E C O N O M I C S   B U L L E T I N

A selection mechanism for the barter equilibrium in the search theoretic monetary model

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Abstract

We modify the standard Kiyotaki-Wright model (1993) in order to add an autarkic option in the agents' choice set. The value of the autarkic option is independent of strategic coordination problems and represents a sort of reservation utility with respect to exchange activity. This allows us to identify the conditions under which we can rule out the barter equilibrium as an exchange coordination outcome. These conditions concern the value of the inter-temporal rate of preference, the total amount of money and the rate of return of the matching technology.

I wish to thank Jean Cartelier and Marcello Messori for their helpful comments. I would also like to thank an anonymous referee for her/his useful suggestions to improve the paper. The usual disclaimers apply.
Citation: Amendola, Nicola, (2008) "A selection mechanism for the barter equilibrium in the search theoretic monetary model." *Economics Bulletin,* Vol. 5, No. 1 pp. 1-10
Submitted: July 23, 2007. **Accepted:** January 19, 2008.
URL: http://economicsbulletin.vanderbilt.edu/2008/volume5/EB-07E00008A.pdf

1 Introduction

One of the main features of the Kiyotaki and Wright [1993] search theoretic monetary model (KW henceforth) is that the value of money crucially depends on the agents' ability to coordinate themselves on a monetary transaction technology. According to Iwai [1996], the search approach implies a *bootstrap theory of money.*

Even if such a property formalizes the widely accepted idea that the use of money depends on its general acceptability, it also produces an uncomfortable result. The monetary equilibrium is always paired with a barter equilibrium which is qualitatively the same: it is a strategic coordination outcome where nobody attaches value to money. The *bootstrap theory of money* is always coupled with a *bootstrap theory of barter* and, paradoxically, the search theory is able to explain why money exists as well as why money does *not* exist[1]. In order to rule out the barter coordination failure, we should impose a constraint on agents' beliefs in money acceptability[2], resorting to institutional and historical elements, that is on elements which are outside search theory and, more in general, outside economic theory[3]. An interesting question is therefore the following one: is it possible to identify a selection mechanism for the barter equilibrium which preserves the bootstrap nature of money but, at the same time, does not call for exogenous elements?

This paper shows that such mechanism can be based on the existence of an autarkic option in the agents' choice set. Since the value of the autarkic choice is independent of strategic coordination problems, agents are always able to compare it with the value of all the other possible exchange-coordination outcomes (money or barter). If the barter exchange system entails a pay-off lower than the autarkic option, the existence of the autarkic option will destroy the barter equilibrium but not necessarily the monetary equilibrium.

Unfortunately the original KW framework is incompatible with the existence of a non trivial autarkic option. In order to allow for autarky, we must remove the assumption of specialization in production and consumption[4], but doing so eliminates any rationale for money[5].

An interesting and easily interpretable way to restore the consistency between money and autarky is to model the endowment's renewal not as an auto-production process, as in KW, but as a complementary relationship between workers and entrepreneurs. In particular, we assume that entrepreneurs produce specialized goods by means of specialized

[1] This problem pertains also to the second generation search theoretic models, where the focus is on monetary price (Trejos and Wright [1995] and Shi [1995]) and/or on monetary policy (Lagos and Wright [2005]); to the extent to which, in these models, preferences are set so as to make barter transactions unfeasible, the problem of the co-existence of barter and monetary equilibria is simply not considered. For an extensive survey of recent developments of the search theoretic approach to monetary theory see Shi [2006]

[2] This is not far from introducing a Clower constraint on individual transactions. However, one of the main aim of the search theoretic approach is just to rise to the challenge advanced by Hellwig [1993]: "....Why should cash-in-advance constraints be imposed?".

[3] This solution would be at odds with the Mengerian perspective, largely concurred by the search theory scholars (see Kiyotaki and Wright [1991], Jones [1976], Iwai [1996] and Gravelle [1996]). In Menger's opinion [1892], the origin of money must have a purely theoretical explanation, independent of historical and institutional elements.

[4] According to the KW notation, this is equivalent to set $x = 1$.

[5] A possible alternative is to assume the existence of an outside option which yelds a utility ηU, with $\eta \in (0, 1) \subset \mathbb{R}^+$, where U is the utility given by a market commodity. However, this is obviously an *ad hoc* solution and the selection mechanism would depend crucially on the arbitrary value η.

labour services provided by workers, where the dynamic system's viability depends on the existence of a production surplus over entrepreneurs' needs, which can be invested to purchase the labour services necessary to start a new production process.

