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Self and Social Signaling in Religious Organizations: A Model and a Study of Calvin''s Reformation in Geneva

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Abstract

As Weber (1904) recognized, Calvinistic beliefs about predestination -perhaps counter-intuitively- constitute a powerful incentive for good works; an individual wishes to receive assurances about her future prospects of salvation, and good works may provide a positive signal about such prospects. These beliefs can in turn create a social pressure to perform, as good works can also signal to others that individuals belong to the Elect, and are therefore likely to behave well in the future. We focus on these self and social signaling incentives, and show how religious organizations that promote them affect levels of cooperation and coordination. We show that religious organizations that promote social signaling through good works provide higher welfare the more information is provided about individuals'' behavior. We contrast these organizations with those that promote social signaling through rituals (as in Levy and Razin (2009)). Our analysis suggests that Calvinistic communities that invest in institutions of information dissemination might promote higher levels of welfare to the community. This accords with the success of Calvin''s Reformation in Switzerland and France, a process characterized by the reduction of rituals along with the creation of institutions to monitor and publicize individuals'' behavior, such as the Consistory.

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Gradualism in dynamic influence games

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Abstract: We analyze a dynamic model in which players compete in each period in an all-pay competition to have their ideal action implemented. The winning policy at each competition is implemented for that period, but only if it is ranked higher than the status quo, according to some exogenous order. We show that in any subgame perfect equilibrium of this game, the dynamic process is gradual, i.e., in each period: (i) there is a substantial probability that a higher ranked action is implemented, but, (ii) the probability that the highest ranked action is implemented is bounded away from one. "Progress" is thus inevitable but relatively slow. In an application to a one-dimensional policy space, we show the existence of a fully gradual equilibrium in which all feasible actions are implemented at some period, before converging to the preferred action of the median voter. Such full gradualism arises when the polity is sufficiently polarized and interested parties have policy -as opposed to winning- motives.

1. Introduction

In this paper we examine the dynamic evolution of political processes in which a policy has to be chosen in every period. Our model has two main features. First, we model the political competition in each period as an all-pay influence game. The all-pay assumption, a traditional assumption in Political Economy, is motivated by the contractual constraints implicit in political transactions, and seems suitable for many environments: In political campaigns, the resources expanded by the players can be interpreted as effort or money used in advocating their policies, whereas in processes such as regime changes, such influence efforts could be interpreted as physical, e.g.

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military, efforts. Starting with the seminal paper by Becker (1983) the literature has mainly focused on static influence games.[2] In this paper we examine the implications of such all-pay influence games when the political process is dynamic.

The second feature of our model is an exogenous constraint on policy changes. Many political dynamic processes follow some particular direction of change: In the case of political campaigns, reforms may be implemented only if they can gather enough electoral support against the status quo. Thus, the preferences of the median voter impose some constraints on which policies might replace the current policy. In the case of regime changes, political constraints may imply that countries can only become more democratic with the extension of the franchise to more and more groups, as withdrawing voting rights once given may prove more difficult.[3]

Naturally, the degree to which the direction of change is constrained may vary in different environments and applications. In this paper we assume that the direction of change is exogenous (in the eyes of the relevant interested parties). Our focus is therefore on the evolution of this decentralized dynamic process in which players compete to influence whether and how fast the process moves. Do policy reforms evolve gradually, i.e., through successive small policy changes, or do they progress in large steps?[4] Does the process get "stuck" on certain policies or does it always move forward?

Specifically, we analyze an infinite-horizon influence game, in which at each period an action is implemented from a fixed set of actions for the duration of that period. A finite number of interested players engage in an all-pay competition in which they exert resources. The winner of this competition implements his ideal policy at this period, but only if it is ranked (weakly) higher than the status quo (the previously implemented policy) according to some exogenous linear order. As this is a dynamic game, players compete to affect the current implemented action as well as the dynamics of the process.

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[2] Some exceptions to the literature are Klumpp and Polborn (2006) who analyze a dynamic competition (primaries) between two candidates, and Konrad and Kovenock (2009) who use influence functions to analyze patent races.

[3] Lizzeri and Persico (2004) analyze the extension of the franchise (but not whether to remove voting rights). Conversely, Acemoglu, Ticchi, and Vindigni (forthcoming) analyze a model in which it is military dictatorships which cannot be reversed.

[4] In a model in which decisions are taken centrally by a government, Dewatripont and Roland (1995) ask whether reforms follow a "bang-bang" path or whether they are gradual. Our analysis asks a similar question albeit in a decentralized process of policy implementation.

Rather than focusing on a particular all-pay mechanism, we analyze a general family of influence functions. This family includes the generalized Tullock functions and all-pay auctions, which are the most common forms of influence functions used in the literature.[5] An important feature of these functions is that a player can always secure a high enough probability of winning by exerting enough resources relative to others'. We use this assumption to characterize our family of all-pay mechanisms.

Our main result, Theorem 2, shows that any subgame perfect equilibrium involves *gradualism*: (i) in each period there is a strictly positive lower bound on the probability that a higher ranked action is implemented, (ii) in each period, the probability that the highest ranked action is implemented is bounded away from one. The bounds above are uniform; they do not depend on the length of periods.

The key step in the proof of our main result is to establish that the willingness to win of players are of comparable magnitude.[6] The proof involves an induction on the state of the game, i.e., the policy that is currently implemented. We establish first that the willingness to win of the player who represents the highest ranked policy is never too low relative to that of others, due to his advantage in the linear order. This allows us to prove, using our assumption about the all-pay competition, that the process must always move forward with a strictly positive probability. Therefore, the game will endogenously terminate, with the implementation of the highest ranked action, in finite time in expectations. But this implies that all players "fight" for only a finite number (in expectations) of instantaneous benefits. Thus, the willingness to win of each player is comparable to that of the others; this allows us to conclude that the player representing the highest ranked action (or any player that represents progress) cannot win at any stage with too high a probability. In other words the dynamic advantage of the highest ranked player does not translate to a quick resolution of the process.

While Theorem 2 provides some bound on the rate of gradualism we consider some applications to measure this rate more precisely. We first analyze a two player dynamic and asymmetric all-pay-auction. We show that this game has a unique subgame perfect equilibrium and we relate the rate of gradualism to the parameters of the model.

[5] See for example Skaperdas (1996) and the survey in Konrad (2009).

[6] With negative externalities, the willingness to win of a player depends on the nature of the particular equilibrium and is thus not straightforward. We define a notion of willingness to win which involves the bilateral willingness to win between pairs of players.

We then consider an application to a political process with one-dimensional policy space, single peaked preferences and a direction of progress according to the preferences of the median voter. This application, using Theorem 2, provides a dynamic foundation to the median voter result as convergence to this policy will arise in finite time in expectations.[7] We show in this application that in some environments there exist equilibria with *full gradualism*: in such equilibria the player representing the best policy for the median voter chooses to compete only once all other policies have been implemented. We show that these equilibria arise irrespective of the length of periods and that the existence of these particularly slow equilibria hinges on negative externalities and the level of polarization in society.

Our paper contributes to the large literature on influence games.[8] We analyze a dynamic influence game and characterize the results for a general set of influence functions. Also related is the literature on political transitions or regime changes, with pioneering contributions by Acemoglu and Robinson (2001, 2008) and recently Acemoglu, Ticchi, and Vindigni (forthcoming). We complement this literature by focusing on the path of convergence to steady states for a general family of influence games.[9] Related to our application is the literature on political entry games, which has mainly focused on static entry games with exogenous fixed cost (see Besley and Coate (1997) or Osborne and Slivinski (1996)).

Several papers have analyzed gradualism in different contexts, albeit stemming from different reasons than the one analyzed in our model. Compte and Jehiel (2004) analyze a bargaining game in which the outside option of the players depends on previous offers. Admati and Perry (1991) show that an agent holds back his payments in contribution games to insure that the other agent contributes his share as well. For gradualism in irreversible repeated games, see Lockwood and Thomas (2002) and

[7] Baron (1996) shows in a dynamic legislative bargaining game in which the status quo is the default policy in the absence of agreement, that as long as the median voter has a strictly positive probability of being recognized to make an offer, then the equilibrium policies converge to the median voter's ideal policy. In contrast, our model has no exogenous probability which allows the median voter or the player who represents his policy to make an offer but on the other hand, this player has the advantage of being able to "terminate" the game.

[8] See Becker (1983) or Grossman and Helpman (1994) which have used contest functions or auctions to model how players can directly affect political outcomes.

[9] Another related literature is that of endogenous agenda formation, where several papers extend bargaining games to include a stage in which agents compete via an all-pay-auction for the right to propose a policy (see for example Zwiebel and Board (2005), Yildrim (2007) and Evans (1997)).

Watson (2002). Finally, in a multistage patent race game among two players, Konrad and Kovenock (2009) show that an agent who is losing in the patent race still does not give up, as long as he can win some strictly positive instantaneous prize.

The rest of the paper is organized as follows. In the next section we present the model. In Section 3 we present our main result of gradualism. Section 4 considers an application to a one-dimensional political economy model, in which we characterize the conditions under which there exists a fully gradual equilibrium. An appendix contains all proofs not in the text.

2. The model

We analyze an infinite-horizon game of complete information with a finite set of players, $M = \{1, 2, ..., m\}$.

Feasible actions and preferences: Each player i represents an action $x_i \in X$. The utility of player i from action x_j is v_{ij}, where $-\infty < v_{ij} < 0$ for $i \neq j$ and $v_{ii} = 0$. This specification includes the possibility of negative externalities, that is, when players' preferences depend on the identity of the winner, or the possibility of pure winning motives (when $v_{ij} = v_{ik}$ for all $j, k \neq i$).

Implemented actions and the direction of the process (irreversibility): At each period $t \in \{1, 2, ...\}$ an action $x^t \in \{x_1, x_2, ..., x_m\}$ is implemented for the length of that period (all periods are of equal length). We assume that there is an exogenous linear (complete and transitive) order on $\{x_1, x_2, ..., x_m\}$. Without loss of generality, we assume that the index of actions represents the order, so that actions with lower index are ranked higher.

Let s^t be the index of the policy that was implemented in period $t - 1$ and fix $s^1 = m$. Our assumption on the direction of the process means that at each period t, only some action x_i with $i \leq s^t$ can be implemented. To this end, we assume that in the beginning of each period t, only players $\{1, 2, .., s^t\}$ engage in an all-pay competition (whose details we specify below). Let the winner of the competition at time t be denoted by i^t. The action x^t evolves according to $x^t = x_{i^t}$. The process is therefore irreversible with the action x_1, once implemented, becoming an absorbing state (as no player will fight anymore).

