alpha}$ and $\beta^\beta(1-\beta)^{1-\beta}$ since they clearly cancel through.

Table A4: Ratio of State-Dependent Indirect Utilities

	Computed Expression
$\mathrm{v^1(1)/v^1(3)}$	$= \mathrm{(p_1(1) - qy_1)/(p_1(3) + cy_1)(p_1(3)/p_1(1))^\alpha}$
$\mathrm{v^1(1)/v^1(2)}$	$= \mathrm{(p_1(1) - qy_1)/(p_1(2) + by_1)(p_1(2)/p_1(1))^\alpha}$
$\mathrm{v^1(2)/v^1(3)}$	$= \mathrm{(p_1(2) + by_1)/(p_1(3) + cy_1)(p_1(3)/p_1(2))^\alpha}$
$\mathrm{v^2(1)/v^2(3)}$	$= \mathrm{(1 + qy_1)/(1 - cy_1)(p_1(3)/p_1(1))^\beta}$
$\mathrm{v^2(1)/v^2(2)}$	$= \mathrm{(1 + qy_1)/(1 - by_1)(p_1(2)/p_1(1))^\beta}$
$\mathrm{v^2(2)/v^2(3)}$	$= \mathrm{(1 - by_1)/(1 - cy_1)(p_1(3)/p_1(2))^\beta}$

My ultimate aim, in whichever case this is feasible, is to arrive at an expression such as: $v^1(1)/v^1(2) = (p_1(1) - qy_1)/(p_1(3) + cy_1)(p_1(3)/p_1(1))^\alpha < (p_1(1) - qy_1)/(p_1(3) + cy_1) < 1$, in other words, an unequivocal bound with respect to the scalar 1.

I work on this in three steps, firstly looking at the final prices ratio terms, then looking to see whether terms of the type $(p_1(1) - qy_1)/(p_1(3) + cy_1)$ are bigger or less than 1, and then in the third step seeing if there are unequivocal chains of inequalities. I summarize the three relevant ratios $p_1(3)/p_1(1), p_1(2)/p_1(1)$ and $p_1(3)/p_1(2)$ in Table A5.

Table A5: Good 1 State Price Ratios

Case	$p_1(3)/p_1(1)$	$p_1(2)/p_1(1)$	$p_1(3)/p_1(2)$
#1	< 1	< 1	> 1
#2	> 1	> 1	< 1
#3	< 1	< 1	< 1
#4	> 1	> 1	> 1
#5	> 1	> 1	< 1
#6	< 1	< 1	> 1
#7	> 1	> 1	> 1
#8	< 1	< 1	< 1
#9	< 1	< 1	< 1
#10	> 1	> 1	> 1
#11	< 1	> 1	< 1
#12	> 1	< 1	> 1
#13	> 1	> 1	> 1
#14	< 1	< 1	< 1
#15	> 1	< 1	> 1
#16	< 1	> 1	< 1

With these relationships in hand I can state, for example, that in Case #1: $v^1(1)/v^1(3) = (p1(1)-qy_1)/(p_1(3)+cy_1)(p_1(3)/p_1(1))^\alpha < (p_1(1)-qy_1)/(p_1(3)+cy_1)$, since $(p_1(3)/p_1(1))^\alpha < 1$.

Continuing with this method, I fill in the case-by-case cells in Table A6, writing an inequality sign to indicate that the indirect utility ratio on the LHS is less than or greater the term on the RHS of the expression in Table A4.