In this new framework there is an exchange coordination problem between workers and entrepreneurs which calls for money, but there is also an autarkic option. As a matter of fact, entrepreneurs can always give up the exchange activity and production opportunities, immediately consuming the production surplus. The opportunity cost of the autarkic option depends on the production and exchange difficulties and, as a consequence, on the efficiency of the exchange technology. If the entrepreneurs' intertemporal rate of preference is sufficiently high, the monetary transaction technology turns out to be essential to leading the economy out of autarky.

The rest of the paper is organized as follows. Section 2 introduces the basic model and analyzes its equilibrium property under the assumption that entrepreneurs decide to invest their production surplus in a new production process. Section 3 makes the autarkic option available and identifies the condition under which the barter equilibrium does not exist. Conclusions follow in section 4.

2 The model

The economy is populated by a continuum of agents with unitary mass and infinite time horizon; time is discrete. There are P entrepreneurs and $1-P$ workers, where $P \in (0,1) \subset \mathbb{R}^+$. Entrepreneurs produce instantaneously two units of an indivisible consumption good using as input a specific labour unit and a consumption good unit necessary to restore productive capacity. Workers produce instantaneously an indivisible labour unit using as input a specific unit of a consumption good necessary to restore working capacity. Because workers and entrepreneurs need, respectively, *specific* consumption goods and labour units, an exchanges coordination problem can arise. Contrary to KW, this problem is placed behind the production process.

The specialization structure is entirely described by a single parameter $x \in (0,1) \subset \mathbb{R}^+$ which identifies the probability that a generic entrepreneur meets a worker suitable for his technology as well as the probability that a generic worker meets an entrepreneur who produces a desired good.

Preferences over goods are homogeneous across agents and are described by a constant utility $U > 0$. Entrepreneurs can directly consume the goods they produce or can store them at no costs. On the other hand, workers can only consume a desired good and no storage is possible. Labour units can be stored only by their original producers[6].

Agents meet at random according to a *Bernoulli process* with parameter $\beta \in (0,1) \subset \mathbb{R}^+$. As in Diamond [1982], and in departure from KW, we assume an *increasing returns matching technology,* i.e. $\beta = \beta(n_s)$ with $\beta' > 0$, where n_s is the measure of *active agents*, who have a positive probability to carry out exchanges. This allows us to capture the effects of positive thick market externalities.

Initially, a randomly selected fraction m of agents is endowed with an indivisible unit of *fiat* money (instead of goods or labour units). We denote with m_P and $m_{1-P} = m - m_P$, respectively, the measure of entrepreneurs and workers initially endowed with *fiat* money.

Agents face the following strategic problems: *(I)* entrepreneurs and workers have to decide whether to accept or to reject goods, labour units and money in all the possible

[6] This can be interpreted as a no-slavery condition.

meetings, given the agents distribution among states and their trading strategies, *(II)* entrepreneurs have to decide weather to invest the second unit of the production good, or to consume the good and move towards a state of autarky.

2.1 Optimality conditions

Let us assume that entrepreneurs always decide to invest the second unit of the production good. Hence, entrepreneurs and workers can be in the following states: *state* 0 = entrepreneur with one unit of good, *state* 1 = worker with one labour unit, *state* 2 = entrepreneur with one unit of money, *state* 3 = worker with one unit of money.

Let V_i, $i = 0, 1, 2, 3$ be the steady state value function of a state i agent. Applying the standard dynamic programming techniques we obtain the following optimality conditions:

$$rV_0 = \beta(n_s) \cdot [\tfrac{n_1}{n_s} x^2 U + \tfrac{n_3}{n_s} x \max \pi_0 (V_2 - V_0)] \quad (1)$$
$$rV_1 = \beta(n_s) \cdot [\tfrac{n_0}{n_s} x^2 U + \tfrac{n_2}{n_s} x \max \pi_1 (V_3 - V_1)] \quad (2)$$
$$rV_2 = \beta(n_s) \cdot \tfrac{n_1}{n_s} x \Pi_1 (V_0 - V_2 + U) \quad (3)$$
$$rV_3 = \beta(n_s) \cdot \tfrac{n_0}{n_s} x \Pi_0 (V_1 - V_3 + U) \quad (4)$$

where $r > 0$ is the inter-temporal preference rate, identical for entrepreneurs and workers, n_i, $i = 0, 1, 2, 3$ is the measure of the active agents in state i, Π_i is the expected probability that a type i agent will accept money and π_i is the individual best reply.

