E C O N O M I C S   B U L L E T I N

Optimal capital investment under uncertainty: An extension

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Abstract

This paper develops a model for optimal capital investment in continuous time when both existing and new capital stocks are subject to uncertainty. The model is generalized to allow for large and infrequent changes in the dynamics of the capital stock, which may arise as a result of natural and man-made disasters.

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1. Introduction

Studies on the issue of optimal capital investment typically assume that only a new capital investment is subject to uncertainty. The uncertainty arises as a result of variations in the cost of capital, financial intermediation, or capital-embodied technological innovation (see the survey by Dixit and Pindyck, 1994). Recently, several studies in the business-cycle literature suggest that the existing capital stock is also subject to uncertainty and may change for reasons other than natural depreciation (Ambler and Paquet, 1994; Cooley *et al*., 1997; Dueker *et al*., 2002). Possible sources of uncertainty are economic obsolescence, physical break-down, compositional shifts, among others. Presumably, some may also affect the portfolio decision on new capital investment concurrently. To take account of this development, the present paper constructs a version of the Ramsey model under the condition that the existing and new capital stocks are subject to different but correlated random fluctuations. We do so in continuous time and are able to derive an analytic solution for the optimal level of new capital investment, which is also in contrast to the studies mentioned above.

Another feature of this paper is a generalization to allow for large but infrequent changes in the existing and new capital stock. These types of changes may be a result of, say, natural and man-made disasters such as earthquakes, tsunamis, terrorism, or wars, which are likely to cause substantial and prolonged damages the physical capital. The effects can be particularly devastating in developing countries, washing out a major portion of their capital stock, as in the 1998 floods in Bangladesh and the 2004 tsunami in Indian Ocean countries. Tol and Leek (1999), Skidmore and Toya (2002), and Okuyama (2003) have examined the economic effects of disasters. In particular, Okuyama shows in a Solow model that investment decisions under disaster conditions are rather complex and different from those made in the absence of disaster. We take one step further by analyzing the effects of such catastrophic events in a stochastic optimizing setting. This extension is important as an increase in the saving rate necessary to return the capital stock to its previous steady state can only be accomplished through a decrease in consumption. In conclusion, some policy implications are drawn for those countries prone to natural and man-made disasters.

2. Optimal capital investment under log-normally distributed uncertainty

We first consider a baseline case: that the random fluctuations of existing and new capital stocks are log-normally distributed. The analytic model is closely related to those of Merton (1975) and Williams (1979), which develop a stochastic version of the Ramsey model in continuous time. Uncertainty in Merton is due to the random dynamics of the labor force, which follows a log-normal distribution. Williams introduces several sources of uncertainty to examine portfolio selection when investing in education and marketable assets. In our model, physical capital investment, either existing or new, is posited to be the source of uncertainty for the reasons discussed in the Introduction.

Consider a one-sector neoclassical model with a constant-returns-to-scale, strictly concave production function, $Y = F(K, L)$, where Y is total output, K the stock of physical capital, and L the stock of labor. The production function can be expressed in intensive form, $y = f(k)$, where $y = Y/L$ is the output per worker and $k = K/L$ is the capital per worker. It is assumed throughout the paper that $f(k)$ is concave (i.e. $f'(k) = f_k > 0$ and $f''(k) = f_{kk} < 0$) and satisfies the Inada conditions (i.e. $\lim_{k\to 0} f_k = \infty$ and $\lim_{k\to\infty} f_k = 0$). We further assume that a linear production function for new capital of the form $s(t)y(t)$, where $s \geq 0$, is the percentage of output that is invested in capital formation at time *t*. The existing capital stock is depreciated at a constant rate δ. In the absence of uncertainty the capital stock thus grows according to the following equation:

$$k(t, t+\Delta t) = k(t) - \delta k(t) + s(t)y(t) \tag{1}$$

We now allow for the impact of random fluctuations on both the existing and the newly installed capital by formulating the following dynamics for the capital stock:

$$k(t, t+\Delta t) = x(t, t+\Delta t)k(t) - \delta k(t) + z(t, t+\Delta t)s(t)y(t) \tag{2}$$

where *x* and *z* are log-normally distributed random variables on the interval [0,1], with means μ_x and μ_z, variance $\sigma_x^2 = \pi_x^{'}\pi_x$ and $\sigma_z^2 = \pi_z^{'}\pi_z$ respectively, and covariance $\sigma_{xz} = \pi_x^{'}\pi_z$. Note that π_x and π_z are two-dimensional vectors. Consequently, the exiting and new capital stocks are subject to different but correlated uncertainties. As in Merton and Williams, the stochastic processes, *x* and *z*, may be interpreted as representing the productivity or efficiency of existing and new capital stocks, respectively. The parameters μ_x (μ_z) and π_x (π_z) measure the expected productivity and uncertainty in the existing (new) capital stock. In a continuous-time setting the dynamics of the capital stock may be represented as

$$\frac{dk}{k} = \left[\frac{sy\mu_z}{k} + \mu_x - \delta\right]dt + \left[\pi_x + \frac{\pi_z sy}{k}\right]^{'} dB \tag{3}$$

subject to $k(0) = k_0 > 0$. Here *B* is a two-dimensional Brownian process. See Williams (1979, p. 526) for a similar expression. Eq. (3) gives the dynamics of the state variable *k*. The control variables are the amount of investment (*s*) and the amount of consumption (*c*). The two control variables are connected through the equation $c(t) = (1 - s(t))y(t)$, where $s \in (0,1)$.