Table A6: Indirect Utilities Ratios Inequalities

Case	$\frac{v^1(1)}{v^1(3)} \sim \frac{p_1(1)-qy_1}{p_1(3)+cy_1}$	$\frac{v^1(1)}{v^1(2)} \sim \frac{p_1(1)-qy_1}{p_1(2)+by_1}$	$\frac{v^1(2)}{v^1(3)} \sim \frac{p_1(2)+by_1}{p_1(3)+cy_1}$	$\frac{v^2(1)}{v^2(3)} \sim \frac{1+qy_1}{1-cy_1}$	$\frac{v^2(1)}{v^2(2)} \sim \frac{1+qy_1}{1-by_1}$	$\frac{v^2(2)}{v^2(3)} \sim \frac{1-by_1}{1-cy_1}$
#1	$<$	$<$	$>$	$<$	$<$	$>$
#2	$>$	$>$	$<$	$>$	$>$	$<$
#3	$<$	$<$	$<$	$<$	$<$	$<$
#4	$>$	$>$	$>$	$>$	$>$	$>$
#5	$>$	$>$	$<$	$>$	$>$	$<$
#6	$<$	$<$	$>$	$<$	$<$	$>$
#7	$>$	$>$	$>$	$>$	$>$	$>$
#8	$<$	$<$	$<$	$<$	$<$	$<$
#9	$<$	$<$	$<$	$<$	$<$	$<$
#10	$>$	$>$	$>$	$>$	$>$	$>$
#11	$<$	$>$	$<$	$<$	$>$	$<$
#12	$>$	$<$	$>$	$>$	$<$	$>$
#13	$>$	$>$	$>$	$>$	$>$	$>$
#14	$<$	$<$	$<$	$<$	$<$	$<$
#15	$>$	$<$	$>$	$>$	$<$	$>$
#16	$<$	$>$	$<$	$<$	$>$	$<$

My next step in Table A7 is to investigate for which of the sixteen generic cases the RHS of the expressions in Table A6 are bigger or smaller than 1.

Table A7: Inequalities in Table A6: RHS Relative Magnitude

- $v^1(1)/v^1(3)$. The ratio $(p1(1) - qy1)/(p1(3) + cy1) > 1$ whenever $(p_1(1) - qy_1) - (p_1(3) + cy_1) > 0$. This holds whenever $(\beta - \alpha - 1)(q + c)y_1 > 0$. In #1 through #8, the first term is negative, the second positive, so we need $y_1 < 0$ which holds only in cases #2, 4, 6 and 8. In #9 through #16 the first is negative, the second positive, and $y_1 < 0$ holds in #10, 12, 14 and 16.

- $v^1(1)/v^1(2)$. The ratio $(p_1(1) - qy_1)/(p_1(2) + by_1) > 1$ whenever $(p_1(1) - qy_1) - (p_1(2) + by_1) > 0$. This holds whenever $(\beta - \alpha - 1)(q + b)y_1 > 0$. In #1 through #8, the first term is negative, the second positive, so we need $y_1 < 0$ which holds only in #1, 3, 5 and 7. In #9 through #16 on the other hand, we need $(b + q)$ and y_1 to be opposite in sign. This holds in #10, 11, 14 and 15.

- $v^1(2)/v^1(3)$. The ratio $(p_1(2) + by_1)/(p_1(3) + cy_1) > 1$ whenever $(p_1(2) + by_1) - (p_1(3) + cy_1) > 0$. This holds whenever $(1-\beta)(b-c)y_1 > 0$. In #1 through #8, this holds whenever $b - c$ and y_1 are of the same sign, which holds in #1, 4, 5 and 8. In #9 through #16 we need y_1 to be negative since $b - c$ is negative. This holds in #10, 12, 14 and 16.

- $v^2(1)/v^2(3)$. The ratio $(1 + qy_1]/[1 - cy_1) > 1$ whenever $(1 + qy_1) - (1 - cy_1) > 0$. This holds whenever $(q + c)y_1 > 0$. In #1 through #8, the first term is positive, so $y_1 > 0$ suffices and this holds in cases #1, 3, 5 and 7. In #9 through #16 we need $y_1 > 0$, which holds in #9, 11, 13 and 15.

- $v^2(1)/v^2(2)$. The ratio $(1 + qy_1]/[1 - by_1) > 1$ whenever $(1 + qy_1) - (1 - by_1) > 0$. This holds whenever $(q + b)y_1 > 0$. In #1 through #8, the first term is positive, so it suffices for $y_1 > 0$ as in #1, 3, 5 and 7. In #9 through #16 we require both to have the same sign, which holds in #9, 12, 13 and 16.