Remark 1 *An alternative and perhaps a more realistic assumption of irreversibility is to allow all M players to compete, and let the dynamic process at time t evolve* *according to $x^t \in \min\{x_{s^t}, x_{i^t}\}$, i.e., allow either the status quo or the winning policy of period t to be implemented at period t, whichever is better according to the linear order. The only difference this assumption would make is that in this specification of the model players with $i > s^t$ might compete to potentially support s^t, which will mainly increase the number of equilibria but will not affect our results. We therefore choose to proceed with the simpler model. Note that also in the simple model free riding and equilibrium multiplicity might arise as players with close enough ideal policies might let the other one fight in their stead.*

Remark 2 *Note that in the above model players can only propose the action they represent. Alternatively one can analyze a model in which players strategically choose policies from a set of (finite) feasible policies. Our main result, Theorem 2, can be generalized to this alternative model.*

The all-pay competition: We now describe the all-pay competition that the players engage in at each period t. Each player $i \leq s^t$ places a bid $b_i \geq 0$ which he must pay regardless of the outcome. Bids are placed simultaneously at the beginning of period t. To ensure equilibrium existence, suppose that (at each stage) b_i is in the compact interval $[0, B]$ for some large finite B. The probability with which player $i \leq s^t$ wins the competition at stage t is determined according to a function $H_i^{s^t}(\mathbf{b})$ where $\mathbf{b}$ is an s^t-dimensional vector of bids.

The political economy literature offers several formulations of influence functions. One such formulation is the parametrized Tullock contest function, with $H_i^{s^t}(\mathbf{b}) = \frac{b_i^r}{\sum_{j \leq s^t} b_j^r}$ for $i \leq s^t$, which for $r = 1$ is referred to as the simple Tullock function, and with $r \to \infty$, approximates the all-pay-auction.[10] Rather than assuming any particular formulation our approach is to analyze a family of influence functions that embody the main features of the existing ones (without imposing for example any symmetry or monotonicity that are implicit above). We therefore assume that the function $H_i^{s^t}(.)$ satisfies the following properties:

H1. *For any $K > 0$, there exists a $K' > 0$ such that if $b_i = \max_j b_j$ and $\frac{b_i}{b_j} > K'$ then $\frac{H_i^{s^t}(\mathbf{b})}{H_j^{s^t}(\mathbf{b})} > K$.*

H2. *There exists a $\kappa < \infty$, such that for every ε, for any player i and bids $\mathbf{b}, \mathbf{b}'$, such that $b_j = b'_j$ for all $j \neq i$, and $H_i^{s^t}(\mathbf{b}), H_i^{s^t}(\mathbf{b}') < \varepsilon$, then $|H_j^{s^t}(\mathbf{b}) - H_j^{s^t}(\mathbf{b}')| \leq$*

[10] We analyze these two particular examples in Section 4. More generally, the Tullock function can be written as $H_i(\mathbf{b}) = \frac{f(b_i)}{\sum_j f(b_j)}$ for an increasing f with $f(0) = 0$.

$\kappa\varepsilon$ *for all* $j \in \{1, 2, .., m\}$.

H3. *At any period* t, $\sum_{i \leq s^t} H_i^{s^t}(\mathbf{b}) = 1$.

Assumption H1 is a weakening of a standard assumption connecting relative probabilities of winning to relative efforts exerted. In H1 this connection only relates to the highest bidder vis a vis other players. In particular, H1 implies that any player can secure a high enough probability of winning if he is the highest bidder and if his bid is high enough relative to other players' bids.

Assumption H2 (together with H3) is a weaker version of the independence assumption often used in the literature.[11] H2 implies that a marginal player, one who has a very small probability of winning a competition, can only have a marginal effect on others' chances and in particular cannot dramatically change the balance of power between other players. Assumption H3 is assumed for expositional purposes.[12]

Utilities, strategies and equilibria: Let $\rho \in (0, 1)$ represent the length of a period along which players incur utility from the implemented action. Without loss of generality, we use $1 - \rho$ as the discount rate between periods,[13] and so for a pure strategy profile, $(\mathbf{b}^{s^t,t})_{t=1}^{\infty}$, the expected utility of player i from the game is:[14]

$$\sum_{t=1}^{\infty}(1-\rho)^{t-1}(\rho\sum_{j=1}^{s^t} H_j^{s^t}(\mathbf{b}^{s^t,t})v_{ij} - b_i^{s^t,t}).$$

For simplicity, we assume that players do not observe the actual bids others place at each stage. Thus the relevant history on which players condition their strategy is $h = (m, i^1, i^2...i^k)$. We can therefore reformulate the state s^t as $s(h) = \min\{i|i \in h\}$.

In what follows to simplify the exposition we drop the subscript from $H_i^{s^t}(.)$ and use $H_i(.)$. If the influence function H is continuous, a Markov Perfect Equilibrium

[11] Specifically, Skaperdas (1996) axiomatizes the generalized Tullock contest by using an independence assumption which implies that a player's bid does not affect the ratio of probabilities of winning of other pairs of players (see also Clark and Riis (1998)). Our assumptions imply that marginal players can only affect such ratio marginally. Alternatively we could impose a stronger assumption of monotonicity.

[12] The results hold as long as there is a fixed lower bound on the probability that some player wins, and with the remaining probability the status quo for example is maintained.

[13] The model has discrete periods with continuos time flows of payments between periods. An alternative more standard formulation is the exponential discounting of utilities; we show in the appendix that this will yield an equivalent analysis.

[14] Where this can be extended to mixed strategies in a straightforward way.

-where the strategies will only depend on the state $s(h)$- exists by Escobar (2008). Assumption H1 implies however that H is not continuous at $\mathbf{b} = 0$. Nonetheless we can show:

Theorem 1 *Suppose that H satisfies* H1, H2, H3, *and is continuous except at* $\mathbf{b} = 0$. *Then there exist an open set of H functions for which a Subgame Perfect Equilibrium exists.*[15]

In the proof we construct a sequence of influence games, similar to the above, with the exception of a strictly positive reservation price. In this game an MPE exists by Escobar (2008). We then take the reservation prices to zero and show that the limit of the sequence of equilibria is an equilibrium in the limit. To do so, we use the properties of equilibria implied by H1, H2 and H3, and show that along the sequence it cannot be that the strategies of all players place a positive measure on bids converging to zero, which allows us to establish continuity in the limit for an open set of specifications of $H(\mathbf{0})$.

Theorem 2 below holds more generally for H functions that are not continuous at other points than zero (for example, the all-pay-auction) and we therefore proceed without imposing continuity.[16]

3. Progress and gradualism

Before introducing our main result we would like to emphasize the role that irreversibility plays in the model. Consider a benchmark model in which there is no irreversibility and any action can be implemented at any stage. This means that at any stage all M players compete and the winning policy of this competition is implemented without any additional constraints.[17] By standard repeated game arguments there exist, for some ρ, SPE in which some particular player can win with probability one in a given period. For example, for some ρ, we can find an equilibrium in which Player 1 wins with probability one in the first period.

In contrast, when the process is irreversible, the set of possible punishments is bounded. As we will show below, this restriction on strategies will imply that we

[15] The proof shows in fact that an MPE exists. Our main result, Theorem 2, applies to all SPE.

[16] In Section 4 we analyze examples for which an MPE exists when H is the all-pay-auction function.

[17] Due to the usual problem of discontinuity around zero, we have to slightly alter the model to allow for such SPE. One option is to assume that players can choose not to compete by yielding. Note that this modification of the game doesn't affect the results presented above.

cannot sustain such equilibria as above and therefore that gradualism will arise in all SPE.

As a first step in our analysis, note that the ratio of players' willingness to win is typically a useful measure of players' success in all-pay competitions. In particular, let us consider two players, i, j and $i, j \leq s(h)$. The players' willingness to win against each other, henceforth WTW, are

$$w_{ij}^{h} \equiv \rho(-v_{ij}) + (1-\rho)(V_i^{(h,i)} - V_i^{(h,j)}),$$
$$w_{ji}^{h} \equiv \rho(-v_{ji}) + (1-\rho)(V_j^{(h,j)} - V_j^{(h,i)})$$

where $V_l^{(h,k)}$ is the continuation utility of player l from the history which follows h and a win by player k.

Note that w_{ij}^{h} is composed from a contemporaneous payoff $\rho(-v_{ij})$ and a continuation payoff $(1-\rho)(V_i^{(h,i)} - V_i^{(h,j)})$. The former is always positive as players always prefer their own policy to be implemented today. In contrast, the future effects of winning today might be positive or negative depending on the difference $V_i^{(h,i)} - V_i^{(h,j)}$. Potentially, the future effects of winning today might offset the contemporaneous benefits. If this is the case, the ratio of the willingness to win of the players might be zero or infinite. In all-pay auctions, we know that if $\frac{w_{ij}^{h}}{w_{ji}^{h}} = \infty$ then player i would win against player j with probability one. This implies that the dynamic process may either get stuck on some $i = s(h)$ or move extremely fast if $i = 1$ for example.

The gist of our analysis is to show that for all players, at all histories, and for all ρ, such a ratio is bounded. This allows us to show that the dynamic process moves forward, but not too quickly, so that even when players have to repeatedly compete to maintain their action as the current state, they never "give up":

Theorem 2 *There exists an $\varepsilon > 0$ such that in any Subgame Perfect Equilibrium, for any history h and state $s(h) > 1$, for any ρ : (i) The probability with which some player $i < s(h)$ wins is larger than ε, (ii) The probability with which Player 1 (or any player $i < s(h)$) wins is lower than $1 - \varepsilon$.*

Theorem 2 implies that society will reach state 1, the absorbing state, in finite time in expectations. Nonetheless, players still fight in a substantial manner relative to one another at any history, so that progress is never immediate; actions other than the highest ranked one will be implemented with a strictly positive probability.

Note the order of qualifiers in the Theorem. The bound on the probabilities of winning is independent of the parameter ρ and the choice of equilibria. This implies that it is only the parameters of the influence function and the distribution of preferences determining the upper and lower bounds on the rate of gradualism.

The basic intuition of the Theorem is as follows. By the irreversibility assumption, by winning, Player 1 can effectively terminate the game, and thus future punishments that can be inflicted on him are bounded. As a result, his willingness to win cannot be too low compared with that of other players, which by H1 implies that he will win any competition he participates in with a strictly positive probability. This in turn implies that the game will end in finite time in expectations or in other words, by induction, future punishments that can be inflicted on other players are also bounded. Thus all players, including Player 1, fight for a finite number of periods. Their willingness to win is therefore comparable, which again by H1 implies that no player can win any period with too high a probability.

To illustrate how we use our assumptions on H and the assumption of irreversibility (the pre-determined direction), we now provide the proof for histories h with $s(h) = 2$ (i.e., we consider only players 1 and 2). We then discuss what is involved in extending the proof to all histories.

In the proof, we often look at sequences of equilibria pertaining to a sequence of parameter choices $\{\rho_n\}_{n=1}^{\infty}$ in which $\rho_n \to \rho^*$ where $\rho^* \in [0,1]$, and their associated values w_{ij}^h. We say that w_{ij}^h is of *order* ρ if $0 < \lim_{\rho_n \to \rho^*} \left| \frac{w_{ij|n}^h}{\rho_n} \right| < \infty$ for any sequence of equilibria computed for a sequence $\{\rho_n\}_{n=1}^{\infty}$ for which $\lim_{\rho_n \to \rho^*} \left| \frac{w_{ij|n}^h}{\rho_n} \right|$ exists.

The first step establishes that, due to his dynamic advantage, the willingness to win (WTW) of Player 1 is never too low compared with that of Player 2.

Step 1: For all ρ, $\frac{w_{12}^h}{w_{21}^h} > \frac{|v_{12}|}{|v_{21}|}$.

