E C O N O M I C S B U L L E T I N

Empirical Test of Affine Stochastic Discount Factor Model of Currency Pricing

Alex Lebedinsky
Western Kentucky University

Abstract

In this note, I conduct an empirical investigation of the affine stochastic discount factor model proposed by Backus, Foresi and Telmer (2001), who showed that such a model might be able to explain the forward premium anomaly. Evidence presented here suggests that the model can reproduce the forward premium anomaly only by placing restrictions on the behavior of spot and forward exchange rates that are not consistent with the data. The model assumes that an increase in the forward premium must be accompanied by an increase in volatilities of the forward premium and of the exchange rate depreciation. I find that this assumption is not supported by the data. When I relax the model by allowing the forward premium to vary without necessarily affecting conditional second moments, the model no longer reproduces the forward premium anomaly. Thus, I conclude that the model can reproduce the anomaly only by forcing a linear heteroskedastic process that is not supported by the data.

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1 Introduction

In this note, I conduct an empirical investigation of the affine stochastic discount factor (SDF) model proposed by Backus, Foresi and Telmer (2001, henceforth BFT), who showed that such a model might be able to explain the forward premium anomaly[1]. BFT also demonstrate that the model has some drawbacks: Whenever it can account for the anomaly, it implies either non-zero probability of negative interest rates or unrealistic values for the price of risk (e.g. implying annual interest rates of 80%).

Evidence presented in this note suggests that these shortcomings are not the only ones associated with affine models. This type of models also assume that conditional second moments are linear functions of the state variables, and this assumptions is crucial for the model to be able to reproduce the forward premium anomaly. This linearity assumption places restrictions on the behavior of spot and forward exchange rates: An increase in the forward premium must be accompanied by an increase in volatility of the forward premium. I find that this type of conditional heteroskedasticity is not supported by the data. When I relax the model by allowing the forward premium to vary without necessarily affecting conditional second moments, the model no longer reproduces the forward premium anomaly. Thus, I conclude that the model can reproduce the anomaly only by forcing a linear heteroskedastic process that is not supported by the data.

The purpose of this note is not to find yet another dimension along which affine SDF models fail but to suggest a direction of future research on currency pricing models. BFT already found that the affine models with few state variables might not be able to account for the anomaly without making unrealistic assumptions. Results presented here suggest that affine structure itself might not be compatible with the data and that increasing the number of state variables will not solve this problem.

In the next section, I describe the nature of the anomaly and how SDF models can be applied to study this issue. In the third section, I present the model by BFT and discuss some of its implications. The fourth and fifth sections contain empirical estimates and tests of the model. I summarize and discuss the results in the conclusion.

2 Asset Pricing Models and Forward Premium Anomaly

The literature on the forward premium anomaly focuses on the regression equation

$$s_{t+1} - s_t = \alpha_1 + \alpha_2 (f_t - s_t) + v_t, \tag{1}$$

where s_t and f_t are logarithms of the spot and the forward exchange rates defined in dollars per unit of foreign currency. Many studies find estimates of α_2 that are negative and significantly different from zero. Most studies reject the null that $\alpha_2 = 1$ which is interpreted as a failure of the hypothesis that the forward rate is an unbiased predictor of the spot rate.

The data sample used here covers the period from January 1975 through December 2001[2]. Regression estimates of equation (1) based on this sample also show that α_2 is negative and

[1]For a reference on the SDF models, see Cochrane (2001), Duffie (2001) and Campbell et al. (1997). Engel (1996) is an extensive survey of the literature on forward premium anomaly.

[2]The dataset is from Bekaert and Hodrick (1993) updated with the data from Datastream.

significantly different from 0 for the yen/dollar exchange rate and significantly different from 1 for the pound/dollar and mark/dollar exchange rates (shown in Table 1).

To conserve space and avoid duplication, I will present only the key insights from BFT, showing how asset pricing models can be used for currency pricing and what kind of restrictions the forward premium anomaly places on asset pricing models. For a detailed explanation, I will refer the readers to the article by BFT.

The starting point is the pricing relationship

$$b_t = E_t \left[c_{t+1} M_{t+1} \right],$$

where b_t is the value of the state-contingent claim c_{t+1}, and M_{t+1} is the stochastic discount factor or pricing kernel. This relationship can be used to price currencies as well. Specifically, BFT show that, assuming no arbitrage opportunities and complete markets, the logarithm of exchange rate depreciation can be expressed as

$$s_{t+1} - s_t = \log M^*_{t+1} - \log M_{t+1}, \tag{2}$$

where the asterisk denotes the pricing kernel for the assets denominated in foreign currency. The forward premium can be written as

$$f_t - s_t = \log E_t \left(M^*_{t+1} \right) - \log E_t \left(M_{t+1} \right). \tag{3}$$

The last two expressions show that the forward premium anomaly can arise due to the differences in behavior of the logarithms of pricing kernels and of the logarithms of expectations of pricing kernels. As BFT pointed out, the slope coefficient in the forward premium regression (which is equal to $Cov(s_{t+1} - s_t, f_t - s_t)/Var(f_t - s_t)$) can become negative only if (1) there is a negative correlation between differences in conditional means of the pricing kernels and differences in higher-order cumulants, and (2) greater variation in the latter.