- $v^2(2)/v^2(3)$. The ratio $(1 - by_1]/[1 - cy_1) > 1$ whenever $(1 - by_1) - (1 - cy_1) > 0$. This holds whenever $(c-b)y_1 > 0$. In #1 through #8, this holds whenever $(b-c)$ and y_1 are of opposite sign which holds in #2, 3, 6 and 7. In #9 through #16 the first term is positive, so we require $y_1 > 0$, which holds in #9, 11, 13 and 15.

My final step is to see when the ratios of state-dependent indirect utilities are unequivocally $>$ or <1. In the Table A8 below, 'na' means not available, and represents instances where, for example in Case #1, fourth column, $v^2(1)/v^2(3) < [1+qy_1]/[1-cy_1] > 1$, which is of little use in bounding $v^2(1)/v^2(3)$ in any direction away from 1.

Table A8: Cases for which Unequivocal Chains of Inequalities Involving Utilities Exist

Case	$\frac{v^1(1)}{v^1(3)}$	$\frac{v^1(1)}{v^1(2)}$	$\frac{v^1(2)}{v^1(3)}$	$\frac{v^2(1)}{v^2(3)}$	$\frac{v^2(1)}{v^2(2)}$	$\frac{v^2(2)}{v^2(3)}$
#1	<1	<1	>1	*na*	*na*	*na*
#2	>1	>1	<1	*na*	*na*	*na*
#3	<1	>1	<1	*na*	*na*	*na*
#4	>1	>1	>1	*na*	*na*	*na*
#5	*na*	*na*	*na*	>1	>1	<1
#6	*na*	*na*	*na*	<1	<1	>1
#7	*na*	*na*	*na*	>1	>1	>1
#8	*na*	*na*	*na*	<1	<1	<1
#9	<1	<1	<1	*na*	*na*	*na*
#10	>1	>1	>1	*na*	*na*	*na*
#11	<1	>1	<1	*na*	*na*	*na*
#12	>1	<1	>1	*na*	*na*	*na*
#13	*na*	*na*	*na*	>1	>1	>1
#14	*na*	*na*	*na*	<1	<1	<1
#15	*na*	*na*	*na*	>1	<1	>1
#16	*na*	*na*	*na*	<1	>1	<1

Combining these chains of inequalities naturally leads to the Table A9 below.

Table A9: Computed Goods Price and Indirect Utility Relationships by Case, Agent

Cases	*Price of Good* 1	*Agent* $1'$*s Indirect Utility*	*Agent* $2'$*s Indirect Utility*
#1	$p_1(1) > p_1(3) > p_1(2)$	$v^1(2) > v^1(3) > v^1(1)$	na
#2	$p_1(2) > p_1(3) > p_1(1)$	$v^1(1) > v^1(3) > v^1(2)$	na
#3	$p_1(1) > p_1(2) > p_1(3)$	$v^1(3) > v^1(2) > v^1(1)$	na
#4	$p_1(3) > p_1(2) > p_1(1)$	$v^1(1) > v^1(2) > v^1(3)$	na
#5	$p_1(2) > p_1(3) > p_1(1)$	na	$v^2(1) > v^2(3) > v^2(2)$
#6	$p_1(1) > p_1(3) > p_1(2)$	na	$v^2(2) > v^2(3) > v^2(1)$
#7	$p_1(3) > p_1(2) > p_1(1)$	na	$v^2(1) > v^2(2) > v^2(3)$
#8	$p_1(1) > p_1(2) > p_1(3)$	na	$v^2(3) > v^2(2) > v^2(1)$
#9	$p_1(1) > p_1(2) > p_1(3)$	$v^1(3) > v^1(2) > v^1(1)$	na
#10	$p_1(3) > p_1(2) > p_1(1)$	$v^1(1) > v^1(2) > v^1(3)$	na
#11	$p_1(2) > p_1(1) > p_1(3)$	$v^1(3) > v^1(1) > v^1(2)$	na
#12	$p_1(3) > p_1(1) > p_1(2)$	$v^1(2) > v^1(1) > v^1(3)$	na
#13	$p_1(3) > p_1(2) > p_1(1)$	na	$v^2(1) > v^2(2) > v^2(3)$
#14	$p_1(1) > p_1(2) > p_1(3)$	na	$v^2(3) > v^2(2) > v^2(1)$
#15	$p_1(3) > p_1(1) > p_1(2)$	na	$v^2(2) > v^2(1) > v^2(3)$
#16	$p_1(2) > p_1(1) > p_1(3)$	na	$v^2(3) > v^2(1) > v^2(2)$