Let τ be a random variable denoting the period in which Player 1 wins in equilibrium. Then

$$V_1^{(h,2)} = \rho v_{12} E_x \sum_{t=1}^{\tau-1} (1-\rho)^{t-1} - E_x \sum_{t=1}^{\tau-1} (1-\rho)^{t-1} \tilde{b}_1^t,$$
$$V_2^{(h,2)} = \rho v_{21} E_x \sum_{t=\tau}^{\infty} (1-\rho)^{t-1} - E_x \sum_{t=1}^{\tau-1} (1-\rho)^{t-1} \tilde{b}_2^t$$

where $\tilde{b}_i^t$ is the random bid of player i in period t. Note that $v_{11} = 0$, and thus:

$$\frac{w_{12}^h}{w_{21}^h} = \frac{\rho(-v_{12}) + (1-\rho)(v_{11} - V_1^{(h,2)})}{\rho(-v_{21}) + (1-\rho)(V_2^{(h,2)} - v_{21})} > \frac{\rho |v_{12}| (1 + (1-\rho)(E_x \sum_{t=1}^{\tau-1} (1-\rho)^{t-1}))}{\rho |v_{21}| (1 + (1-\rho)(E_x \sum_{t=1}^{\tau-1} (1-\rho)^{t-1}))} = \frac{|v_{12}|}{|v_{21}|}$$

where in the inequality we ignore the expected bids of Player 1 (showing in the numerator with a positive sign) and the expected bids of Player 2 (showing in the denominator with a negative sign).□

Note that in Step 1 what is important is that Player 1 by winning can terminate the game and achieve the highest possible utility $v_{11} = 0$, i.e., our assumption of irreversibility. Note further that the bound does not depend on ρ.

Step 1 allows us to show, using H1, that at $s(h) = 2$, Player 1 does not lose with a probability converging to one:

Step 2: *There exists an $\bar{\varepsilon} > 0$ such that in any Subgame Perfect Equilibrium, for any h with $s(h) = 2$, for any ρ, Player 1 wins with a probability larger than $\bar{\varepsilon}$.*

Suppose by way of contradiction that we can construct a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ converging to zero, with a corresponding sequence $\{\rho_n\}_{n=1}^{\infty}$ *with* $\rho_n \to \rho^*$ and equilibria, such that along the sequence Player 1 wins with probability smaller than ε_n.

As $E(\Pr(\text{Player 1 wins})) \equiv \bar{H}_1(\tilde{b}) < \varepsilon_n$, then

$$\bar{H}_1(b_1, \tilde{b}_2) < k\varepsilon_n$$

for a measure of at least $1 - \frac{1}{k}$ of bids in the support of Player 1. Choosing a sequence of $k_n \to_{n\to\infty} \infty$ and $k_n\varepsilon_n \to_{n\to\infty} 0$ this implies that for a measure $1 - \frac{1}{k_n} \to_{n\to\infty} 1$ of bids, b_1^*, in the support of Player 1,

$$\bar{H}_1(b_1^*, \tilde{b}_2) < k_n\varepsilon_n.$$

We now compare the utility from each such bid in the support of Player 1 with that from a bid of zero. Given the strategy of Player 2, Player 1 is better off using b_1^* rather than zero only if:

$$b_1^* \leq w_{12}^h(\bar{H}_2(0, \tilde{b}_2) - \bar{H}_2(b_1^*, \tilde{b}_2))$$

Thus, for almost all b_1^*, by H3, $b_1^* < k_n\varepsilon_n w_{12}^h$.

Consider Player 2. A possible strategy for him is a sequence of bids $\gamma_n b_1^*$ where $\gamma_n \to \infty$ and $\gamma_n k_n \varepsilon_n \to 0$ so that his bids converge to zero. By H1, this guarantees winning with probability converging to one and thus recouping almost all w_{21}^h. We now use Step 1 to claim that such strategy represents the maximum bids he will use: As $\frac{w_{12}^h}{w_{21}^h}$ is bounded from below, only bids b with $\frac{b}{w_{12}^h} \to_{n\to\infty} 0$ will allow him to recoup almost all w_{21}^h which is the best he can get.[18] Player 2 will therefore not use bids of a higher order and thus the equilibrium strategy of Player 2 must involve a maximal bid $b_2^{\max *}$ with $\frac{b_2^{\max *}}{w_{12}^h} \to_{n\to\infty} 0$.

We now reach a contradiction, as Player 1 can deviate from his equilibrium strategy. To see this, consider a sequence of bids b_1' such that $\frac{b_1'}{b_2^{\max *}} \to_{n\to\infty} \infty$ and $\frac{b_1'}{w_{12}^h} \to_{n\to\infty} 0$. By H1, his (relative) gain is almost w_{12}^h while his (relative) cost is infinitely smaller than w_{12}^h, yielding a strictly positive benefit far enough into the sequence. Therefore, we have shown that there exists an $\bar{\varepsilon} > 0$, such that for all ρ, $\bar{H}_1(\tilde{b}) > \bar{\varepsilon}$.□

We now show that Step 2 implies that both players only fight for instantaneous benefits and that their WTW are comparable to one another:

Step 3: *For any sequence of parameter choices* $\{\rho_n\}_{n=1}^{\infty}$ *with* $\rho_n \to \rho^*$ *and corresponding equilibria, for any history h with s(h)=2,* w_{21}^h and w_{12}^h *are of order* ρ *and all bids are of order* ρ *or lower.*

Let ε be given by Step 2. Consider first Player 2. Note that an upper bound on the continuation utility following history $(h, 2)$ will be given if we set Player 1's probability of winning at its lowest level, ε, and abstract away from bidding costs:

$$V_2^{(h,2)} \leq \frac{\varepsilon v_{21}}{1-(1-\rho)(1-\varepsilon)}$$

Hence

$$w_{21}^h \leq \frac{\rho(-v_{21})}{1-(1-\rho)(1-\varepsilon)}.$$

Note also that the minimum continuation value following history $(h, 2)$ is v_{21} as player 2 can bid zero throughout and guarantee that value. Thus

$$\rho(-v_{21}) \leq w_{21}^h \leq \frac{\rho(-v_{21})}{1-(1-\rho)(1-\varepsilon)},$$

which implies that w_{21}^h is of order ρ. This implies that the bids of Player 2 in equilibrium are at most of order ρ.

Let us now consider the WTW of Player 1, w_{12}^h. Consider his continuation utility for some bid $b_1^{h'} = K^{h'}\rho$ for a large enough $K^{h'}$ so that, by H1 and the fact that the

[18] If $\frac{w_{12}^h}{w_{21}^h} \to 0$, then player 2 might be able to get closer to w_{21}^h with bids that are in the order of w_{12}^h.

bids of Player 2 are of at most order ρ, the probability that Player 2 wins is smaller than a half at every period. Let $K = \max_{h'} K^{h'}$. As this strategy is not necessarily optimal, we have:

$$V_1^{(h,2)} \geq \frac{\frac{1}{2}\rho v_{12} - K\rho}{1 - (1-\rho)\frac{1}{2}}$$

and hence

$$\rho(-v_{12}) \leq w_{12}^h \leq \rho(-v_{12}) + (1-\rho)(-(\frac{\frac{1}{2}\rho v_{12} - K\rho}{1 - (1-\rho)\frac{1}{2}}))$$

where the rhs is of order ρ and the left inequality follows as $V_1^{(h,2)} \leq 0$. We therefore find that the WTW of Player 1 is of order ρ. The equilibrium bids of Player 1 are therefore of at most order ρ.□

The above step allows us to deduce that the ratio $\frac{w_{21}^h}{w_{12}^h}$ is also bounded from below. This implies that similarly to Step 2, we can use it and H1 to show that also Player 2 cannot lose with too high a probability:

Step 4: *There exists an $\bar{\varepsilon} > 0$ such that in any Subgame Perfect Equilibrium, for any h with s(h)=2, for any ρ, Player 2 wins with a probability larger than $\bar{\varepsilon}$.*

The proof is analogous to that of Step 2, with the labels of the players reversed.■

We use an induction on $s(h)$ to generalize the above arguments to any history h; we simultaneously show that an option ranked higher than the status quo always wins with a strictly positive probability, that the WTW of players is at most of order ρ, and exactly of order ρ for some. As we establish that the WTW of players is comparable, we can conclude (again, using H1) that Player 1 (or any player representing a policy that is ranked higher than the status quo) cannot win with probability converging to one.

Extending the arguments above to more than two players involves considering several issues, specifically in Step 2. First, to bound the magnitude of losing players' bids in equilibrium we use H2 to guarantee that withdrawing their bids does not affect dramatically the probabilities of winning of other players. Second, by withdrawing his bid, such a player might shift (even a small probability) to the player he fears most. This implies that we need to consider the worst case scenario for each player, or more generally *all* bilateral comparisons between players. To do this we use the irreversibility assumption which implies the inductive structure of the game.

As an illustration of the implications of Theorem 2 consider the following example of a two-player, asymmetric, all-pay auction:

Example 1: *Consider two players, 1 and 2. Assume that* $v_{12} = v_{21} = v$, *and that the competition function is an asymmetric all-pay auction, i.e., Player 1 wins the stage game whenever* $b_1 > \alpha b_2$ *for some* $0 < \alpha \leq 1$ *(the case of* $\alpha = 1$ *is the standard symmetric all-pay auction). We then have:*

Proposition 1 *There exists a unique SPE, which is stationary. In equilibrium, the probability that Player 1 wins at any stage equals* $1 - \frac{\alpha}{2(1+\alpha(1-\rho))}$. *For all* ρ, *this probability is bounded in:*

$$[1 - \frac{\alpha}{2}, 1 - \frac{\alpha}{2(1+\alpha)}]$$

Theorem 2 establishes gradualism for a general environment, but does not tell us how slow (or quick) is the rate of gradualism. The example above allows us to capture the rate of gradualism in a SPE for a two-player environment. For example, when $\alpha = 1$, for all ρ, the probability that Player 1 wins is bounded in $[\frac{1}{2}, \frac{3}{4}]$. Moreover, the example shows how the bounds on gradualism depend on the competition function. For any parameter ρ, the probability that Player 1 wins is decreasing in α, and both bounds are also decreasing in α.

The example also illustrates the necessity of assumption H1. When α is strictly positive, the assumption holds and therefore the bounds are strictly within $(0, 1)$. When $\alpha = 0$, assumption H1 is violated. For any sequence of models in which α converges to zero, the probability that Player 1 wins any stage in equilibrium will converge to one.

In a two-player game it is quite straightforward to measure the rate of gradualism. More generally, in games with more than two players and other competition functions, it might be a harder task. We next consider an application of the model to a standard political set up. We focus on the existence of a particular set of equilibria, equilibria with *full gradualism*, in which all policies are implemented in the political process at some stage.

4. Convergence to the median voter and full gradualism

We now consider a specific political economy model in which the median voter's preferences determine in which direction policies can change.

4.1 A one-dimensional model: extremists vs. moderates

Consider the model of Section 2 in which player i -a lobbyist, or a politician- represents an action $x_i \in X = [-1, 1]$. The utility of player i from action x_j is $v_{ij} = -|x_i - x_j|$ and thus players have an interest in the outcome of the political process. For a concrete direction of progress, assume that at any period t, the median voter is the decisive voter in society, with an ideal policy at zero and single-peaked preferences which are symmetric around zero. Let $x_1 = 0$, and $|x_i| < |x_{i+1}|$, so that policies with smaller indices are better for the median voter. New policies or reforms are therefore adopted at each election only if they are better than the current policy for the median voter.