3 BFT Model

One of the models that can allow for negative correlation between the forward premium and exchange rate depreciation is an affine SDF model similar to the bond pricing model by Cox, Ingersoll and Ross (1985). Affine SDF models have two components: state equations and kernel[3] equations. The BFT model specifies *state equations* in the following way:

$$\mathbf{z}_{t+1} = (I - \Phi)\, \theta + \Phi \mathbf{z}_t + V \left(\mathbf{z}_t \right)^{\frac{1}{2}} \boldsymbol{\varepsilon}_{t+1}, \tag{4}$$

where $\mathbf{z}_t$ and $\boldsymbol{\varepsilon}_t$ are vectors of state variables and innovations respectively. Vector $\mathbf{z}_t$ consists of n state variables $z_{i,t}$, $i = \{1, 2, ..., n\}$. Vector $\boldsymbol{\varepsilon}_{t+1}$ consists of n innovations $\varepsilon_{i,t+1}\tilde{}N(0,1)$.

[3] The structure of the kernel equations is slightly different from the setup by BFT, who did not include i.i.d. innovation η. This random error allows for an additional source of innovations to the pricing kernel (and exchange rates) that does not have any impact on contemporaneous or future short rates. Campbell et. al (1997, Ch. 11) discuss a similar setup of affine models as I present here. I include this innovation term because, without it, in the single-state model the exchange rate depreciation must exhibit as much persistence as the forward premium. Table 8 shows that this is not true – autocorrelation of the exchange rate depreciation is close to zero, and the forward premium is very persistent.

Matrix $V(\mathbf{z}_t)$ is diagonal with $v_{i,i} = \sigma_i^2 z_{i,t}$, Matrix Φ is stable with positive roots. Vector θ contains only positive elements.

Kernel equations are

$$\begin{aligned} -m_{t+1} &= \delta + \boldsymbol{\gamma}'\mathbf{z}_t + \boldsymbol{\lambda}'V(\mathbf{z}_t)^{\frac{1}{2}}\boldsymbol{\varepsilon}_{t+1} + \eta_{t+1}, \\ -m_{t+1}^* &= \delta^* + \boldsymbol{\gamma}^{*\prime}\mathbf{z}_t + \boldsymbol{\lambda}^{*\prime}V(\mathbf{z}_t)^{\frac{1}{2}}\boldsymbol{\varepsilon}_{t+1} + \eta_{t+1}^*, \end{aligned} \tag{5}$$

where $\boldsymbol{\gamma}, \boldsymbol{\gamma}^*, \boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^*$ are $n \times 1$ vectors of constants, δ, δ^* are scalars and innovations η and η^* are normal random variables with zero means and variances σ_η^2 and σ_η^{*2} respectively.

Using the lognormality assumption, the logarithm of the expectation of pricing kernel is

$$\log E_t\left(M_{t+1}^*\right) = E_t\left[m_{t+1}\right] + \frac{1}{2}Var_t\left[m_{t+1}\right].$$

Substituting kernel equations (5) into this expression and using equation (3), we can write the forward premium as a function of state variables:

$$f_t - s_t = (\delta - \delta^*) - \frac{\sigma_\eta^2 - \sigma_{\eta^*}^2}{2} + \left[(\boldsymbol{\gamma} - \boldsymbol{\gamma}^*) - (\boldsymbol{\tau} - \boldsymbol{\tau}^*)\right]\mathbf{z}_t,$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\tau}^*$ are vectors whose ith elements are $\tau_i = \lambda_i^2\sigma_i^2/2$ and $\tau_i^* = \lambda_i^{*2}\sigma_i^2/2$. This equation shows that, in an affine model, the forward premium is a linear function of the state variables.

Depreciation of the spot exchange rate also can be expressed in terms of the state variables. From equations (2) and (5), exchange rate depreciation is

$$s_{t+1} - s_t = (\delta - \delta^*) + (\boldsymbol{\gamma} - \boldsymbol{\gamma}^*)\mathbf{z}_t + (\boldsymbol{\lambda} - \boldsymbol{\lambda}^*)' V(\mathbf{z}_t)^{\frac{1}{2}}\varepsilon_{t+1} + \eta_{t+1} - \eta_{t+1}^*. \tag{6}$$

To reproduce the forward premium anomaly, the model must be able to generate negative covariance between exchange rate depreciation and the forward premium. In the BFT model, this covariance is

$$Cov\left(f_t - s_t, s_{t+1} - s_t\right) = \left[(\boldsymbol{\gamma} - \boldsymbol{\gamma}^*) - (\boldsymbol{\tau} - \boldsymbol{\tau}^*)\right]' V(\mathbf{z}_t)(\boldsymbol{\gamma} - \boldsymbol{\gamma}^*). \tag{7}$$