There exists a benevolent utility-maximizing agent with utility function, $U(c(t))$ or $U((1-s(t))y(t)$. By assumption, the instantaneous utility function *U* is continuously differentiable, strictly increasing, and strictly concave in consumption (i.e. $U_c > 0$ and $U_{cc} < 0$). We also assume that $lim_{c\to 0}U_c = \infty$ and that $lim_{c\to\infty}U_c = 0$. These conditions guarantee a positive rate of consumption and a positive savings function. Thus the optimization problem can be summarized as follows:

$$max_{\{s\}}\ E_O\left\{\int_0^T U(c(t))dt\right\} \tag{4}$$

subject to Eq. (3), where $c \geq 0$, $s \geq 0$, and *E* is the conditional expectation operator. Define the indirect utility function:

$$J[k(t), t] = max\ E_t\left\{\int_t^T U[c(\tau)]d\tau\right\}. \tag{5}$$

Note that $\partial J / \partial k = J_k > 0$ and $\partial^2 J / \partial k^2 = J_{kk} < 0$. The former can be obtained by taking the derivative of Eq. (5) with respect to *k* and acknowledging that $y = f(k)$, $f_k > 0$ and $U_c > 0$. The latter follows from the assumption that $f_{kk} < 0$.

Using theory of stochastic optimal control we define the Hamilton-Jacobi-Bellman (HJB) equation as follows

$$0 = max_{\{s\}} \left\{ U(c) + J_t + \left(\frac{sy\mu_z}{k} + \mu_x - \delta \right) kJ_k + \frac{1}{2} \left(\sigma_x^2 + \sigma_z^2 \left(\frac{sy}{k} \right)^2 + 2\sigma_{xz} \frac{sy}{k} \right) k^2 J_{kk} \right\} \qquad (6)$$

where J_t is the partial derivative of the indirect utility function with respect to *t*. Taking the derivative of Eq. (6) with respect to *s* yields the first-order condition for a regular interior maximum as:

$$(-y)U_c + yu_z J_k + \sigma_{xz} ykJ_{kk} + \sigma_z^2 sy^2 J_{kk} = 0 . \qquad (7)$$

Simplifying and solving for *s* gives an expression for the optimal capital investment.

$$s^* = \frac{U_c - \mu_z J_k}{\sigma_z^2 yJ_{kk}} - \frac{\sigma_{xz} k}{\sigma_z^2 y} = \frac{1}{\sigma_z^2} \left(\left(-\frac{J_k}{kJ_{kk}} \right) \left(\mu_z - U_c / J_k \right) - \sigma_{xz} \right) \frac{k}{y} \qquad (8)$$

or,

$$s^* = \left(\left(-\frac{J_k}{kJ_{kk}} \right) \left(\frac{\mu_z - U_c / J_k}{\sigma_z^2} \right) - \frac{\sigma_{xz}}{\sigma_z^2} \right) \frac{k}{y} \qquad (9)$$

Several comments are in order. First, the term $-J_k / kJ_{kk}$ represents the Arrow-Pratt index of relative risk aversion for gambles with existing capital. Note that $(-J_k / kJ_{kk}) > 0$ because $J_k > 0$ and $J_{kk} < 0$ (see discussion below Eq. (5)). As it rises and hence the investor becomes more risk averse to the existing capital, the optimal level of new investment rises. Second, the term U_c / J_k measures the marginal rate of substitution of capital accumulation for consumption. The difference between the expected productivity of new capital (μ_z) and the marginal rate of substitution of capital for consumption (U_c / J_k) divided by the variance of the uncertainty in new capital (σ_z^2) can be interpreted as the risk-adjusted excess return on new capital investment. As the risk-adjusted excess return on new capital investment increases, there is a rise in the optimal level of new capital investment.

Third, the greater the uncertainty associated with new capital stock (σ_z), the lower the amount of optimal investment in new capital. This can be more easily seen from the last expression Eq. (8) by noting that the figure in the largest parentheses should be positive to ensure that $s^* > 0$. Fourth, the greater the expected productivity of the new capital stock (μ_z), the greater the amount of optimal investment in new capital. Finally, the greater the covariance between the uncertainty of existing and new capital (σ_{xz}), the lower the amount of optimal investment in new capital if σ_{xz} >0 and the greater the amount of optimal investment in new capital if σ_{xz} <0. This result may be a reflection of the investor’s desire to spread the risks across the existing and new capital investments.