The specific formulation of players' ideal policies and utilities on the one-dimensional policy space highlights a tension between moderates and extremists. Extremists have a higher intensity to win, and as such, have an advantage when ρ is large and few competitions determine implemented policies. Moderates, with ideal policies close to zero, represent better policies for the median voter, an advantage manifested for small values of ρ, i.e., when others need to win numerous competitions to crowd out moderates. An alternative specification of the utilities is to consider only winning motives, i.e., when $v_{ij} = v$ for any $i \neq j$. By comparing the moderates-extremists formulation to the case of pure winning motives we will be able to highlight the role that negative externalities play in the dynamic political process.

At any period, the all-pay competition or influence game represents political campaigns, primaries or lobbying, which determine at each stage a reform or a policy to be implemented (or a new candidate to be elected). The cost of entry into the political arena is endogenous, and depends on who else competes. For concreteness, we assume that political competition can be one of two: either the all-pay auction, or the simple Tullock contest function (where $H_i(b) = \frac{b_i}{\sum_j b_j}$ if $\exists b_j > 0$ and $H_i(\mathbf{0})$ otherwise). These functions differ in their level of competitiveness, a feature which we will examine below.

The model described above is a model of a dynamic entry game among candidates or policies.[19] By Theorem 2, convergence to the median voter will arise in finite time

[19] Notable political entry games are the citizen candidate or town meeting models (Besley and

in expectations. Our model provides then dynamic foundations to the static median voter results.[20] As Theorem 2 implies that convergence to the median voter's favorite policy is not immediate, we will use the above model to better understand the path of gradualism, how it depends on the distribution of preferences, the assumption of negative externalities, and the competitiveness of the H function. For tractability, we will focus below on Markov Perfect Equilibria in which players' strategies only depend on the state $s(h)$.

4.2 Full gradualism and the role of negative externalities

We study a particular equilibrium, a *fully gradual equilibrium*, in which all policies are implemented in some period. This implies that each player can win with a strictly positive probability only after all other policies which are worse than the one he represents have already been implemented. Formally, we say that a player is *active* if the measure of non-zero bids in his support is strictly positive. Then:

Definition 1: In a *fully gradual equilibrium,* at any state $s > 1$, only players s and $s-1$ are: (i) active, and (ii) win with a strictly positive probability.

For simplicity, we focus on $m = 3$, with $x_1 = 0$, $x_2 > 0$ and $|x_3| > x_2$. In a fully gradual Markov Perfect Equilibrium, at $s = 2$, only players 1 and 2 compete whereas at $s = 3$, only players 2 and 3 compete. The binding condition for a fully gradual MPE, as we show in the appendix, is that at $s = 3$, given that players 2 and 3 compete, Player 1 prefers not to be active.

Note, for example, that when $x_3 < 0$ and ρ is large, it is easy to sustain a fully gradual equilibrium. In such a case, there are few competitions, and extreme players with strong intensity to win have a relative advantage compared with moderate ones. Players 2 and 3's WTW vis a vis each other is roughly $|x_3| + |x_2|$ whereas that of Player 1 is at most $|x_3|$ and he is therefore priced out of the competition. We show however that full gradualism arises also for small values of ρ as well as when $x_3 > 0$ (a "one-sided" polity):

Proposition 2 *(i) For sufficiently small values of ρ, there exists a $\gamma > 0$ such that a fully gradual MPE exists when $\frac{|x_3|}{x_2} \geq \gamma$. (ii) The cutoff γ is larger under*

Coate (1997) and Osborne et al (2004)). For models that consider mechanisms such as the simple Tullock function or the all-pay auction to determine the results of election or primaries, see Kaplan and Sela (2008), Mattozzi and Merlo (2010), and Klumpp and Polborn (2008).

[20] For a similar dynamic foundation in a different model, see also Baron (1996).

the simple Tullock function than under the all-pay-auction. (iii) For a fixed $|x_3|$, the cutoff γ is larger in a "one-sided" polity (when $x_3 > 0$) compared with a "two-sided" polity (when $x_3 < 0$). (iv) If players only have winning motives, then for all ρ, there is no fully gradual MPE.

We find that it is the relative values of x_2 and $|x_3|$ that are important in characterizing the equilibria above. Somewhat counter-intuitively, Player 1 stays out if Player 3 - which represents his worst policy outcome, is located far enough from him. The reason is that the distance between these players affects mainly the WTW of Player 3, who knows that in expected finite time Player 1 will win (by Theorem 2). On the other hand, Player 1's willingness to become active is affected also by his distance from Player 2, who wins at $s = 3$ with a strictly positive probability. In the case of the all-pay auction, ensuring that Player 1's willingness to become active is lower than the WTW of Player 3 (or the highest bid in the game, which is the best deviation for Player 1) yields the condition on $\frac{|x_3|}{x_2}$.

In the simple Tullock function, Player 3 has to be even further away for Player 1 to stay out for two reasons. First, fixing equilibrium behavior as in the all-pay auction, this competition is less aggressive and Player 1 can beneficially deviate even with a small bid and thus is more likely to do so. Second, as actual equilibrium behavior is less aggressive than in the all-pay auction, Player 3 wins more often at $s = 3$ which encourages Player 1 to become active. For similar reasons, Player 1 is more motivated to deviate when the polity is "one-sided". In this case Player 3 is not as disadvantaged as he is on the same side of Player 2 so he wins more often.

Finally, we show that negative externalities are important in sustaining full gradualism. If politicians or interest groups care only about winning, in all MPE, Player 1 is active in every stage and wins every stage with a strictly positive probability.[21] Intuitively, the absence of negative externalities sharpens the advantage of player 1 and so convergence to the median voter arises in this case in the first stage of the game, or in any stage of the game, with a strictly positive probability.

5. Conclusion

We have presented a model of a dynamic process in which society can only move forward in one direction and players compete to influence whether and how fast the

[21] In the appendix we establish that indeed an MPE exists in the all-pay auction.

process moves. Our main assumption about the competition function at each period is that a player can guarantee a high enough probability of winning whenever his bid is high enough relative to that of other players. We show that society moves forward with a strictly positive probability and that players fight only for instantaneous benefits. Thus, even if these are marginal, players still compete, implying that the process is gradual. We have presented an application to a dynamic political entry game in which society converges to the median voter's ideal policy, which also allowed us to shed some light on the rate of gradualism. In particular it has highlighted the role of negative externalities in sustaining full gradualism.

6. Appendix

6.1 Notation and preliminaries

Let V_i^h be the continuation utility of player i following some history $h = (m, i^1, ..i^k)$. Let $s(h) = \min\{i|i \in h\}$ be the state at history h. Let u_{ij}^h be the utility of player i when player j wins at history h, abstracting from the possible payments made by player i right after this history. That is, $u_{ij}^h = \rho(-v_{ij})+(1-\rho)V_i^{(h,j)}$. Let $w_{ij}^h = u_{ii}^h - u_{ij}^h$ denote the willingness to win (WTW) of player i against player j in history h.

We will often look at sequences of equilibria and then the relevant willingness to win will be $w_{ij|_n}^h$. We will often suppress the notation for n in these expressions, writing w_{ij}^h for $w_{ij|_n}^h$. Often these sequences will relate to a sequence of parameter choices $\{\rho_n\}_{n=1}^{\infty}$ in which $\rho_n \to \rho^*$ where $\rho^* \in [0,1]$.

In what follows we use the following terms to refer to the magnitudes of variables computed for a sequence of parameters $\{\rho_n\}_{n=1}^{\infty}$ and equilibrium strategies. We say that w_{ij}^h is of *order* ρ if $0 < \lim_{\rho_n \to \rho^*} \left| \frac{w_{ij|n}^h}{\rho_n} \right| < \infty$ for any sequence of equilibria computed for a sequence $\{\rho_n\}_{n=1}^{\infty}$. Similarly we say that w_{ij}^h is of *order* ρ *or lower* if $0 \leq \lim_{\rho_n \to \rho^*} \left| \frac{w_{ij|n}^h}{\rho_n} \right| < \infty$ and of an order ρ *or higher* if $0 < \lim_{\rho_n \to \rho^*} \left| \frac{w_{ij|n}^h}{\rho_n} \right| \leq \infty$. We will also sometimes write $x(\rho) \approx y(\rho)$ for two functions $x(\rho)$ and $y(\rho)$ to imply that $\lim_{\rho_n \to \rho^*} \frac{x(\rho_n)}{y(\rho_n)} = 1$.

Finally, note that the WTW of player 1 vis a vis any other player is strictly positive and of order ρ or higher. To see this note that,

$$w_{1i}^h = \rho(-v_{1i}) + (1-\rho)(v_{11} - V_1^{(h,i)}) \geq \rho(-v_{21}),\ i = 2, ..., m,$$

where $v_{11} = 0$, $-v_{1i} > 0$ and $V_1^h \leq 0$ for all h. The same is true for the WTW of Player 2 vis a vis Player 1:

$$w_{21}^h = \rho(-v_{21}) + (1-\rho)(V_2^{(h,2)} - v_{21}) \geq \rho(-v_{21}),$$

as after any history h with $s(h) = 2$ (as is $(h, 2)$), Player 2 could choose not to bid and for the rest of the game, so that $V_2^h \geq v_{21}$ for any h with $s(h) = 2$.

A note about the utility specification:

Let r be the discount rate and $1/k$ the length of each period. The utility function can be written as $\sum_{t=1}^{\infty} e^{\frac{-rt}{k}}(\int_0^{\frac{1}{k}} e^{-\frac{rx}{k}}dx \sum_{j=1}^{s^t} H_j^{s^t}(\mathbf{b}^{s^t,t})v_{ij\}} - b_i^{s^t,t})$. In this formulation the equivalent of ρ, or the weight on period t payoffs vis a vis bids, is $\beta(k,r) = \int_0^{\frac{1}{k}} e^{-\frac{rx}{k}}dx$. The equivalent of $1-\rho$, or the discount factor on period t, is $\gamma(k,r) = e^{\frac{-rt}{k}}$. For any $\rho > 0$, there exist (k,r) such that the two models are equivalent.

Proof of Theorem 1:

We construct a sequence of games that converge to the game analyzed. Let $\{r_n\}_{n=1}^{\infty}$ be a decreasing sequence of real numbers in $(0, B]$ such that $r_n \to 0$. For any n, the action set for each player is $[r_n, B] \cup \{0\}$. We assume that for any $r > 0$, H is continuous on $([r, B] \cup \{0\})^m$. By Escobar (2008) there exists a behavior strategy Markov Perfect equilibrium for any n. We follow a sequence of equilibrium strategies, s^n, that converges to some strategy profile s. We now show that s is an equilibrium in the limit game in which the set of action profiles is $([0, B])^m$.

Suppose to the contrary that s is not an equilibrium in the limit game. Therefore there exist a player i and a strategy s'_i such that $U_i(s'_i, s_{-i}) > U_i(s)$.

Step 1:

$$U_i(s^n) \to U_i(s)$$

We show that H is continuous on the equilibrium strategies.[22] Note that H is continuous on all bidding strategies but for the case in which all players bid zero. We will now prove that strategy profiles which put a strictly positive measure on all players bidding zero do not arise in equilibrium in the limit.