To account for the anomaly, state variables must affect pricing kernels asymmetrically: The anomaly requires that at least one state variable that has a greater effect on the level of one kernel (e.g. $\gamma_i > \gamma_i^*$) and the variance of another ($\tau_i < \tau_i^*$). This requirement imposes a number of restrictions on the data. To understand the nature of these restrictions, consider the following two expression that describe innovations to exchange rate depreciation and forward premium. Innovation to the exchange rate depreciation is

$$\psi_{t+1} \equiv (s_{t+1} - s_t) - E_t(s_{t+1} - s_t) = (\boldsymbol{\lambda} - \boldsymbol{\lambda}^*) V(\mathbf{z}_t)^{\frac{1}{2}}\varepsilon_{t+1} + \eta_{t+1} - \eta_{t+1}^*,$$

and the innovation to the forward premium is

$$\xi_{t+1} \equiv (f_{t+1} - s_{t+1}) - E_t(f_{t+1} - s_{t+1}) = \left[(\boldsymbol{\gamma} - \boldsymbol{\gamma}^*) - (\boldsymbol{\tau} - \boldsymbol{\tau}^*)\right]' V(\mathbf{z}_t)^{\frac{1}{2}}\varepsilon_{t+1}.$$

Conditional variance of the exchange rate depreciation $Var_t(s_{t+1} - s_t) = Var_t(\psi_{t+1})$ can be expressed as a linear function of the state variables:

$$Var_t(\psi_{t+1}) = (\boldsymbol{\lambda} - \boldsymbol{\lambda}^*)' V(\mathbf{z}_t)(\boldsymbol{\lambda} - \boldsymbol{\lambda}^*) + Var(\eta - \eta^*). \tag{8}$$

The anomaly requires that $(\boldsymbol{\tau} - \boldsymbol{\tau}^*) \neq \mathbf{0}$, which implies that $(\boldsymbol{\lambda} - \boldsymbol{\lambda}^*) \neq \mathbf{0}$. Therefore, conditional variance of $s_{t+1} - s_t$ must be correlated with the forward premium because the latter is also a linear function of the state variables.

Similarly conditional variance of the forward premium $Var_t(f_{t+1} - s_{t+1})$ is

$$Var_t(\xi_{t+1}) = [(\boldsymbol{\gamma} - \boldsymbol{\gamma}^*) - (\boldsymbol{\tau} - \boldsymbol{\tau}^*)]' V(\mathbf{z}_t)[(\boldsymbol{\gamma} - \boldsymbol{\gamma}^*) - (\boldsymbol{\tau} - \boldsymbol{\tau}^*)] \tag{9}$$

and it is also a linear function of state variables. Therefore, $Var_t(\xi_{t+1})$ must be correlated with the forward premium.

The covariance of the two innovations is also a linear function of state variables

$$Cov_t(\psi_{t+1}, \xi_{t+1}) = (\boldsymbol{\lambda} - \boldsymbol{\lambda}^*)' V(\mathbf{z}_t)[(\boldsymbol{\gamma} - \boldsymbol{\gamma}^*) - (\boldsymbol{\tau} - \boldsymbol{\tau}^*)] \tag{10}$$

and, like the other two moments, must be correlated with forward premium.

4 Single-state Model

In this section, I present estimates of the single-state model. Using these estimates, I calculate implied conditional moments described in equations $(8-10)$ and test whether these implied moments fit the data.

If there is only one state variable, the state equation (4) becomes

$$z_{t+1} = \theta(1-\phi) + \phi z_t + \sigma z_t^{\frac{1}{2}} \varepsilon_{t+1}, \tag{11}$$

and the expressions for the forward premium and exchange rate depreciation can be written as

$$\begin{aligned} f_t - s_t &= (\delta - \delta^*) - \frac{\sigma_\eta^2 - \sigma_{\eta^*}^2}{2} + [(\gamma - \gamma^*) - (\tau - \tau^*)] z_t, & (12) \\ s_{t+1} - s_t &= (\delta - \delta^*) + (\gamma - \gamma^*) z_t + (\lambda - \lambda^*) \sigma z_t^{\frac{1}{2}} \varepsilon_{t+1} + \eta_{t+1} - \eta_{t+1}^*. & (13) \end{aligned}$$

For the empirical test of the model, it is not important to identify all of the structural parameters in the above two equations. To simplify the task of estimation and presentation of the results, I demean the data and estimate these reduced form equations:

$$\begin{aligned} f_t - s_t - \overline{(f_t - s_t)} &= (\nabla\gamma - \nabla\tau) z_t, & (14) \\ s_{t+1} - s_t - \overline{(s_{t+1} - s_t)} &= \nabla\gamma z_t + \nabla\lambda\sigma z_t^{\frac{1}{2}} \varepsilon_{t+1} + \nabla\eta_{t+1,} & (15) \end{aligned}$$

where ∇ denotes the difference between "domestic" and "foreign" parameters.