Given the result of Lemma 1, I am justified in replacing the orderings in those cells in which no conclusive chain of inequalities was found by the inverse of the other agent's ordering. This leads to Table A10.

Table A10: Implied Indirect Utility Relationships by Case, Agent

Cases	*Agent 1's Indirect Utility*	*Agent 2's Indirect Utility*
#1	$v^1(2) > v^1(3) > v^1(1)$	$v^2(1) > v^2(3) > v^2(2)$
#2	$v^1(1) > v^1(3) > v^1(2)$	$v^2(2) > v^2(3) > v^2(1)$
#3	$v^1(3) > v^1(2) > v^1(1)$	$v^2(1) > v^2(2) > v^2(3)$
#4	$v^1(1) > v^1(2) > v^1(3)$	$v^2(3) > v^2(2) > v^2(1)$
#5	$v^1(2) > v^1(3) > v^1(1)$	$v^2(1) > v^2(3) > v^2(2)$
#6	$v^1(1) > v^1(3) > v^1(2)$	$v^2(2) > v^2(3) > v^2(1)$
#7	$v^1(3) > v^1(2) > v^1(1)$	$v^2(1) > v^2(2) > v^2(3)$
#8	$v^1(1) > v^1(2) > v^1(3)$	$v^2(3) > v^2(2) > v^2(1)$
#9	$v^1(3) > v^1(2) > v^1(1)$	$v^1(1) > v^1(2) > v^1(3)$
#10	$v^1(1) > v^1(2) > v^1(3)$	$v^1(3) > v^1(2) > v^1(1)$
#11	$v^1(3) > v^1(1) > v^1(2)$	$v^1(2) > v^1(1) > v^1(3)$
#12	$v^1(2) > v^1(1) > v^1(3)$	$v^1(3) > v^1(2) > v^1(1)$
#13	$v^1(3) > v^1(2) > v^1(1)$	$v^2(1) > v^2(2) > v^2(3)$
#14	$v^1(1) > v^1(2) > v^1(3)$	$v^2(3) > v^2(2) > v^2(1)$
#15	$v^1(3) > v^1(1) > v^1(2)$	$v^2(2) > v^2(1) > v^2(3)$
#16	$v^1(2) > v^1(1) > v^1(3)$	$v^2(3) > v^2(1) > v^2(2)$

Finally, given that the risk aversion transformation is strictly concave, the orderings of the first derivatives is in the opposite direction to those of the indirect utilities, as shown in Table A11.

Table A11: Risk Aversion Transformation Gradients Relationships by Case, Agent

Cases	*Agent 1's Risk Aversion Gradients*	*Agent 2's Risk Aversion Gradients*
#1	$h'(1) \geq h'(3) \geq h'(2)$	$g'(2) \geq g'(3) \geq g'(1)$
#2	$h'(2) \geq h'(3) \geq h'(1)$	$g'(1) \geq g'(3) \geq g'(2)$
#3	$h'(1) \geq h'(2) \geq h'(3)$	$g'(3) \geq g'(2) \geq g'(1)$
#4	$h'(3) \geq h'(2) \geq h'(1)$	$g'(1) \geq g'(2) \geq g'(3)$
#5	$h'(1) \geq h'(3) \geq h'(2)$	$g'(2) \geq g'(3) \geq g'(1)$
#6	$h'(2) \geq h'(3) \geq h'(1)$	$g'(1) \geq g'(3) \geq g'(2)$
#7	$h'(1) \geq h'(2) \geq h'(3)$	$g'(3) \geq g'(2) \geq g'(1)$
#8	$h'(3) \geq h'(2) \geq h'(1)$	$g'(1) \geq g'(2) \geq g'(3)$
#9	$h'(1) \geq h'(2) \geq h'(3)$	$g'(3) \geq g'(2) \geq g'(1)$
#10	$h'(3) \geq h'(2) \geq h'(1)$	$g'(1) \geq g'(2) \geq g'(3)$
#11	$h'(2) \geq h'(1) \geq h'(3)$	$g'(3) \geq g'(1) \geq g'(2)$
#12	$h'(3) \geq h'(1) \geq h'(2)$	$g'(2) \geq g'(1) \geq g'(3)$
#13	$h'(1) \geq h'(2) \geq h'(3)$	$g'(3) \geq g'(2) \geq g'(1)$
#14	$h'(3) \geq h'(2) \geq h'(1)$	$g'(1) \geq g'(2) \geq g'(3)$
#15	$h'(2) \geq h'(1) \geq h'(3)$	$g'(3) \geq g'(1) \geq g'(2)$
#16	$h'(3) \geq h'(1) \geq h'(2)$	$g'(2) \geq g'(1) \geq g'(3)$