3. Optimal capital investment with catastrophic losses

Now, the model is generalized to allow for large and infrequent changes as an additional source of uncertainty in the existing and newly installed capitals. We do so by assuming that the arrival of such catastrophic events follows a Poisson process. Then, the capital accumulation in discrete time can be defined as follows:

$$k(t,t+\Delta t)= x(t,t+\Delta t)k(t)-\delta k(t)+z(t,t+\Delta t)s(t)y(t)$$

$$-l(t+\Delta t)[k(t)+s(t)y(t)]N(t,t+\Delta t) \quad (10)$$

where $N(t,t+\Delta t)$ is an increment of the Poisson process. The random variable $l \notin (0,1)$ is the loss rate associated with the catastrophic event. The dynamics of capital stock in continuous time may be represented as:

$$dk=\left[\frac{sy\mu_z}{k}+\mu_x-\delta\right]kdt+\left[\pi_x+\pi_z\frac{sy}{k}\right]kdB-l\left(1+\frac{sy}{k}\right)kdN \quad (11)$$

where *dN* is an increment of the Poisson process. Thus *dN*=1 if there is a catastrophic event and *dN*=0 otherwise. The process N_t can also be represented as $N_t=\sum_{j=0}^{\infty}1_{\{\tau_j\le t\}}$, where τ_j represent the arrival of the j^{th} catastrophe. When the Poisson process has such a representation, the probability of having *j* events in the time interval $(0,t]$ is given by

$$\text{Prob}(N_t=j)=\frac{(\lambda t)^j}{j!}\exp(-\lambda t)$$

where λ is the intensity of arrival of a catastrophe, which is constant.

The expected utility is now maximized subject to Eq. (11), $c\ge 0$, and $s\ge 0$. Thus the HJB equation is given as

$$0= max_{\{s\}}\left\{U(c)+J_t+\left(\frac{sy}{k}\mu_z+\mu_x-\delta\right)kJ_k+\frac{1}{2}\left(\sigma_x^2+2\sigma_{xz}\frac{sy}{k}+\sigma_z^2\left(\frac{sy}{k}\right)^2\right)k^2J_{kk}\right.$$

$$\left.+\lambda J\left(k-l\left(1+\frac{sy}{k}\right)k,t\right)-J(k,t)\right\}. \quad (12)$$

See Merton (1990, Section 5.8) for a similar derivation of this HJB equation. Taking the derivative of the HJB equation with respect to *s* gives

$$(-y)U_c+yu_zJ_k+(\sigma_{xz}ky+\sigma_z^2sy^2)J_{kk}+\lambda J_k(-ly)=0. \quad (13)$$

Simplifying and solving for *s* yields the optimal capital investment as

$$s^* = \frac{U_c - \mu_z J_k}{\sigma_z^2 y J_{kk}} + \frac{\lambda l J_k}{\sigma_z^2 y J_{kk}} - \frac{\sigma_{xz} k}{\sigma_z^2 y}$$

or, in a more intuitive form

$$s^* = \left(\left(-\frac{J_k}{kJ_{kk}} \right) \left(\frac{\mu_z - U_c / J_k - \lambda l}{\sigma_z^2} \right) - \frac{\sigma_{xz}}{\sigma_z^2} \right) \frac{k}{y} . \tag{14}$$

Eq. (14) is similar to the form derived earlier in Eq. (9). In fact, it encompasses all the properties implied by Eq. (9). It is also worth noting that the risk-adjusted excess return on new capital investment is now reduced by $\lambda l / \sigma_z^2$. The terms λl can be regarded as the expected fractional loss rate of capital when the model allows for large and infrequent changes. A rise in λl will make the excess return on new capital investment smaller and hence the optimal level of new capital investment lower. In addition, Eq. (14) offers two interesting results in relation to the parameters λ and l. They are: (1) the greater the likelihood of a catastrophe (measured by λ), the lower the amount of optimal investment in new capital, and (2) the greater the loss rate (l), the lower the amount of optimal investment in new capital. Hence, those countries that are prone to greater likelihood of natural or man-made disasters will have a lower capital investment than countries that are not, *ceteris paribus*.

This model prediction may be used to shed some light on the current issue of why capital accumulation in the African countries has been chronically low, often blamed as a major cause of their dismal economic performance. Some studies find that geography, climate, ethnic fractionalization, political turbulence, and civil war are the key factors (see Bloom and Sachs, 1998). Devarajan *et al.* (2003) conclude that such a low level of capital accumulation is an inevitable and rational outcome after all. Arising from the same factors, some countries in this region would be exposed to a greater possibility of natural and man-made disasters. They may also face a greater expected loss rate due to lack of infrastructure and hazard management. In our model, these will bring the optimal capital investment to a lower level. Low capital investment accumulation may hence be an optimal response to such catastrophic risks, along the lines of Bloom and Sachs and Devarajan *et al.* An implication is that policy makers should be more careful about calling for an investment boom to obtain a resumption of growth. Unless some or all of the underlying factors that made investment unattractive in the past are addressed, the results may be disappointing.

4. Conclusion

This paper derives an analytic solution for the optimal level of capital investment in continuous time when both existing and new capital stocks are subject to uncertainty. Two forms of uncertainty are considered. One is the log-normal specification, as typically assumed in previous studies. The other one is a Poisson distribution to allow for large and infrequent changes that may arise as a result of natural and man-made disasters. Several implications are drawn, including that countries prone to catastrophic risks will have lower capital investment and hence lower economic growth.

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