The model described in the above to equations and equation (11) is estimated using Maximum Likelihood. The results are presented in Table (2). Reduced form parameter $\nabla\tau$

represents the difference in the effect of the state variable on the conditional variances of the pricing kernels. Its estimates are negative and significant in all cases. Parameter $\nabla\gamma$ measures the effect of the state variable on the pricing kernel. The estimates are positive and significantly different from zero. These results support the idea behind the BFT model that the anomaly is due to the asymmetric effects of the state variable on the level and the conditional variance of the pricing kernels.

To find out whether the parameter estimates of the model actually reproduce the forward premium anomaly, I compute the slope coefficient of the forward premium regression using the parameter estimates of the model. Table 3 shows that, in all cases, the model successfully reproduces the anomaly. In all cases, estimates of the implied slope coefficient are very close to the OLS estimates of equation (1).

To test the implications of the model for the conditional second moments, I construct innovations ψ_{t+1} and ξ_{t+1} defined earlier. Using parameter estimates of the model, we can calculate $E_t(s_{t+1} - s_t)$ and $E_t(f_{t+1} - s_{t+1})$ and compare these expected values with realizations of $s_{t+1} - s_t$ and $f_{t+1} - s_{t+1}$. The differences are innovations ψ_{t+1} and ξ_{t+1}, respectively. Using these innovations I construct realizations of the conditional second moments: The square of these innovations is the realization of conditional variance and the product of ψ_{t+1} and ξ_{t+1} is the realization of their covariance. Then, I regress the difference between the realizations of the second moments and their implied values on the forward premium. For example, for the exchange rate depreciation, I estimate the following regression:

$$\psi_{t+1}^2 - \widehat{Var}_t(s_{t+1} - s_t) = \kappa_0 + \kappa_1 (f_t - s_t) + v_t, \tag{16}$$

where $\widehat{Var}_t(s_{t+1} - s_t)$ is the expected variance implied by the parameters estimates. If the model correctly characterizes conditional second moments then the regression intercept κ_0 and the slope κ_1 must not be significantly different from zero. Estimates of those coefficients are shown in Table 4.

These results show that, for each exchange rate, at least one of the these conditional moments is not characterized correctly. In the case of the pound/dollar exchange rate, the estimates reject characterization of the covariance and variance of the forward premium. For the mark/dollar rate, conditional variance of the forward premium is modeled incorrectly. For the yen/dollar rate, the conditional variance of both the exchange rate depreciation and the forward premium are modeled incorrectly.

5 Two-state Model

Certainly, it would be naive to think that a simple single-state model would be able to capture *all* of the features of the data. After all, the BFT model even with a single state variable does reproduce the forward premium anomaly, although second moments seem to be misspecified. However, these results do raise a question: would the model still reproduce the forward premium anomaly if it did not assume that *all* changes in forward premia will lead to changes in conditional moments? To explore this issue, I modify the model by adding a homoskedastic state variable. With the addition of this state variable, I can decompose the forward premium into two components: one that has an effect on the conditional variances and one that doesn't. This model also can generate the forward premium anomaly but only if the heteroskedastic state is responsible for most of the variation in the forward premium. If it turns out that most variation in the forward premium is due to the homoskedastic state variable, then we'll be able to conclude that affine models can reproduce the anomaly only by making invalid assumptions about conditional second moments.

The setup of the model is as follows:

$$\begin{aligned} -m_{t+1} &= \delta + \gamma z_t + \mu x_t + \lambda \sigma_z z_t^{\frac{1}{2}} \varepsilon_{t+1} + \eta_{t+1}, & (17) \\ -m_{t+1}^* &= \delta^* + \gamma^* z_t + \lambda^* \sigma_z z_t^{\frac{1}{2}} \varepsilon_{t+1} + \eta_{t+1}^*, & (18) \\ z_{t+1} &= \theta_z \left(1 - \phi_z\right) + \phi_z z_t + \sigma_z z_t^{\frac{1}{2}} \varepsilon_{t+1}, & (19) \\ x_{t+1} &= \theta_x \left(1 - \phi_x\right) + \phi_x x_t + \sigma_x \zeta_{t+1}. & (20) \end{aligned}$$

The two state variables are x_t and z_t. Subscripts on the parameters of the state equations indicate the parameters of the corresponding equations. As before, innovations are assumed to be uncorrelated with each other and normally distributed. The state variable x_t appears only in one of the kernel equations. I arbitrarily chose the domestic kernel[4].