Suppose that under s there is a strictly positive measure on all players bidding zero. Let α_{-i} be the measure of the event in which all players besides player i place zero bids under s. For any s_i^n take $\hat{b}_i^n = \inf b_i^n$ such that b_i^n is in the support of s_i^n. We take a sequence of bids $b_i^{m,n}$ in the support of s_i^n such that $b_i^{m,n} \underset{m\to\infty}{\to} \hat{b}_i^n$. Compute $\alpha_i \equiv \lim_n \lim_m \bar{H}_i(b_i^{m,n}, s_{-i}^n)$.

[22]Given the compactness of strategies and utilities, the expected utility from the dynamic game will be continuous whenever H is continuous.

If $\alpha_1 < \alpha_{-1}$, then for any ε, there exists an n_ε large enough such that $\lim_m \bar{H}_1(b_1^{m,n}, s_{-1}^n) < \alpha_1 + \varepsilon$ and the measure of all bids smaller than ε of all others is larger than $\alpha_{-1} - \varepsilon$. If we take ε to converge to zero, by H1, we can find a deviation of player 1 to a bid which converges to zero that will capture the gap $\alpha_{-1} - \alpha_1$. As shown above his bilateral WTW (and hence is total willingness to win) is strictly positive and hence he would rather capture this gap. Moreover by H1 and H2, we know that this deviation will alter only a measure zero of any other probabilities of winning in the game.

Suppose $\alpha_1 \geq \alpha_{-1}$. This implies $\alpha_2 = 0$. We can repeat the same deviation for Player 2 whose WTW against Player 1 is strictly positive. We have therefore reached a contradiction to the proposed strategies.

Step 2: For any t_i there exists some sequence $t_i^n \to t_i$ such that

$$\lim U(t_i^n, s_{-i}^n) \geq U(t_i, s_{-i})$$

If $t_i > 0$, $U(t_i^n, S_{-i}^n) \to U(t_i, S_{-i})$ as H is continuos in this case. If $t_i = 0$, and there is another player at least whose strategy has a measure zero on bids converging to zero, then H is continuous on s_{-i}^n and t_i^n. Suppose that this is not the case. Then we take some sequence $t_i^n \to 0$ and compute the probabilities of winning of each player in each sequence. We then assume this as the tie breaking rule for H when all players bid zero. Note that there exist then an open set around this particular H that would satisfy that $\lim U(t_i^n, S_{-i}^n) \geq U(0, .., 0)$.

We can now conclude the proof. By Step 1,

$$U_i(s^n) \to U_i(s)$$

and by Step 2, for s_i', there exists some sequence $s_i'^n \to s_i'$ such that

$$\lim_{\substack{s_i'^n \to s_i' \\ s_{-i}^n \to s_{-i}}} U(s_i'^n, s_{-i}^n) \geq U(s_i', s_{-i}).$$

Therefore, there must exist some n for which $U(s_i'^n, s_{-i}^n) > U_i(s^n)$, a contradiction to the sequence s^n being a sequence of equilibria.$\square$

6.3 Proof for Section 3

Proof of Theorem 2:

We first prove the following Lemma.

Lemma A1: *(i) There exists an* $\bar{\varepsilon} > 0$*, such that for all* ρ*, the probability that some player* $i < s(h)$ *wins is larger than* $\bar{\varepsilon}$*. (ii) For any sequence of parameter choices* $\{\rho_n\}_{n=1}^{\infty}$ *with* $\rho_n \to \rho^*$ *and corresponding equilibria: a. for any* $j \neq 1, 2$*,* w_{ji}^h *is of order* ρ *or lower, b.* w_{21}^h and w_{1j}^h *are of order* ρ, c. *all bids are of order* ρ *or lower.*

Proof of Lemma A1:

We will prove the Lemma by induction on the state of a history.

Step 1: Proving the Lemma for histories h such that $s(h) = 2$.

This was proven in Section 3.□

Step 2: The induction hypothesis: *Assume that for all histories* h *with* $s(h) \leq l - 1$*: (i) There exists an* $\bar{\varepsilon} > 0$*, such that for all* ρ*, the probability that some player* $i < s(h)$ *wins is larger than* $\bar{\varepsilon}$*. (ii) For any sequence of parameter choices* $\{\rho_n\}_{n=1}^{\infty}$ *with* $\rho_n \to \rho^*$ *and corresponding equilibria: a. for any* $j \neq 1$*,* w_{ji}^h *is of order* ρ *or lower and* w_{21}^h *is of order* ρ*, b.* w_{1j}^h *is of order* ρ *and all bids are of order* ρ *or lower.*

Step 3: Proving the Lemma for histories h with $s(h) = l$.

First we prove that $\frac{\max_{j<l} \max_{i \leq l} w_{ji}^h}{\max_{j<l} w_{lj}}$ is bounded from below. Note first that $\max_{j<l} \max_{i \leq l} w_{ji}^h \geq w_{1l}$.

Suppose that $\max_{j<l} w_{lj} = w_{lk}$. Let τ^h be a random variable of the period at which Player 1 wins wins in some history h.

$$\frac{w_{1l}}{w_{lk}} = \frac{\rho(-v_{1l}) + (1-\rho)(-V_1^{(h,l)})}{\rho(-v_{lk}) + (1-\rho)(V_l^{(h,l)} - V_l^{(h,k)})} > \frac{\rho(-v_{1l}) + (1-\rho)(E_x[\rho(A)\sum_{t=1}^{\tau^{(h,l)}-1}(1-\rho)^{t-1}]}{\rho(-v_{lk}) + (1-\rho)E_x[\rho(-C)\sum_{t=1}^{\tau^{(h,l)}-1}(1-\rho)^{t-1} + \rho\sum_{t=\tau^l}^{\infty}(1-\rho)^{t-1}v_{l1} - (\rho(-B)\sum_{t=1}^{\tau^{(h,k)}-1}(1-\rho)^{t-1} + \rho\sum_{t=\tau^k}^{\infty}(1-\rho)^{t-1}v_{l1})]}$$

Where the strict inequality comes from ignored bids in the numerator and ignored bids in the denominator for $V_l^{(h,l)}$, and A, B and C are strictly positive numbers.

By setting $C = 0$ we further decrease this expression and we get,

$$\frac{w_{1l}}{w_{lk}} > \frac{\rho(-v_{1l}) + (1-\rho)(E_x[\rho A \sum_{t=1}^{\tau^{(h,l)}-1}(1-\rho)^{t-1}]}{\rho(-v_{lk}) + (1-\rho)E_x[\rho \sum_{t=\tau^l}^{\tau^{(h,k)}}(1-\rho)^{t-1}v_{l1} - (\rho(-B)\sum_{t=1}^{\tau^{(h,k)}-1}(1-\rho)^{t-1}]}$$

with the convention that $\sum_{t=\tau^{(h,l)}}^{\tau^{(h,k)}} = -\sum_{t=\tau^{(h,k)}}^{\tau^{(h,l)}}$.

Note that

$$\frac{(E_x[\rho A \sum_{t=1}^{\tau^{(h,l)}-1}(1-\rho)^{t-1}]}{E_x[\rho \sum_{t=\tau^l}^{\tau^{(h,k)}}(1-\rho)^{t-1}v_{l1} - (\rho(-B)\sum_{t=1}^{\tau^{(h,k)}-1}(1-\rho)^{t-1}]} > \frac{A}{B\frac{E_x[\sum_{t=1}^{\tau^{(h,k)}-1}(1-\rho)^{t-1}]}{E_x[\sum_{t=1}^{\tau^{(h,l)}-1}(1-\rho)^{t-1}]}}$$

where $\frac{E_x[\sum_{t=1}^{\tau^{(h,k)}-1}(1-\rho)^{t-1}]}{E_x[\sum_{t=1}^{\tau^{(h,l)}-1}(1-\rho)^{t-1}]}$ is bounded by the induction. Therefore, there exists some $\gamma > 0$ such that $\frac{\max_{j<l}\max_{i\le l} w_{ji}^h}{\max_{j<l} w_{lj}} \ge \frac{w_{1l}}{w_{lk}} \ge \gamma > 0$.

(i) We now show that the probability that some player $i < l$ wins is bounded away from zero.

For some $\rho_n > 0$ and an equilibrium, let ε_n be the probability that at least one player $j < l$ wins. We will show that there cannot be a sequence of parameter choices, $\{\rho_n\}_{n=1}^{\infty}$, and equilibria such that ε_n converges to zero. By arguments similar to above, almost all bids of players $j < l$ must satisfy,

$$b_j^* \le \sum_{i\ne j, i\le l} w_{ji}^h(\bar{H}_i(0,\tilde{b}_{-j}) - \bar{H}_i(b_j^*,\tilde{b}_{-j}))$$

As $\bar{H}_j(0,\tilde{b}_{-j}) - \bar{H}_j(b_j^*,\tilde{b}_{-j}) < \varepsilon_n$,[23] by H2, $\bar{H}_i(0,\tilde{b}_{-j}) - \bar{H}_i(b_j^*,\tilde{b}_{-j}) < \kappa\varepsilon_n$ and we have,

$$b_j^* < (l-1)\kappa\varepsilon_n \max_{i\le l} w_{ji}^h.$$

The above implies that for all most any configuration of bids of players other than j, the maximal bid is bounded by $b_{-l}^* \equiv (l-1)\kappa\varepsilon_n \max_{j<l}\max_{i\le l} w_{ji}^h$.

Consider now Player l. A possible strategy for him is a sequence of bids $\gamma_n b_{-l}^*$ where $\gamma_n \to \infty$ and $\gamma_n\kappa\varepsilon_n \to 0$ so that such bids converge to zero. By H1, such a sequence guarantees him winning with probability converging to one while recouping almost all of his WTW. As $\frac{\max_{j<l}\max_{i\le l} w_{ji}^h}{\max_{j<l} w_{lj}}$ is bounded from below, the equilibrium

[23] Note that this holds by $H1$ and given that $\bar{H}_j(b_j^*,\tilde{b}_{-j}) < \varepsilon_n$ for $b_j^* > 0$.

strategy of Player l must involve bids that do not surpass a sequence of bids $b_l^{\max *}$ with $\frac{b_l^{\max *}}{\max_{j<l} \max_{i \leq l} w_{ji}^h} \rightarrow_{n \rightarrow \infty} 0$.

Assume that $\max_{j<l} \max_{i \leq l} w_{ji}^h = w_{j^* i^*}^h$. Note that as $w_{j^* i^*}^h \geq w_{1l}^h$, it is of order ρ or higher. If $i^* < l$, then by induction, $w_{j^* i^*}^h$ is of order ρ. In this case, Player 1 can deviate to a sequence b_1^* such that $\frac{b_1^*}{w_{j^* i^*}^h} \rightarrow 0$ and guarantees a probability one of winning. His gain will be $(1-\varepsilon) w_{1l}^h + \varepsilon \bar{w}_{1i}^h - b_1^*$ which is strictly positive (as $\bar{w}_{1i}^h$ is of order ρ by the induction), a contradiction.