Finally, in Table A12, it is clearly seen that in every one of the eight cases, there is a contradiction found whether the determinant is positive or negative. The highlighted results identify the contradictions in each case.

Table A12: Contradictions Generated by Tests of the Eight Generic Cases in the Two-Asset Economy

Case	(12*a*)	(12*b*)	(12*c*)	(13*a*)	(13*b*)	(13*c*)
1	×		×		×	
2		×		×		×
3			×	×	×	
4	×	×				×
5	×		×		×	
6		×		×		×
7			×	×	×	
8	×	×				×
9	×				×	×
10		×	×	×		
11	×		×		×	
12		×		×		×
13	×				×	×
14		×	×	×		
15	×		×		×	
16		×		×		×

For example, let us consider the Test (9a) with Case #2 in the second row of Table 5. Test (9a) requires $bc[(h'(2)/h'(3))(g'(3)/g'(2))] > bc[((p_1(3)/p_1(2))^{\beta\text{-}\alpha}]$, where $bc > 0$. The orderings computed earlier imply that $((p_1(3)/p_1(2))^{\beta\text{-}\alpha} \leq 1^{\beta\text{-}\alpha} \leq 1$ since $\alpha < \beta$ within the Case 2 partition. But $(h'(2)/g'(2))(g'(3)/h'(3)) = (h'(2)/h'(3))(g'(3)/g'(2)) \geq 1$ given the derivative orderings computed earlier. Clearly this is consistent with the particular test, and we are not able to elicit a contradiction with this test (9a).

However with test (9b) we require $c[(h'(3)/g'(3))\ (g'(1)/h'(1))] < c[(p(3)/p(1))^{\alpha-\beta}]$ the situation is different. Now $(h'(3)/g'(3))(g'(1)/h'(1)) = (h'(3)/h'(1))(g'(1)/g'(3)) \geq 1$, yet $(p_1(3)/p_1(1))^{\alpha-\beta} = (p_1(3)/p_1(1))^{-(\beta-\alpha)} \leq 1$ since the price ratio is greater than one. The contradiction is clear.

Similar results can be computed for the other fifteen cases and in Table A12, it is clearly seen that in every one of the sixteen cases, there is at least one contradiction found whether the determinant is positive or negative.

The proposition is thus proved. Note well that I did not use the failure of Leontief's transfer paradox to exist, in this simple demonstration of the impossibility of the existence of SSE in the two-asset context. The only tests I have used are ones which guarantee state probability positivity given the parameters chosen for the case. This is clearly a more primitive and certainly different condition than the existence of the transfer paradox.

Appendix 3: Proof of Lemma 2

The proof of Lemma 1 relied on the unique determination of residual allocations to the remaining second agent, once the first agent's allocations were determined. Intuitively this link between indirect utilities and allocations is lost when more than two agents are considered.

To see that the equilibrium orderings of indirect utility across the sunspot states for two agents do not uniquely determine the ordering of state utility for the third agent, it suffices to consider the following counterexample. In this example, three agents, three goods each in total supply of 1, and four equi-probable states exist. For convenience I focus on just the first three states.