As with the single-state model, the model is estimated using demeaned data in the following reduced form:

$$\begin{aligned} s_{t+1} - s_t - \overline{(s_{t+1} - s_t)} &= \nabla\gamma z_t + \mu x_t + \nabla\lambda \sigma_z z_t^{\frac{1}{2}} \varepsilon_{t+1} + \nabla\eta_{t+1}, & (21) \\ f_t - s_t - \overline{(f_t - s_t)} &= \left(\nabla\gamma - \nabla\tau\right) z_t + \mu x_t. & (22) \end{aligned}$$

To be able to identify the model described in equations (19-22), we need more information than just the exchange rate depreciation and the forward premium because the model contains a new factor x_t. To identify the two-state model, I use three-month forward premium in addition to the one-month forward premium. Because affine SDF models can be used to express long-term interest rates as linear functions of state variables, we can use this property of the model to express the forward premium with maturities beyond one month as a function of state variables. Thus, the three-month forward premium can be written as

$$f_t^3 - s_t - \overline{(f_t^3 - s_t)} = \mathbf{B} z_t + \mathbf{C} x_t,$$

where f_t^3 is the logarithm of the three-month forward rate and $\mathbf{B}$ and $\mathbf{C}$ are reduced form affine coefficients (derived in appendix).

Parameter estimates of the model are presented in table 5. These results show that state variables z and x exhibit different levels of persistence. The estimates of the autoregressive parameters ϕ_z and ϕ_x indicate that the homoskedastic state x_t is very persistent (ϕ_x is between 0.92 and 0.97), and the heteroskedastic state shows little autocorrelation (ϕ_z is between 0 and 0.17.)

Table 6 shows that, in all cases, the model does not reproduce the forward premium anomaly. Implied slope coefficients in the forward premium regression are between 0.77 and 1.24. The standard errors are small enough to reject the null hypothesis that the implied coefficients are negative.

[4]The choice is innocuous: if x_t were included into both kernel equations, the forward premium and the exchange rate depreciation equations would still have the same reduced form.

Table 7 presents the result of the specification tests of the model. As in the single-state version, I compute the innovations and test whether the difference between realizations and the expectations of conditional second moments is i.i.d. The results show that one cannot reject the hypothesis that the model correctly characterizes conditional moments for all three exchange rates.

To see which state variable explains more variation in the forward premia, I regress forward premia on x_t. The R^2 of these regressions are 0.88, 0.88 and 0.73 for pound, mark and yen exchange rates respectively. The homoskedastic state variable explains most of the variation in the forward premia[5]. This might explain why the forward premium anomaly is not reproduced by this model. Conditional heteroskedasticity is essential for the model's ability to reproduce the anomaly, and stochastic behavior of the forward premium for these three exchange rates is best explained by a homoskedastic state variable.

Thus, the estimates of the two-state version of the model show that it fails to reproduce the anomaly for all three countries. The tests indicate that one cannot reject the hypothesis that the conditional second moments are modeled correctly in the two-state version. Therefore, it appears that the model is not capable of simultaneously delivering the forward premium anomaly and correctly characterizing conditional second moments.

6 Conclusion

The object of this note is to contribute to the literature aimed at developing unified currency and asset pricing models. As BFT themselves pointed out, affine SDF models, although capable of reproducing the forward premium anomaly, do so only at the expense of making some unrealistic assumptions. Although the results presented here demonstrate another dimension where affine models fail, the main point is not to criticize further the model that was already to shown to have several shortcomings but to explore future directions of research.

The main finding presented here is that the data don't seem to fit the structure of the affine SDF models. This finding implies that a possible candidate for an SDF model that can reproduce the forward premium anomaly might be found outside of the class of affine models. Bansal (1997), for example, developed a general framework of SDF models that nests affine models. He showed that it is possible to reproduce the forward premium anomaly with a symmetric (state variables affect countries in the same way) model that does not imply a linear relationship between the conditional variance of the forward premium and forward premium itself. Such model and other models that allow either non-linear relationship between the states and the conditional moments (e.g. Constantinides (1992) or Ahn et al. (2002)) or variability in higher moments of the kernel and state equations likely will be able to fit the data and generate the forward premium anomaly.

[5] The sample correlations between the state variables x and z are 0.03 or less. Therefore, regression on the heteroskedastic variable would be a mirror image of this regression, because forward premia are exact functions of the two state variables

Appendix

Deriving Parameters of the Affine Model

Here I show how the parameters **B** and **C** are derived. The state and kernel equations are

$$\begin{aligned} -m_{t+1} &= \delta + \gamma z_t + \mu x_t + \lambda \sigma_z z_t^{\frac{1}{2}} \varepsilon_{t+1} + \eta_{t+1}, \\ z_{t+1} &= \theta_z (1 - \phi_z) + \phi_z z_t + \sigma_z z_t^{\frac{1}{2}} \varepsilon_{t+1}, \\ x_{t+1} &= \theta_x (1 - \phi_x) + \phi_x x_t + \sigma_x \zeta_{t+1}. \end{aligned}$$

From these equations one can find the domestic short rate:

$$\begin{aligned} r_{1,t} &= -\log E_t M_{t+1} = -E_t m_{t+1} - \frac{1}{2} Var(m_t) \\ &= \delta + \left(\gamma - \frac{\lambda^2 \sigma_z^2}{2} \right) z_t + \mu x_t - \frac{\sigma_\eta^2}{2}. \end{aligned}$$

To get the solution for an n-period interest rate, guess $r_{n,t} = A_n + B_n z_t + C_n x_t$.