If $i^* = l$, Player j^* can deviate to a sequence b_{j^*} such that $\frac{b_{j^*}}{w_{j^* l}^h} \rightarrow 0$ and guarantees a probability one of winning. His gain will be $(1-\varepsilon) w_{j^* l}^h + \varepsilon \bar{w}_{j^* i}^h - b_{j^*}$ which is strictly positive (as $w_{j^* l}^h$ is of order ρ or higher and $\bar{w}_{j^* i}^h$ of order ρ by the induction), a contradiction.

We have therefore proved that there exists a $\bar{\varepsilon} > 0$ such that $\varepsilon_n > \bar{\varepsilon} > 0$ for all sequences, $\{\rho_n\}_{n=1}^{\infty}$, and equilibria.

(ii) *a.* Note first that $w_{21}^h = w_{21}^{h'}$ for some history h' with $s(h') = 2$. By Step 1 this is of order ρ. Similarly, note that by the induction hypothesis w_{ji}^h for $j, i < s(h)$ is of order ρ or lower. We are left to show that w_{ji}^h is of order ρ or lower for all $j \neq 1$, where either $j = l$ and $i < l$ (case 1) or $j < l$ and $i = l$ (case 2).

Case 1: Let $w_{li}^h = \max_k w_{lk}^h$ for $k < l$.

In this case,

$$w_{li}^h = \rho(-v_{li}) + (1-\rho)(V_l^{(h,l)} - V_l^{(h,i)})$$

By similar arguments to the analysis for $s(h) = 2$, we have (where $t(h,l)$ is the period following history (h,l)):

$$V_l^{(h,l)} = Ex[\rho(-A) \sum_{t=1}^{\tau^{(h,l)}-1} (1-\rho)^t + \rho \sum_{t=\tau^{(h,l)}}^{\infty} (1-\rho)^t v_{l1} - \sum_{t=1}^{\tau^{(h,l)}-1} (1-\rho)^t \tilde{b}_l^{t(h,l)+t}]$$

where $A > 0$ and by (i) we know that $\tau^{(h,l)}$ has a finite expectation.

$$V_l^{(h,i)} = Ex[\rho(-B) \sum_{t=1}^{\tau^{(h,i)}-1} (1-\rho)^t + \rho \sum_{t=\tau^{(h,i)}}^{\infty} (1-\rho)^t v_{l1}]$$

where $B > 0$ and by induction we know that $\tau^{(h,i)}$ has a finite expectation.

We now have

$$
\begin{aligned}
V_l^{(h,l)} - V_l^{(h,i)} &= Ex[\rho(-A)\sum_{t=1}^{\tau^{(h,l)}-1}(1-\rho)^t + \rho\sum_{t=\tau^{(h,l)}}^{\infty}(1-\rho)^t v_{l1} \\
&\quad -[\rho(-B)\sum_{t=1}^{\tau^{(h,i)}-1}(1-\rho)^t + \rho\sum_{t=\tau^{(h,i)}}^{\infty}(1-\rho)^t v_{l1}] - \sum_{t=1}^{\tau^{(h,l)}-1}(1-\rho)^t\tilde{b}_l^{t(h,l)+t}] \\
&= Ex[\rho(-A)\sum_{t=1}^{\tau^{(h,l)}-1}(1-\rho)^t - \rho(-B)\sum_{t=1}^{\tau^{(h,i)}-1}(1-\rho)^t \\
&\quad +\rho\sum_{t=\tau^{(h,i)}}^{\tau^{(h,l)}}(1-\rho)^t v_{l1} - \sum_{t=1}^{\tau^{(h,l)}-1}(1-\rho)^t\tilde{b}_l^{t(h,l)+t}]
\end{aligned}
$$

If $\sum_{t=1}^{\tau^{(h,l)}-1}(1-\rho)^t\tilde{b}_l^{t(h,l)+t}$ is strictly positive and of order higher than ρ, then $w_{li}^h = \max_{k<l} w_{lk}^h$ must be negative, which is a contradiction as it must be that the bids of player l are lower than w_{li}^h. This implies that bids are of at most order ρ and as a result w_{lk}^h for all $k<l$ is of order ρ or lower.

Case 2: w_{jl}^h for $j<l$.

In this case, we have

$$
w_{jl}^h = \rho(-v_{jl}) + (1-\rho)(V_j^{(h,j)} - V_j^{(h,l)})
$$

We now consider some possible strategy for player j. In all periods in which the state is still l, we choose the bid b for j in the following way. If in equilibrium, following a zero bid, the probability that the state remains l is less than 1, we choose $b=0$. If it is not, then there must exist some K large enough, so that by H1, if a bid $b=K\rho$ is placed in equilibrium, the probability that the state remains l is strictly less than 1, as player l places bids of at most order ρ by the above. With the chosen b, for any $i<l$, we have that

$$
V_j^{(h,l)} \geq Ex[\rho(-A)\sum_{t=1}^{\tau'^{(h,l)}-1}(1-\rho)^t + \rho\sum_{t=\tau'^{(h,l)}}^{\infty}(1-\rho)^t v_{j1} - \sum_{t=1}^{\tau'^{(h,l)}-1}(1-\rho)^t K\rho]
$$

for some $A>0$, where $\tau'^{(h,l)}$ has finite expectation by part (i) and by construction.

We then have

$$V_j^{(h,j)} - V_j^{(h,l)} \leq Ex[\rho(-B) \sum_{t=1}^{\tau^{(h,j)}-1} (1-\rho)^t - \rho(-A) \sum_{t=1}^{\tau'^{(h,l)}-1} (1-\rho)^t$$
$$+\rho \sum_{t=\tau'^{(h,l)}}^{\tau^{(h,j)}} (1-\rho)^t v_{j1} - \sum_{t=1}^{\tau^{(h,j)}-1} (1-\rho)^t \tilde{b}_j^{t(h,j)+t} + \sum_{t=1}^{\tau'^{(h,l)}-1} (1-\rho)^t K\rho]$$

for $A, B > 0$, where $\tilde{b}_j^{t(h,j)+t}$ for histories following from (h, j) are of order ρ or lower and so the whole expression is of order ρ or lower.

To compute an upper bound on the continuation value, we maintain the equilibrium probabilities (as if player j is using his equilibrium strategy) but set player $j's$ bid to zero, so that:

$$V_j^{(h,l)} \leq Ex[\rho(-A) \sum_{t=1}^{\tau^{(h,l)}-1} (1-\rho)^t + \rho \sum_{t=\tau^{(h,l)}}^{\infty} (1-\rho)^t v_{j1}] \implies$$
$$V_j^{(h,j)} - V_j^{(h,l)} \geq Ex[\rho(-B) \sum_{t=1}^{\tau^{(h,j)}-1} (1-\rho)^t - \rho(-A) \sum_{t=1}^{\tau'^{(h,l)}-1} (1-\rho)^t$$
$$+\rho \sum_{t=\tau'^{(h,l)}}^{\tau^{(h,j)}} (1-\rho)^t v_{j1} - \sum_{t=1}^{\tau^{(h,j)}-1} (1-\rho)^t \tilde{b}_j^{t(h,j)+t}]$$

which is again of order ρ as by part (i) $\tau^{(h,j)}$ has finite expectation and $\tilde{b}_j^{t(h,j)+t}$ for histories following (h, j) is of at most order ρ. Therefore, w_{jl}^h is of order ρ or lower.

(ii) b. We now consider the magnitude of the bids and show that w_{1l}^h is of order ρ. At state l, for all players other than player 1, w_{ij}^h at any history is of at most order ρ and thus their bids will be of at most order ρ. We can then find some sequence of bids, as long as Player 1 hasn't won, $b = K\rho$, which wins against all other bids with probability bounded from zero. Let τ'^h be the random variable of the period in which Player 1 wins. By construction τ'^h has finite expectation. We then have:

$$0 \geq V_1^h \geq Ex[\rho(-B) \sum_{t=1}^{\tau'^h-1} (1-\rho)^t + \rho \sum_{t=\tau'^h}^{\infty} (1-\rho)^t v_{j1} - \sum_{t=1}^{\tau'^h-1} (1-\rho)^t K\rho]$$

As the right hand side is of order ρ we conclude that w_{1l}^h is of order ρ.

This completes the proof of Step 3 and of Lemma A1.■

We can now prove the Theorem. Lemma A1 establishes (i). We now show (ii). Suppose by way of contradiction that at some history h with $s(h) = l$, there is a sequence of parameter choices and equilibrium strategies such that Player 1 wins with probability $1-\varepsilon$ converging to one. Similar arguments as in Lemma A1 imply that for all $j \neq 1$, for almost all bids b_j^* in the support of j, $\frac{b_j^*}{\max_{i \leq l} w_{ji}^h} \to 0$ and thus all losing bids are bound by $\kappa(l-1)\varepsilon_n \max_j \max_{i \leq l} w_{ji}^h$. As $\frac{\max_j \max_{i \leq l} w_{ji}^h}{\max w_{1i}} \geq \frac{w_{21}^h}{\max w_{1i}} \nrightarrow 0$, Player 1's bids must satisfy $\frac{b_1^*}{\max_j \max_{i \leq l} w_{ji}^h} \to 0$.

Now consider Player 2 for whom $w_{21}^h > 0$ and is of order ρ by Lemma A1. Player 2 can deviate to some bid b_2' with $\frac{b_2'}{\max_j \max_{i \leq l} w_{ji}^h} \to 0$ and $\frac{b_2'}{b_1^*} \to \infty$ which will guarantee winning with probability converging to one, and therefore, relative to his equilibrium strategy, a gain of at least $(1-\varepsilon)w_{21}^h + \varepsilon \bar{w}_{2j}^h - b_2'$ which is strictly positive as $\bar{w}_{2j}^h$ is of order ρ or lower, a contradiction.

Now suppose that at some history h with $s(h) = l$, there is a sequence of parameter choices and equilibrium strategies such that player $k < l, k \neq 1$ wins with probability $1-\varepsilon$ converging to one. Again we know that $\frac{\max_j \max_{i \leq l} w_{ji}^h}{\max w_{ki}} \nrightarrow 0$ as it includes that of either Player 2 vis a vis Player 1 or that of Player 1 vis a vis everyone. We can then conclude that all equilibrium bids are infinitely smaller than $\max_j \max_{i \leq l} w_{ji}^h$. We can repeat the same analysis as above with Player 1 now deviating, for whom $w_{1k}^h > 0$ and is of order ρ and thus a deviation to overcome player k will provide him a strictly positive gain, a contradiction.

This completes the proof of Theorem 2.■

Proof of Proposition 1:

We first solve for a one shot asymmetric all-pay-auction assuming different valuations, v_1 and v_2, satisfying $v_2 < v_1$. It is then easy to see, in the spirit of Hillman and Riley (1989), that in the unique equilibrium, Player 2 places a bid of zero with measure $1-\alpha\frac{v_2}{v_1}$, and with measure $\alpha\frac{v_2}{v_1}$ mixes uniformly on $(0, v_2)$. For Player 1 the largest bid is αv_2 and hence he mixes uniformly on $(0, \alpha v_2)$. The wining probability of Player 1 is $1-\alpha\frac{v_2}{v_1} + \int_0^{\alpha v_2} \frac{b}{v_1}\frac{1}{\alpha v_2} db = 1 - \frac{\alpha v_2}{2v_1}$.