Equations $(1) - (4)$ have the usual interpretation. For instance, according to (1) the flow return to an entrepreneur endowed with one unit of good is the sum of two terms: the first term is the probability with which he meets a worker holding one unit of labour $[\beta(n_s) \frac{n_1}{n_s}]$ times the probability that both want to trade $[x^2]$ times the gain from trade, produce and go back to the previous state $[U]$; the second term is the probability that he meets a worker holding money and willing to trade $[\beta(n_s) \frac{n_3}{n_s} x]$ times the gain from accepting money with probability π_0, where π_0 is chosen optimally. The other conditions have a similar interpretation.

Equations $(1) - (4)$ define a correspondence from (Π_0, Π_1) to the best replies (π_0, π_1). The set of equilibria for the economy is the set of fixed points of this correspondence. However optimal strategies depend on agents' distribution and we have to prove that the steady state distribution is uniquely determined by the couple (Π_0, Π_1).

2.2 Steady state

In steady state the in-flow must equal the out-flow state i, for $i = 0, 1, 2, 3$. First of all we observe that: out-flow state $0 \equiv$ in-flow state 2, out-flow state $2 \equiv$ in-flow state 0, out-flow state $1 \equiv$ in-flow state 3 and out-flow state $3 \equiv$ in-flow state 1. As a consequence, the flow equilibrium conditions can be reduced to:

$$\tfrac{n_3 n_0}{n_s} x \Pi_0 = \tfrac{n_1 n_2}{n_s} x \Pi_1 \quad (5)$$

where the LHS is the out-flow state 0 [in-flow state 1] and the RHS is the in-flow state 0 [out-flow state 1].

To be consistent with respect to the exogenous distribution of agents between types and to the exogenous amount of money, the following constraints must be added:

$$n_0 + n_2 = P \quad (6)$$

$$n_1 + n_3 = 1 - P \quad (7)$$

$$n_3 + n_2 = m \quad (8)$$

$$n_i \geq 0 \quad (9)$$

The following results hold true:

Proposition 1 *For every $m, P, x \in (0,1) \subset \mathbb{R}^+$, $m_P \in [0, \min\{P, m\}] \subset \mathbb{R}^+$ and $\Pi_0, \Pi_1 > 0$, the system $(5)-(9)$ has a unique solution in the variables n_i, $i = 0, 1, 2, 3$.*

Proof. see the appendix. ■

Proposition 2 *For every $m, P, x \in (0,1) \subset \mathbb{R}^+$, $m_P \in [0, \min\{P, m\}] \subset \mathbb{R}^+$, if $\Pi_0 = \Pi_1 = 0$, the system $(5)-(9)$ has a unique solution in the variables n_i, $i = 0, 1, 2, 3$.*

Proof. see the appendix. ■

Proposition 1 and proposition 2 state that a unique steady state distribution is associated with every meaningful couple (Π_0, Π_1)[7].

2.3 Equilibria

A symmetric steady state Nash equilibrium for the economy is a collection of values $[n_i$, V_i, $i = 0, 1, 2, 3$; $\Pi_0, \Pi_1]$ such that $(5) - (9)$ and $(1) - (4)$ are satisfied with $\pi_0 = \Pi_0$ and $\pi_1 = \Pi_1$.

Proposition 3 *The above economy has three symmetric steady state Nash equilibria: i) a pure strategies equilibrium, with $\Pi_0 = \Pi_1 = 0$ (barter equilibrium B); ii) a mixed strategies equilibrium, with $\Pi_0 = \Pi_1 = x$ (partial acceptability equilibrium MS); iii) a pure strategies equilibrium, with $\Pi_0 = \Pi_1 = 1$ (pure monetary equilibrium M).*

Proof. see the appendix. ■

Not surprisingly, the content of Proposition 3 is consistent with the KW results. The main feature which distinguishes our model from KW is, indeed, the existence of an autarkic option; but this last one is precluded here by assumption.

3 The autarkic option

If entrepreneurs are allowed to directly consume the second unit of good produced, we have to take into account the strategic decision problem *(II)*. As a matter of fact, entrepreneurs can now give up future production opportunities in favour of immediate consumption. The permanence in state 0 yields the expected discounted utility V_0 while immediate consumption yields an expected utility U. As a consequence, entrepreneurs will decide to consume their real endowments if and only if $V_0 \leq U$[8]. A sort of participation constraint for the exchange equilibria sustainability must be imposed. Let us define V_0^J; $J = B, MS, M$ as the value function of a state 0 entrepreneur in a type J equilibrium.