Under log-normality assumption, it should be true that

$$b_{n,t} = E_t \left[m_{t+1} + b_{n-1,t+1} \right] + (1/2) Var_t \left[m_{t+1} + b_{n-1,t+1} \right],$$

where $b_{n,t}$ is the price at time t of the bond maturing at $t+n$. Consider the two components of this expression separately:

$$\begin{aligned} E_t \left[\log m_{t+1} + b_{n-1,t+1} \right] &= -\delta - \gamma z_t - \mu x_t - A_{n-1} \\ &\quad - B_{n-1} (1 - \phi_z) \theta_z - B_{n-1} \phi_z z_t - \\ &\quad - C_{n-1} (1 - \phi_x) \theta_x - C_{n-1} \phi_x x_t, \end{aligned}$$

$$Var_t \left[\log m_{t+1} + b_{n-1,t+1} \right] = \lambda^2 \sigma_z^2 z_t + B_{n-1}^2 \sigma_z^2 z_t + 2 B_{n-1} \sigma_z^2 \lambda z_t + C_{n-1}^2 \sigma_x^2 x_t + \sigma_\eta^2.$$

Therefore, the negative of the price of an n-period bond (which is equal to $r_{n,t}$) is

$$\begin{aligned} -b_{n,t} &= \left(B_{n-1} \phi_z + \gamma - \frac{1}{2} B_{n-1}^2 \sigma_z^2 - B_{n-1} \sigma_z^2 \lambda - \frac{1}{2} \lambda^2 \sigma_z^2 \right) z_t + \\ &\quad \left(C_{n-1} \phi_x + \mu - \frac{1}{2} C_{n-1}^2 \sigma_x^2 \right) x_t + \\ &\quad \delta + A_{n-1} + B_{n-1} \theta_z (1 - \phi_z) + C_{n-1} \theta_x (1 - \phi_x) - \frac{1}{2} \sigma_\eta^2 \end{aligned}$$

From the last expression one can get the recursive solutions for A_n and B_n and C_n

$A_n = \delta + A_{n-1} + B_{n-1} \theta_z (1 - \phi_z) + C_{n-1} \theta_x (1 - \phi_x) - \frac{1}{2} \sigma_\eta^2,$

$B_n = B_{n-1} \phi_z + \gamma - \frac{1}{2} B_{n-1}^2 \sigma_z^2 - B_{n-1} \sigma_z^2 \lambda - \frac{1}{2} \lambda^2 \sigma_z^2,$

$C_n = C_{n-1} \phi_x + \mu - \frac{1}{2} C_{n-1}^2 \sigma_x^2.$

B_3, C_3 can be found solving recursively and using the starting values $B_1 = \gamma, C_1 = \mu$.

Finally, **B** $= B_3 - B_3^*$ and **C** $= C_3$.

Estimation and Identification

Parameters $(\gamma - \gamma^*), \lambda, \lambda^*, \sigma_z$ and σ_x with μ in the two-state model are identified only up to a factor of proportionality. Normalizing σ_x and σ_z at 0.1 allows to identify the rest of model parameters. Changing these two parameters simply rescales the state variables without having a real effect on the model. To see why these parameters are identified only up to a factor of proportionality, consider the state equation from the single-state model

$$z_{t+1} = (1-\phi)\,\theta + \phi z_t + \sigma z_t^{\frac{1}{2}} \varepsilon_{t+1}.$$

Define $w_{t+1} = \frac{z_{t+1}}{\sigma^2}$ and substitute it into the state equation:

$$\sigma^2 w_{t+1} = (1-\phi)\,\theta + \phi \sigma^2 w_t + \sigma \left(\sigma^2 w_t\right)^{\frac{1}{2}} \varepsilon_{t+1}.$$

Dividing by σ^2 yields

$$w_{t+1} = (1-\phi)\,\theta' + \phi w_t + w_t^{\frac{1}{2}} \varepsilon_{t+1}$$

where $\theta' = \frac{\theta}{\sigma^2}$.

Turning to the kernel equation:

$$\begin{aligned} -m_{t+1} &= \delta + \gamma \sigma^2 w_t + \lambda \left(\sigma^2 w_t\right)^{\frac{1}{2}} \varepsilon_{t+1}, \\ -m_{t+1} &= \delta + \gamma \sigma^2 w_t + \lambda \sigma \left(w_t\right)^{\frac{1}{2}} \varepsilon_{t+1}, \\ -m_{t+1} &= \delta + \gamma' w_t + \lambda' \left(w_t\right)^{\frac{1}{2}} \varepsilon_{t+1}. \end{aligned}$$

Therefore, after rescaling the state equation, the kernel equation retains the same form.