Now consider the dynamic game with equal per-period valuation v. At some period t, we can compute the willingness to pay of both players (where with some abuse of notation we let w_{ij}^h and V_i^h depend on the history only through the time period t which is the only relevant parameter when $s(h) = 2$):

$$w_{12}^t = \rho(-v) + (1-\rho)(-V_1^{t+1})$$
$$w_{21}^t = \rho(-v) + (1-\rho)(V_2^{t+1} - v)$$

Note that as $V_1^{t+1} + V_2^{t+1} \leq v$ we have,

$$w_{12}^t \geq w_{21}^t$$

By the above, Player 2, if at all, places an atom on a bid of zero. Therefore, for any t, $V_2^{t+1} = v$ and we have,

$$\omega^t \equiv \frac{w_{21}^t}{w_{12}^t} = \frac{\rho(-v)}{\rho(-v) + (1-\rho)(-V_1^{t+1})}$$

An equilibrium generates a sequence $\{\frac{w_{21}^t}{w_{12}^t}\}_{t=1}^{\infty}$. Let

$$\begin{aligned}\omega^* &= \sup_t \frac{w_{21}^t}{w_{12}^t} \text{ and } \omega_* = \inf_t \frac{w_{21}^t}{w_{12}^t} \\ v^* &= \sup_t(-V_1^t) \text{ and } v_* = \inf_t(-V_1^t)\end{aligned}$$

Note that:

$$\begin{aligned}\omega^* &\leq \frac{\rho(-v)}{\rho(-v) + (1-\rho)v_*} \\ \omega_* &\geq \frac{\rho(-v)}{\rho(-v) + (1-\rho)v^*}\end{aligned}$$

Computing the continuation value of Player 1,

$$-V_1^t = \rho \sum_{m=t}^{\infty} (1-\rho)^{m-t} \alpha \omega^m \prod_{l=t}^{m} \omega^l$$

we get,

$$\begin{aligned}v_* &\geq \rho \sum_{m=t}^{\infty} (1-\rho)^{m-t} \alpha \omega_* \prod_{l=t}^{m} \alpha \omega_* (-v) = \frac{\alpha \omega_* \rho(-v)}{1-(1-\rho)\alpha\omega_*} \\ v^* &\leq \rho \sum_{m=t}^{\infty} (1-\rho)^{m-t} \alpha \omega^* \prod_{l=t}^{m} \alpha \omega^* (-v) = \frac{\alpha \omega^* \rho(-v)}{1-(1-\rho)\alpha\omega^*}\end{aligned}$$

But the above inequalities imply that,

$$\omega^* = \omega_* = \frac{1}{1+\alpha(1-\rho)}$$

Implying that the unique equilibrium probability that Player 1 wins is $1 - \frac{\alpha}{2(1+\alpha(1-\rho))}$. ■

6.4 Proofs for Section 4

We start with useful results about MPE in which only two players are active. With some abuse of notation, we denote the WTW (continuation value) of player i vis a vis player j as w^s_{ij} (V^s_i), i.e., replace the history superscript with that of the state, which is the only information relevant for an MPE.

Lemma A2: *Consider all-pay auctions. (i) Suppose m=2. In the unique MPE, at any state s,* $\min\{w^s_{ij}, w^s_{ji}\} > 0$ *and the players' cumulative distribution over bids* F_i *and* F_j *are determined by (for* $\min\{w^s_{ij}, w^s_{ji}\} = w^s_{ij}$*):*

$$F_j(b) = \frac{b}{w^s_{ij}};\ F_i(b) = \frac{w^s_{ji} - w^s_{ij} + b}{w^s_{ji}} \text{ for all } b \in [0, w^s_{ij}]$$

(ii) For any m, suppose that in equilibrium only players i and j are active and behave as in (i). Then for any other player k, if $w^s_{ki} + w^s_{kj} > 0$, *then* $\bar{H}_i(\tilde{b}_i, \tilde{b}_j, \mathbf{b}_{-\{i,j\}})w^s_{ki} + (1 - \bar{H}_i(\tilde{b}_i, \tilde{b}_j, \mathbf{b}_{-\{i,j\}}))w^s_{kj} \leq w^s_{ij}$ *where* $\mathbf{b}_{-\{i,j\}} = \mathbf{0}$.

Proof of Lemma A2: (i) Consider the first order conditions for player i and j :

$$f_j(b)w^s_{ij} = 1$$
$$f_i(b)w^s_{ji} = 1.$$

These imply the form of the distribution function above, with an atom on zero for F_i.[24] (ii) For some player k, any utility maximizing bid must satisfy the first order condition $f_iF_jw^s_{ki} + f_jF_iw^s_{kj} - 1 = 0$. The second order condition, using (i), is $f_if_j(w^s_{ki} + w^s_{kj}) \geq 0$. Hence utility maximizing bids are either 0 or the maximum bid which is w^s_{ij}. So for player k not to enter, we must have that his utility from a bid of zero is higher than the utility from the maximum bid, which implies the condition in the Lemma.■

Lemma A3: *Consider the simple Tullock function. Suppose that at some state s, only two players, i and j, are active in a pure strategy MPE. Then -for* $\mathbf{b}_{-\{i,j\}} = \mathbf{0}$– *(i)* $H_i(b_i, b_j, \mathbf{b}_{-\{i,j\}})w^s_{ji} = H_j(b_i, b_j, \mathbf{b}_{-\{i,j\}})w^s_{ij} = b_i + b_j$, *and (ii) for any other player k,* $H_j(b_i, b_j, \mathbf{b}_{-\{i,j\}})w^s_{kj} + (1 - H_j(b_i, b_j, \mathbf{b}_{-\{i,j\}}))w^s_{ki} \leq H_j(b_i, b_j, \mathbf{b}_{-\{i,j\}})w^s_{ij}$.

Proof of Lemma A3: The conditions in (i) are the first order conditions for i and j that must hold in equilibrium. For player k, expected utility from the

[24]Standard arguments from all-pay-auction (see Hillman and Riley 1989) imply the continuity, non atomness and same support of the distribution functions used in equilibrium.

equilibrium, given some bid b is

$$H_i(b_k, b_i, b_j, 0, .., 0)(-w^s_{ki}) + H_j(b_k, b_i, b_j, 0, .., 0)(-w^s_{kj}) - b_k$$

and the first order condition is

$$\frac{H_i(b_k, b_i, b_j, 0, .., 0)}{(b_k + b_i + b_j)} w^s_{ki} + \frac{H_j(b_k, b_i, b_j, 0, .., 0)}{(b_k + b_i + b_j)} w^s_{kj} - 1$$

Note that if the first order condition is positive at some point, then it also must be positive for $b_k = 0$. Thus, to check a possible deviation, it is sufficient to check that the condition is positive at $b_k = 0$. This together with the conditions in (i) of players i and j imply the condition in (ii).■

Proof of Proposition 2:

We first show when a fully gradual MPE exists for both the all-pay-auction and the simple Tullock function, both when $x_3 < 0$ and when $x_3 > 0$. We then consider the case of no negative externalities.

I. All-pay-auction:

We compute first the equilibrium for $s = 2$. The analysis follows Lemma A2: We conjecture that in equilibrium player 2 has a lower willingness to win[25], and is therefore the one who places an atom on zero of size $1 - \frac{w^2_{21}}{w^2_{12}}$, where

$$w^2_{21} = \rho x_2 + (1-\rho)(V^2_2 - V^1_2),$$
$$w^2_{12} = \rho x_2 + (1-\rho)(V^1_1 - V^2_1).$$

By Lemma A2,

$$V^1_2 = -x_2;\ V^2_1 = \frac{w^2_{21}}{w^1_{12}}(-\rho x_2 + (1-\rho)V^2_1)$$

and plugging for these values, we can solve for the ratio $\frac{w^2_{21}}{w^2_{12}} = \frac{1}{2-\rho}$, implying that the atom is of size $\frac{1-\rho}{2-\rho}$. Thus, $V^2_1 = -\rho x_2$, $V^2_2 = -x_2$.

We now consider the case of $x_3 < 0$. In this case, $V^2_3 = \frac{-3|x_3| - x_2\rho + |x_3|\rho}{3-\rho}$.

Now consider the equilibrium in which players 2 and 3 only are active at $s = 3$ and both ignore Player 1. The willingness to win of each player is:

$$w^3_{23} = \rho(|x_3| + x_2) + (1-\rho)(V^2_2 - V^3_2);$$
$$w^3_{32} = \rho(|x_3| + x_2) + (1-\rho)(V^3_3 - V^2_3);$$

[25] Indeed, conjecturing the opposite leads to a contradiction.

Conjecture that the atom on zero is on player 3 (the opposite cannot arise). Let the size of the atom be δ. Then:

$$\begin{aligned}
V_2^3 &= \delta(1-\rho)(-x_2) + (1-\delta)(-\rho(|x_3|+x_2) + (1-\rho)V_2^3 \\
V_2^3 &= \frac{\delta(1-\rho)(-x_2) - (1-\delta)\rho(|x_3|+x_2)}{1-(1-\delta)(1-\rho)} \\
V_2^2 - V_2^3 &= \frac{\rho(|x_3|(1-\delta) - \delta x_2)}{\delta(1-\rho)+\rho} \\
V_3^3 - V_3^2 &= -\rho(|x_3| + x_2 + V_3^2)
\end{aligned}$$

We can solve for $\delta = 1 - \frac{w_{32}^3}{w_{23}^3}$ to find:

$$\delta(\rho) = \frac{3|x_3| + 3x_2\rho - 4|x_3|\rho - 5x_2\rho^2 + 2x_2\rho^3 + |x_3|\rho^2}{6|x_3| + 7x_2\rho - 5|x_3|\rho - 7x_2\rho^2 + 2x_2\rho^3 + |x_3|\rho^2}$$

Note that $\delta(\rho) \underset{\rho\to 0}{\to} \frac{1}{2}$.

Now assume that Player 1 is not active, but is part of the game. When Player 2 considers placing a bid of zero, given the atom of Player 3 on zero, Player 1 may now win with some probability. Note however that it is not optimal for Player 2 to place a bid on zero, as a marginal bid wins against both other players, a profitable deviation for Player 2, as his WTW against each is positive. This implies that the analysis above is valid for the case in which Player 1 exists but is not active.

Note that $\delta(\rho) \geq 0$ for all ρ. This allows to compute

$$\begin{aligned}
&\frac{1+\delta(\rho)}{2} w_{12}^3 + \frac{1-\delta(\rho)}{2} w_{13}^3 \\
&= \rho\left(\frac{1+\delta(\rho)}{2} x_2 + \left(\frac{1-\delta(\rho)}{2}\right)|x_3|\right) + \\
&\quad (1-\rho)\left(-\frac{1+\delta(\rho)}{2} V_1^2 - \left(\frac{1-\delta(\rho)}{2}\right) V_1^3\right)
\end{aligned}$$

where

$$V_1^3 = -\frac{\frac{1+\delta}{2}(\rho x_2 + (1-\rho)\rho x_2) + \frac{1-\delta}{2}\rho|x_3|}{1-(1-\rho)\frac{1-\delta}{2}}.$$

Recall that the willingness to win of player 1 vis a vis any other player is positive. We can therefore use the condition in Lemma A2. To check that $\frac{1+\delta(\rho)}{2} w_{12}^3 + \frac{1-\delta(\rho)}{2} w_{13}^3 - w_{32}^3 < 0$ we note that the lhs is maximal for $\rho \to 0$. We therefore compute $\lim_{\rho\to 0}[\frac{1+\delta(\rho)}{2} w_{12}^3(\rho) + \frac{1-\delta(\rho)}{2} w_{13}^3(\rho) - w_{32}^3(\rho)] = 3x_2 - |x_3|$ to get the condition of $\frac{|x_3|}{x_2} > 3$.