[7] We have not considered the cases $\Pi_0 > 0, \Pi_1 = 0$ and $\Pi_0 = 0, \Pi_1 > 0$. However, these can never be equilibrium strategies.

[8] We are ruling out mixed strategies.

Definition 1 A type J equilibrium is sustainable if and only if $U - V_0^J < 0$

We can now characterize the sustainability condition for all three possible equilibria. In order to do that we have to: *(i)* substitute the equilibrium values $\Pi_0^J = \Pi_1^J$, for $J = B, MS, M$, in $(1) - (4)$ and $(5) - (9)$, *(ii)* solve for the steady state distribution n_i^J for $i = 0, 1, 2, 3$; $J = B, MS, M$; *(iii)* solve for the value function V_0^J for $J = B, MS, M$ and *(iv)* apply Definition 1 for $J = B, MS, M$. The *(i)*-*(iv)* steps procedure identifies a γ_J threshold value for the inter-temporal preference rate r characterizing the sustainability of a type J equilibrium.

Barter Equilibrium: a $J = B$ equilibrium is sustainable iff:

$$r < \beta(1-m) \cdot \left[\frac{1-P-m_{1-P}}{1-m}\right] x^2 = \gamma_B \tag{10}$$

Mixed Strategies Equilibrium: a $J = MS$ equilibrium is sustainable iff:

$$r < \beta(1) \cdot (1-P)(1-m) x^2 = \gamma_{MS} \tag{11}$$

Monetary Equilibrium: a $J = M$ equilibrium is sustainable iff:

$$r < \beta(1) \cdot (1-P)(1-m) x^2 \Delta = \gamma_M = \gamma_{MS}\Delta \tag{12}$$

where $\Delta = \left\{ \frac{r+\beta(1)\cdot[(1-m)(1-P)]x+\beta(1)\cdot[m(1-P)]}{r+\beta(1)\cdot[(1-m)(1-P)]x+\beta(1)\cdot[m(1-P)]x} \right\} > 1$

Conditions $(10) - (12)$ have a quite intuitive interpretation: if entrepreneurs choose autarky, their choice yields immediate positive utility U. As an alternative, they can give up present consumption in favour of future consumption. The last option will be preferred to autarky if only if the inter-temporal rate r is sufficiently low. Moreover, because the value of future consumption depends on the exchange process, the threshold level for r depends on the efficiency of the prevailing exchange system. The value γ_j can also be interpreted as the *internal rate of return* of the transaction technology j. The investment in production implies an opportunity cost U, and future revenues with values dependent on the transaction technology. The threshold γ_j is the rate of discount which equates opportunity costs and revenues. A more efficient transaction technology implies a higher value of γ, i.e. a higher internal rate of return[9]

3.1 Selection of equilibria

Entrepreneurs cannot maximize γ_J over J, but they can rule out the exchanges coordination systems with an internal rate of return lying below r. In other words, if $\gamma_M > \gamma_{MS} > \gamma_B$, there are values of r such that the barter equilibrium, or even the mixed strategies equilibrium, can be ruled out as exchange coordination outcomes. The next analytical step is therefore to look for the conditions under which $\gamma_M > \gamma_{MS} > \gamma_B$.

From (12) we know that $\gamma_M > \gamma_{MS}$ for every admissible parameters set. Let us now focus on the relationship between γ_B and γ_{MS}. In order to accomplish this task it is necessary to specify a functional form for the increasing matching technology. In

[9] In the KW framework we cannot define the internal rate of return of the transaction technology just because of the absence of opportunity costs with respect to the exchange activity.

particular, we assume $\beta(n_s) = kn_s^\delta$ with $\delta \geq 1$ and $0 < k < 1$ real numbers; $\delta = 1$ describes a *linear increasing returns* matching technology, while $\delta > 1$ identifies a *convex increasing returns* matching technology. It should be noted, however, that with a linear increasing matching technology, meeting probabilities are constant across equilibria and thick market externalities do not produce any effect. This is because in a barter equilibrium the number of active agents decreases, compensating for the negative thick market externalities due to the increasing returns of the matching technology. In this sense, a linear increasing returns matching technology is equivalent to a *constant return matching technology* where all the agents are active, independently of their exchange opportunities. More precisely, all the results obtained under the assumption of a linear increasing matching technology could also be obtained with a constant return matching technology (i.e. $\delta = 0$) assuming that money traders are always active[10].