I show that knowledge of the orderings of indirect state utilities for two of the agents does not conclusively determine the orderings of the third agent.

Let the agents' have Cobb Douglas utility (and the identity risk aversion transformation) given by the following matrix, where rows correspond to agents and columns to the parameters $\{\alpha,\, \beta,\, \gamma\}$:

$$\begin{pmatrix} 0.34 & 0.31 & 0.35 \\ 0.31 & 0.35 & 0.34 \\ 0.35 & 0.34 & 0.31 \end{pmatrix}$$

The following state utilities represent equilibrium values, where rows correspond to agents and columns to states 1 and 2:

$$\begin{pmatrix} 0.217 & 0.524 & 0.212 \\ 0.289 & 0.285 & 0.488 \\ 0.495 & 0.192 & 0.301 \end{pmatrix}$$

Here I note that $v^1(2) > v^1(1) > v^1(3)$, $v^2(3) > v^2(1) > v^2(2)$, and $v^3(1) > v^3(3) > v^3(2)$. These utilities are supported by the following allocations, where rows correspond to agents and columns to goods in state 1, 2, and 3 below:

$$\begin{pmatrix} 0.220 & 0.200 & 0.231 \\ 0.265 & 0.300 & 0.300 \\ 0.515 & 0.500 & 0.469 \end{pmatrix}$$

$$\begin{pmatrix} 0.530 & 0.500 & 0.540 \\ 0.270 & 0.300 & 0.285 \\ 0.200 & 0.200 & 0.175 \end{pmatrix}$$

$$\begin{pmatrix} 0.210 & 0.200 & 0.225 \\ 0.465 & 0.500 & 0.497 \\ 0.325 & 0.300 & 0.278 \end{pmatrix}$$

To complete this counterexample, I need to show a different set of allocations which lead to the same ordering of state utilities for agents 1 and 2, but a different ordering for agent 3.

Indeed, I obtain the following state utilities

$$\begin{pmatrix} 0.318 & 0.537 & 0.110 \\ 0.389 & 0.280 & 0.402 \\ 0.295 & 0.184 & 0.489 \end{pmatrix}$$

where $v^1(2) > v^1(1) > v^1(3)$, $v^2(3) > v^2(1) > v^2(2)$, but $v^3(3) > v^3(1) > v^3(2)$. These utilities are supported by the following new allocations, where rows correspond to agents and columns to goods in state 1, 2, and 3 below:

$$\begin{pmatrix} 0.320 & 0.300 & 0.331 \\ 0.365 & 0.400 & 0.400 \\ 0.315 & 0.300 & 0.269 \end{pmatrix}$$

$$\begin{pmatrix} 0.550 & 0.510 & 0.550 \\ 0.260 & 0.300 & 0.280 \\ 0.190 & 0.190 & 0.170 \end{pmatrix}$$

$$\begin{pmatrix} 0.114 & 0.100 & 0.117 \\ 0.378 & 0.410 & 0.417 \\ 0.508 & 0.490 & 0.467 \end{pmatrix}$$

This constructive counterexample[4] establishes the lemma in the case of $n = 3$ agents and goods. This clearly generalizes to all cases in which $n \geq 3$.

[4] I have omitted the results showing that in both examples, all respective marginal rates of substitution are constructed to be equalized at the value of 1 across agents within in each state.

Appendix 4: Constructive 3 Asset Counterexample

I provide a simple constructive (computer generated) example below of a fully specified economy, due to Hens, Mayer & Pilgrim (2004, p27-29), with a unique deterministic equilibrium, nonetheless exhibiting sunspot equilibria that matter.