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Tables

Table 1: Forward Premium Regression Estimates

	GBP/USD	**DEM/USD**	**JPY/USD**
α_1	-0.0032 (0.0022)	0.0010 (0.0021)	0.0065** (0.0025)
α_2	-0.8751 (0.6790)	-0.5896 (0.6763)	-1.4889** (0.6369)
R^2	0.005	0.002	0.016

Table reports estimates of the regression equation $s_{t+1} - s_t = \alpha_1 + \alpha_2 (f_t - s_t) + v_t$. The numbers in parentheses are standard errors. Double asterisk indicates significance at 5% level. The estimates were obtained using OLS.

Table 2: Estimates of the Single-State Model

	GBP/USD		**DEM/USD**		**JPY/USD**	
ϕ	0.8258**	(0.0203)	0.8953**	(0.0183)	0.6929**	(0.0212)
θ	0.9041**	(0.1847)	0.5702**	(0.0757)	0.9070**	(0.2405)
$\nabla\gamma$	-0.0209**	(0.0035)	-0.0171**	(0.0028)	-0.4810**	(0.0102)
$\nabla\lambda$	-0.0570**	(0.0241)	-0.0393	(0.0339)	-0.0458	(0.0327)
$\nabla\tau$	-0.0435**	(0.0051)	-0.0432**	(0.0035)	-0.0814**	(0.0139)
$Var(\eta - \eta^*)$	-0.0020**	(0.0001)	0.0021**	(0.0005)	0.0024**	(0.0002)

Estimates are obtained using MLE. The numbers in parentheses are standard errors associated with each coefficient. Standard errors are computed using the outer product of gradients of the likelihood function. Single * indicates significance at 10% level, and ** indicates significance at 5% level.

Table 3: Single-State Model: Comparison of the OLS Estimates and Implied Slope of the Forward Premium Regression

	GBP/USD		DEM/USD		JPY/USD	
α_2-Implied	-0.9294	(0.6114)	-0.6521	(0.6765)	-1.4425**	(0.6318)
α_2-OLS estimate	-0.8751	(0.6790)	-0.5896	(0.6763)	-1.4889**	(0.6369)

The numbers in parentheses are standard errors. The implied slope coefficient of the forward premium regression equation is $\alpha_2 = \frac{\nabla\gamma}{\nabla\tau - \nabla\gamma}$. Sandard errors for the implied coefficients are computed from the parameter estimates using the delta method. Single * indicates significance at 10% level, and ** indicates significance at 5% level.

Table 4: Testing the Implications of the Single-State model

1. Equation: $\psi_{t+1}^2 - \widehat{Var}_t(s_{t+1} - s_t) = \gamma_0 + \gamma_1 (f_t - s_t) + \upsilon_t$						
	GBP/USD		DEM/USD		JPY/USD	
γ_0	-0.0006**	(0.0001)	0.0010**	(0.0001)	-0.0001	(0.0002)
γ_1	-0.0219	(0.0512)	-0.0404	(0.0395)	0.2312**	(0.0575)
R^2	0.001		0.002		0.048	
2. Equation: $\xi_{t+1}^2 - \widehat{Var}_t(f_{t+1} - s_{t+1}) = \gamma_0 + \gamma_1 (f_t - s_t) + \upsilon_t$.						
	GBP/USD		DEM/USD		JPY/USD	
γ_0	-4×10^{-6}**	(1×10^{-6})	-1×10^{-6}	(1×10^{-6})	-1×10^{-6}	(2×10^{-6})
γ_1	0.0006**	(0.0002)	-0.0009**	(0.0002)	-0.0014**	(0.0005)
R^2	0.027		0.037		0.023	
3. Equation: $\psi_{t+1}\xi_{t+1} - \widehat{Cov}_t(f_{t+1} - s_{t+1}, s_{t+1} - s_t) = \gamma_0 + \gamma_1 (f_t - s_t) + \upsilon_t$						
	GBP/USD		DEM/USD		JPY/USD	
γ_0	1.8×10^{-5}**	(4×10^{-6})	$6\times10^{-6}*$	3×10^{-6}	1.4×10^{-5}	(1×10^{-5})
γ_1	-0.0059**	(0.0012)	-0.0017*	0.0009	-0.0025	(0.0025)
R^2	0.064		0.010		0.003	

The numbers in parentheses are OLS standard errors associated with each coefficient. Using parameter estimates of the model, I construct innovations ψ_{t+1} and ξ_{t+1} of the exchange rate depreciation and the forward premium respectively: Squaring these innovations gives realizations of the variance, and computing their product gives the covariance of ψ_{t+1} and ξ_{t+1}. Then, I regress the difference between the realizations of the second moments and their conditional expectations on the forward premium which is a linear function of the state variable z_t. Single * indicates significance at 10% level, and ** indicates significance at 5% level.