Proposition 4 *For every* $m, P, x \in (0,1) \subset \mathbb{R}^+$, $m_P \in [0, \min\{P, m\}] \subset \mathbb{R}^+$ *and* $\delta \geq 1$, *a value* $m^*(\delta) \leq 1$ *exists such that for* $m < m^*(\delta)$ *it is* $\gamma_{MS} > \gamma_B$, *where* $\partial m^*(\delta)/\partial\delta > 0$ *for* $\delta < 2$ *and* $m^*(\delta) = 1$ *for* $\delta \geq 2$.

Proof. let us define $\gamma_{MS} - \gamma_B = kx^2[(1-m)(1-P) - (1-m)^{\delta-1}(1-P-m_{1-P})] = g(\delta, m)$. Note that $g(\delta, m)$ is a continuous function of m and, as the quantity of money initially distributed to workers can not exceed the total amount of money, g is defined on $m \in [m_{1-P}, 1)$. Moreover, g has the following properties:

a) if $1 \leq \delta < 2$ then $g(\delta, m) > 0$ if and only if $m < m^*(\delta) = 1 - \left(\frac{1-P-m_{1-P}}{1-P}\right)^{\frac{1}{2-\delta}}$

b) if $\delta \geq 2$ then $g(\delta, m) > 0$ for every $m \in [m_{1-P}, 1)$

c) $g(\delta, m_{1-P}) > 0$ for every $\delta \geq 1$

Property *(a)* states that if $1 \leq \delta < 2$, and provided that the total amount of money is not too large, i.e. $m < m^*(\delta)$, $\gamma_{MS} > \gamma_B$. Property *(c)*, together with continuity of g, make sure that $m^*(\delta) > m_{1-P}$. According to property *(b)*, if $\delta \geq 2$, $\gamma_{MS} > \gamma_B$ for every admissible value of m. Eventually, it is a simple matter of algebra to verify that $\partial m^*(\delta)/\delta > 0$.■

Proposition 4 identifies sufficient conditions for an increasing monotonic ordering of the internal rate of return γ_J from $J = B$ to $J = M$. These conditions concern the total amount of money and the rate of return of the matching technology. If we impose a *linear increasing returns* matching technology, meeting probabilities are constant across equilibria. This is because the thick market externalities effect is always exactly compensated by the change in the number of active agents n_s. At the same time, we have that an increase in m reduces production opportunities in a monetary economy. Hence, in order to have $\gamma_{MS} > \gamma_B$, the total amount of money must be sufficiently low, i.e. $m \leq m^*$. However, as we move toward a *convex return* technology, i.e. δ increases, the monetary exchange system appears to be more favourable, in terms of meeting probabilities, that the barter system. As a consequence, the constraint on m becomes less binding. In particular, if the thick market externalities effect is sufficiently large ($\delta \geq 2$), the constraint on m ceases to be binding and the monotonic ordering of γ_J turns out to be a general property.

[10] This is the same assumption introduced by KW where, in a barter equilibrium, money traders actively search for trading partners even if money will be never accepted.

Moreover, it is interesting to note that $\partial\gamma_M/\partial\delta = \Delta\partial\gamma_{MS}/\partial\delta = 0$ and, if $m < m^*(\delta)$, $\partial g/\partial\delta > 0$; the larger the thick market externalities effect, the higher is the gap between the internal rates of return of a monetary and of a barter transaction technologies.

We are now able to prove our main result:

Proposition 5 *For every $P, x \in (0,1) \subset \mathbb{R}^+$ and $m_P \in [0, \min\{P, m\}] \subset \mathbb{R}^+$, if $m < m^*(\delta)$ a value $r > 0$ exist such that the pure monetary equilibrium is the only sustainable exchange equilibrium.*

Proof. it follows immediately from (12) and Proposition 4 ■

If $m < m^*(\delta)$, the monetary exchange system exhibits the highest internal rate of return while the barter system has the lowest rate: the barter technology appears to be more fragile and less efficient than the monetary technology. As a consequence, a sufficiently high inter-temporal preference rate destroys the barter exchange equilibrium while the monetary exchange equilibrium may still be feasible. In other words, if entrepreneurs are too much impatient, they will never decide to invest their surplus in production and exchange activity, unless a monetary transaction technology is available. Moreover, as δ increases, the gap between γ_M and γ_B also increases and the parameter set which allows for the barter equilibrium selection enlarges.