Let there be three agents with Cobb Douglas utility functions

$$u^i(x^i) = \exp \sum_{l=1}^{3} \alpha_l^i \ln(x_l^i)$$

with the following parameters: $\alpha^1 = (0.4072, 0.5112, 0.0816)$, $\alpha^2 = (0.0269, 0.9301, 0.0430)$, and $\alpha^3 = (0.0088, 0.0019, 0.9892)$. Following the same model structure as in the body of my paper, let strictly concave transformations h apply to the underlying Cobb Douglas utility functions. Below the degree of risk aversion at the equilibrium allocations is calculated. Also suppose endowments initially are as follows: $\omega^1 = (0.1093, 0.9045, 0.1008)$, $\omega^2 = (0.0715, 0.0672, 0.3781)$, and $\omega^3 = (0.8192, 0.0283, 0.5211)$.

Let there be four sunspot states $s = 1, 2, 3, 4$, and let the third good be the numeraire. Let the following matrix represent the three assets payoffs in each of the four states:

$$\begin{pmatrix} 410.0959 & 726.8543 & 384.2747 \\ -19.8893 & -35.3459 & -18.6341 \\ -386.5250 & -685.5971 & -362.1722 \\ 0.1150 & 0.8172 & 0.0893 \end{pmatrix}$$

In Hens et al's computer generated example, the following equilibrium asset prices obtain, $q = (1.0108, 1.7917, 0.9472)$, and Hens et al obtain the following matrix of agent asset holdings at equilibrium, in which the ij^{th} entry represents the i^{th} asset and j^{th} agent:

$$\begin{pmatrix} 0.9911 & -0.1125 & -0.8787 \\ -0.0544 & 0.0777 & -0.0233 \\ -0.9549 & -0.0269 & 0.9818 \end{pmatrix}$$

In each of the induced spot market equilibria (one for each state), there is a vector of prices. In the following matrix, each row corresponds to one such vector, and the final entry is 1, corresponding to the numeraire good:

$$\begin{pmatrix} 0.4358 & 0.9623 & 1.0000 \\ 0.4324 & 0.9518 & 1.0000 \\ 0.4141 & 0.8954 & 1.0000 \\ 0.4638 & 1.0489 & 1.0000 \end{pmatrix}$$

At each of these equilibria, there is an equilibrium allocation of the three goods across the three agents. In the following matrices, corresponding to x^1, x^2, x^3 respectively, the ij^{th} entry represents the i^{th} state and j^{th} good:

$$\begin{pmatrix} 0.9525 & 0.5415 & 0.0832 \\ 0.9525 & 0.5432 & 0.0825 \\ 0.9527 & 0.5531 & 0.0791 \\ 0.9522 & 0.5285 & 0.0885 \end{pmatrix}$$
$$\begin{pmatrix} 0.0291 & 0.4567 & 0.0204 \\ 0.0289 & 0.4550 & 0.0200 \\ 0.0278 & 0.4449 & 0.0184 \\ 0.0307 & 0.4698 & 0.0228 \end{pmatrix}$$
$$\begin{pmatrix} 0.0184 & 0.0018 & 0.8965 \\ 0.0185 & 0.0018 & 0.8974 \\ 0.0194 & 0.0020 & 0.9025 \\ 0.0171 & 0.0017 & 0.8887 \end{pmatrix}$$

The penultimate description of the equilibria is the first derivative of the risk aversion transformation h' evaluated at the equilibrium allocation of goods, for each state. The following matrix lists the value of h' by agent and by state, where again the ij^{th} entry represents the i^{th} state and j^{th} agent:

$$\begin{pmatrix} 0.6414 & 0.6519 & 0.9985 \\ 0.9980 & 0.6583 & 0.9985 \\ 0.6513 & 0.6930 & 0.9983 \\ 0.6282 & 0.5979 & 0.9986 \end{pmatrix}$$

Finally, the following matrix summarizes the total indirect utility by agent and by state, where again the ij^{th} entry represents the i^{th} state and j^{th} agent:

$$\begin{pmatrix} 0.7176 & 0.4248 & 0.8651 \\ 0.7183 & 0.4229 & 0.8661 \\ 0.7225 & 0.4122 & 0.8714 \\ 0.7123 & 0.4388 & 0.8570 \end{pmatrix}$$

This constructive example thus demonstrates the non utility-equivalence of sunspot equilibria in the case in which the underlying economic fundamentals admit a unique equilibrium, thus falsifying Mas-Colell's conjecture.