Table 5: Estimates of the Two-State Model

	GBP/USD		**DEM/USD**		**JPY/USD**	
ϕ_z	0.1738**	(0.0306)	0.0015	(0.0128)	0.0120	(0.0251)
ϕ_x	0.9539**	(0.0054)	0.9727**	(0.0067)	0.9245**	(0.0115)
θ_z	0.1003**	(0.0032)	0.2053**	(0.0018)	2.2570**	(0.0201)
$\nabla\gamma$	0.1260**	(0.0068)	-0.0314**	(0.0034)	0.0007	(0.0014)
$\nabla\lambda$	-0.0054*	(0.0029)	-0.0021	(0.0024)	0.0006	(0.0014)
$\nabla\tau$	0.0982**	(0.0070)	-0.0517**	(0.0032)	0.0111**	(0.0013)
μ	-0.0099**	(0.0001)	-0.0075**	(0.0001)	0.0115**	(0.0002)
B_3	0.0175**	(0.0016)	0.0004	(0.0011)	-0.0030**	(0.0004)
$Var\,(\eta - \eta^*)$	0.0320**	(0.0004)	0.0331**	(0.0006)	0.0354**	(0.0006)

The numbers in parentheses are standard errors computed using the outer product of the gradients. Single * indicates significance at 10% level, and ** indicates significance at 5% level.

Table 6: Two-State Model: Comparison of the OLS Estimates and Implied Slope of the Forward Premium Regression

	GBP/USD		**DEM/USD**		**JPY/USD**	
α_2-Implied	1.2412**	(0.0753)	0.8101**	(0.0882)	0.7759**	(0.0752)
α_2-OLS Estimate	-0.8751	(0.6790)	-0.5897	(0.6763)	-1.4889**	(0.6369)

The numbers in parentheses are standard errors. The implied coefficient of the forward premium regression is $\alpha_2 = 1 + \frac{\nabla\tau(\nabla\tau - \nabla\gamma)Var(z)}{(\nabla\tau-\nabla\gamma)^2 Var(z) + \mu^2 Var(x)}$. Standard errors for the implied coefficients are computed from the parameter estimates using the delta method. Single * indicates significance at 10% level, and ** indicates significance at 5% level.

Table 7: Testing the Implications of the Two-State model

1. Equation: $\psi_{t+1}^2 - \widehat{Var}_t(s_{t+1} - s_t) = \gamma_0 + \gamma_1 z_t + \upsilon_t$.						
	GBP/USD		**DEM/USD**		**JPY/USD**	
γ_0	-1.5×10^{-4}	(-1.4×10^{-4})	-4×10^{-5}	(1.1×10^{-4})	-0.0003	(1.6×10^{-4})
γ_1	-0.0823*	(0.0448)	0.0264	(0.0360)	0.1141	(0.4050)
R^2	0.010		0.002		0.002	
2. Equation: $\xi_{t+1}^2 - \widehat{Var}_t(f_{t+1} - s_{t+1}) = \gamma_0 + \gamma_1 z_t + \upsilon_t$.						
	GBP/USD		**DEM/USD**		**JPY/USD**	
γ_0	0.0000	(1×10^{-6})	0.0000	(1×10^{-6})	-1×10^{-6}	(2×10^{-6})
γ_1	8.3×10^{-5}	(1.9×10^{-4})	1.9×10^{-4}	(2.0×10^{-4})	2.7×10^{-4}	(4.8×10^{-4})
R^2	0.001		0.002		0.001	
3. Equation: $\psi_{t+1}\xi_{t+1} - \widehat{Cov}_t(f_{t+1} - s_{t+1}, s_{t+1} - s_t) = \gamma_0 + \gamma_1 z_t + \upsilon_t$.						
	GBP/USD		**DEM/USD**		**JPY/USD**	
γ_0	-4×10^{-4}	(0.0001)	-3.6×10^{-6}	(-2.0×10^{-5})	1×10^{-6}	(4×10^{-6})
γ_1	-0.0012	(0.0008)	-2.7×10^{-4}	(6.5×10^{-4})	-0.0014	(0.0010)
R^2	0.003		0.001		0.006	

The numbers in parentheses are standard errors associated with each coefficient.
Using parameter estimates, I construct innovations ψ_{t+1} and ξ_{t+1} of the exchange rate depreciation and the forward premium respectively. Squaring these innovations gives realizations of the variance, and computing their product gives the covariance of ψ_{t+1} and ξ_{t+1}. Then, I regress the difference between the realizations of the second moments and their conditional expectations on the state variable z_t. Single * indicates significance at 10% level, and ** indicates significance at 5% level.

Table 8: Descriptive Statistics of the Data

	GBP/USD	**DEM/USD**	**JPY/USD**
$E(s_t - s_{t-1})$	-0.0015	0.0001	0.0025
$E(f_t - s_t)$	-0.0019	0.0015	0.0026
$Var(s_t - s_{t-1})$	0.0010	0.0011	0.0012
$Var(f_t - s_t)$	6.8×10^{-6}	7.3×10^{-6}	9.4×10^{-6}
$Corr(s_{t+1} - s_t, s_t - s_{t-1})$	0.0664	0.0095	0.0509
$Corr(f_{t+1} - s_{t+1}, f_t - s_t)$	0.8067	0.8443	0.6682