4 Conclusions

We modify the standard KW model in order to add an autarkic option to the agents' choice set. The essential modification concerns the renewal process of endowments, which is modelled as a complementary relationship between entrepreneurs and workers.

While the rationale and the nature of money appears to be the same as in KW, the presence of an autarkic option makes operative a selection mechanism for the barter equilibrium.

In the original KW framework there are no alternatives to exchange activity and the only strategic decision concerns the exchange transaction technology. To the extent to which agents may be unable to coordinate themselves on a monetary transaction technology, the barter coordination failure always exists. In our framework the participation in exchange activity itself becomes a strategic decision variable. Because of the decentralized structure of transactions, exchange and production are time consuming activities. To the extent to which agents discount future, they may prefer to stay in autarky, where this last decision depends on the efficiency of the prevailing transaction technology.

According to Proposition 4, money accelerates exchanges relative to barter and, provided that agents are sufficiently impatient, it may be *essential* to sustain an exchange equilibrium (Proposition 5). If this happens, the barter coordination failure disappears, and the only alternative to money is autarky, i.e. a no-exchanges equilibrium. With partial reference to the classical Smith's argument, money turn out to be the essential instrument that allows the transition from an autarkic economy to a market economy, i.e. to an economy characterized by increasing specialization in production and consumption. From this point of view the modified model supports a bootstrap theory of monetary exchanges but *not* a bootstrap theory of barter exchanges.

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Appendix

Proof of Proposition 1: The system $(5)-(8)$ can be reduced to $\varphi(n_3) = \Pi_1/\Pi_0 \in (0, +\infty)$, where φ is a monotonically strictly increasing continuous function. Moreover, because of the non-negativity constraints (9), φ is defined on $(a, b) \subset \mathbb{R}^+$, where $a = a(m, P)$ and $b = b(m, P)$. It is easy to show that $\forall\, m, P \in (0, 1) \subset \mathbb{R}^+$, $\lim_{n_3 \to a} \varphi(n_3) = 0$ and $\lim_{n_3 \to b} \varphi(n_3) = +\infty$. By continuity of φ it follows that $\exists!\ n_3^* \in (a, b)$ such that $\varphi(n_3^*) = \Pi_1/\Pi_0$. The steady state solution for n_i, $i = 0, 1, 2.$ can be obtained from $(6)-(8)$ with $n_3 = n_3^*$.■

Proof of Proposition 2: If $\Pi_1 = \Pi_0 = 0$, agents endowed with money have no exchange opportunity. It follows that $n_2 = n_3 = 0$ and $n_s = 1 - m$. Only entrepreneurs initially endowed with one unit of good and workers initially endowed with a labour unit will be active, so that $n_0 = P - m_P$ and $n_1 = 1 - P - m_{1-P}$.■

Proof of Proposition 3: The optimal choices π_1 and π_0 depend on $sign(V_2 - V_0)$ and $sign(V_3 - V_1)$. From (1) $-$ (4), we have:

$$V_2 - V_0 = \frac{\beta(n_s) \cdot [n_1/n_s]\, xU\, [\Pi_1 - x]}{r + \beta(n_s) \cdot \{[n_1/n_s]\, x\Pi_1 + [n_3/n_s]\, x \max \pi_0\}} \tag{13}$$

$$V_3 - V_1 = \frac{\beta(n_s) \cdot [n_0/n_s]\, xU\, [\Pi_0 - x]}{r + \beta(n_s) \cdot \{[n_0/n_s]\, x\Pi_0 + [n_2/n_s]\, x \max \pi_1\}} \tag{14}$$

Equation (13) represents the gain of an entrepreneur moving from state 0 to state 2 as a function of workers' strategy. Similarly, equation (14) represents the gain of a worker moving from state 1 to state 3 as a function of entrepreneurs' strategy.

From (13) we observe that if $\Pi_1 > x$, i.e. if workers' probability of accepting money is higher than workers' probability of accepting goods, $V_2 - V_0 > 0$. This implies, according to (1), that $\max \pi_0 = 1$; entrepreneurs always take advantage of accepting money. Because of symmetry $\Pi_0 = 1$; but if $\Pi_0 = 1 > x$, from (14) we have $V_3 - V_1 > 0$ and $\max \pi_1 = 1$, i.e. $\Pi_1 = 1$. Therefore $\Pi_0 = \Pi_1 = 1$ is a pure strategies equilibrium. Following the same procedure, the existence of the other two equilibria can be proven.■