For the case of $x_3 > 0$, we follow the same strategy as above and find that the fully gradual equilibrium holds for all ρ iff $4 < \frac{x_3}{x_2}$ and that player 3 wins the stage game when $s = 3$ with a higher probability than when $x_3 < 0$.

II. Simple Tullock function:

We compute the equilibrium when $s = 2$ and players 1 and 2 compete. The expected utility of player 1 from bid b_1 is

$$\frac{b_1}{b_1+b_2}u_{11}^2 + \frac{b_2}{b_1+b_2}u_{12}^2 - b_1$$

where the first order condition is

$$\frac{b_2}{(b_1+b_2)^2}u_{11}^2 - \frac{b_2}{(b_1+b_2)^2}u_{12}^2 - 1 = 0 \tag{1}$$

which together with the *foc* for player 2 implies that

$$\frac{w_{12}^2}{w_{21}^2} = \frac{b_1}{b_2} \tag{2}$$

where,

$$\begin{aligned}
w_{12}^2 &= \rho x_2 + (1-\rho)(-V_1^2) \\
w_{21}^2 &= \rho x_2 + (1-\rho)(V_2^2 + x_2) \\
V_1^2 &= \frac{\frac{b_2}{b_1+b_2}(-\rho x_2) - b_1}{1 - \frac{b_2}{b_1+b_2}(1-\rho)} \\
V_2^2 + x_2 &= \frac{-b_2 + x_2\frac{b_2}{b_1+b_2}\rho}{1 - \frac{b_2}{b_1+b_2}(1-\rho)}.
\end{aligned}$$

Solving the system of the first order equations (1) and (2), we find that:

$$\begin{aligned}
b_1 &= \rho b_2 \frac{x_2}{-2b_2 + 2\rho b_2 + \rho x_2} \underset{\rho\to 0}{\approx} \frac{\rho x_2}{2} \\
b_2 &= x_2 \frac{\sqrt{-\rho+\rho^2+1} - 1}{2\rho - 2} \underset{\rho\to 0}{\approx} \frac{\rho x_2}{4}
\end{aligned}$$

In the limit, as $\rho \to 0$:

$$\begin{aligned}
V_1^2 &\approx -\frac{5\rho x_2}{4}, \\
V_2^2 + x_2 &\approx -\frac{3\rho x_2}{8}
\end{aligned}$$

Again we first consider the case of $x_3 < 0$. For this case, $V_3^2 \approx (-|x_3|) + \frac{1}{2}\rho(-|x_3| - x_2)$. Now consider the game between players 2 and 3 at $s = 3$. Let d_i be the equilibrium bid of player i. First order conditions for 2 and 3 are:

$$\begin{aligned}
\frac{w_{23}^3}{w_{32}^3} &= \frac{d_2}{d_3} \\
\frac{d_3}{(d_2+d_3)^2}w_{23}^3 &= 1
\end{aligned}$$

where,

$$
\begin{aligned}
w_{23}^3 &= \rho(x_2+|x_3|)+(1-\rho)(V_2^2-V_2^3)\\
w_{32}^3 &= \rho(x_2+|x_3|)+(1-\rho)(V_3^3-V_3^2)\\
V_2^2-V_2^3 &= \frac{\frac{-b_2-\frac{b_1}{b_1+b_2}x_2}{1-\frac{b_2}{b_1+b_2}(1-\rho)}\rho+d_2+\frac{d_3}{d_2+d_3}\rho(x_2+|x_3|)}{1-\frac{d_3}{d_2+d_3}(1-\rho)}\\
V_3^3-V_3^2 &= \frac{-d_3-\frac{d_2}{d_2+d_3}\rho(x_2+|x_3|)-\rho\frac{\frac{b_1}{b_1+b_2}(-|x_3|)+\frac{b_2}{b_1+b_2}\rho(-|x_3|-x_2)}{1-(1-\rho)\frac{b_2}{b_1+b_2}}}{1-\frac{d_3}{d_2+d_3}(1-\rho)}
\end{aligned}
$$

With these expressions we solve the set of first order conditions above in the limit as $\rho \to 0$ we find that

$$
\begin{aligned}
d_2 &\approx \frac{1}{2}|x_3|\rho\\
d_3 &\approx \frac{1}{4}|x_3|\rho
\end{aligned}
$$

and thus

$$
\begin{aligned}
V_1^3 &\approx \rho(-\frac{1}{2}|x_3|-\frac{9}{4}x_2)\\
V_2^2-V_2^3 &\approx \rho(\frac{5}{4}|x_3|-x_2)
\end{aligned}
$$

We now consider the condition in Lemma A3 to allow for full gradualism:

$$
\frac{d_2}{d_2+d_3}w_{12}^3+\frac{d_3}{d_2+d_3}w_{13}^3 < \frac{d_3}{d_2+d_3}w_{23}^3 \iff
$$

$$
\begin{aligned}
&\frac{d_3}{d_2+d_3}(\rho|x_3|+(1-\rho)(-V_1^3))+\frac{d_2}{d_2+d_3}(\rho x_2+(1-\rho(-V_1^2))\\
&<\frac{d_3}{d_2+d_3}(\rho(x_2+|x_3|)+(1-\rho)(V_2^2-V_2^3)
\end{aligned}
$$

When $\rho \to 0$, this amounts to:

$$
\begin{aligned}
\frac{1}{3}(|x_3|+\frac{1}{2}|x_3|+\frac{9}{4}x_2)+\frac{2}{3}(x_2+\frac{5x_2}{4}) &= \frac{1}{3}(x_2+|x_3|+1.25|x_3|-x_2) \iff\\
x_3 &> 9x_2
\end{aligned}
$$

and so an equilibrium will exist iff $\frac{|x_3|}{x_2} > 9$.

Now consider $x_3 > 0$. For this case, $V_3^2 \approx -\frac{1}{\rho+2}\left(2x_3-\rho x_2+\rho x_3\right)$. Now consider the game between players 2 and 3 at $s=3$. Solving for the first order conditions in the limit as $\rho \to 0$ we find that

$$d_2 \approx \rho \frac{1}{4}\sqrt{2}\frac{16x_3 - 32x_2}{24x_3 - 32x_2}$$
$$d_3 \approx \rho\sqrt{2}\frac{x_3}{6x_3 - 8x_2}$$

In the limit, the probability that Player 2 wins is $\lim_{\rho\to 0} \frac{d_2}{d_2+d_3} = \frac{2x_2 - x_3}{2x_2 - 2x_3}$, which holds when $x_3 > 2x_2$. Again using the condition in Lemma A3 as above, in the limit, amounts to showing that:

$$\frac{9x_2}{4} + \frac{d_3}{d_2}x_3 < \frac{d_2}{d_2+d_3}x_3 \iff$$
$$x_3 > x_2\frac{9}{4}\frac{1}{\frac{d_2}{d_2+d_3} - \frac{d_3}{d_2}} \iff$$
$$x_3 > x_2\frac{9}{4}\frac{1}{\frac{2x_2-x_3}{2x_2-2x_3} - \frac{x_3}{x_3-2x_2}}$$

When $x_3 < 0$, full gradualism MPE exists whenever $|x_3| > 9x_2$. To show that when $x_3 > 0$ it exists for less parameters, it is sufficient to show that $\frac{9}{4}\frac{1}{\frac{2x_2-x_3}{2x_2-2x_3} - \frac{x_3}{x_3-2x_2}} > 9$. This indeed holds for all $x_3 > 2x_2$ (a necessary condition for this equilibrium to hold).

We have therefore proven (i),(ii) and (iii).

To prove (iv) we first prove another Lemma. Let α be the probability that player 2 wins in equilibrium in $s = 3$ when only players 2 and 3 are active and only they win with a strictly positive probability:

Lemma A4: *Suppose that Player 1 is not active and wins with a zero probability at $s = 3$. Then for any H, $\alpha w_{12}^3 + (1-\alpha)w_{13}^3 > w_{23}^3$.*

Proof of Lemma A4:

(i) $w_{12}^2 \geq w_{21}^2$:

$$w_{12}^2 = \rho(-v) + (1-\rho)(-V_1^2) \geq \rho(-v) + (1-\rho)(V_2^2 - v) = w_2^2$$

as $V_1^2 + V_2^2 \leq v$.

(ii) $w_{21}^2 > w_{23}^3$ as $V_2^3 \geq v$:

$$w_{21}^2 = \rho(-v) + (1-\rho)(V_2^2 - v) \geq \rho(-v) + (1-\rho)(V_2^2 - V_2^3) = w_{23}^3.$$

(iii) $w_{13}^3 > w_{12}^2$:

$$w_{13}^3 = \rho(-v) + (1-\rho)(-V_1^3) > \rho(-v) + (1-\rho)(-V_1^2) = w_{12}^2$$

as $V_1^2 > V_1^3$: in state $s = 3$, as long as the game remains in this state, player 1's instantaneous utility is v, and $V_1^2 \geq v$.

From (i), (ii), and (iii), $\alpha w_{12}^3 + (1-\alpha) w_{13}^3 > w_{23}^3$.□

By Lemmata A2, A3 and A4, there exists a deviation in both the all-pay auction and the simple Tullock function whenever player 1 wins with probability converging to zero. This proves (iv). To render the result meaningful, we now show that an MPE exist in the all-pay-auction for N players; for convenience let us normalize the payoff so that the payoff from winning is $v > 0$ and the payoff from losing is 0.

Note that players' continuation values are at least 0 at any stage game. Second, consider $s = 2$ and note that the only players that may potentially submit strictly positive bids are 1 and 2. The atom must be on 2 and the solution involves $V_1^2 = v(1-\rho)$, $w_{12}^2 = \rho v(2-\rho)$, $V_2^2 = 0$ and $w_{21}^2 = \rho v$. Suppose, by way of induction, that for every state $l < s$, the equilibrium is as in Baye et al (1996) in which player 1 has the highest willingness to pay. In particular this implies that there is an atom on bid zero for all players beside Player 1 and so $v > V_1^l > 0$ and $V_i^l = 0$ for all $1 < i \leq l$, implying that $w_{1l}^l > \rho v$ and that $w_{ij}^l = \rho v$ for all $1 < i \leq l$, $j \leq l$.

Suppose we are now at state s. Note that players $i > s$ do not participate. For the remaining players, consider the family of equilibria as in Baye et al (1996)[26] in which (i) $\Pi_i F_i(0) = \frac{1-\rho}{2-\rho}$ for all $i \neq 1$ that participate, and so $F_i(0) > 0$ for any such i, (ii) player 1 participates in every stage and he wins with a higher probability than any other. We then have that $w_{ij}^s = \rho v + (1-\rho)(V_i^i - V_i^j) = \rho v$ for all $i \neq 1 < s, j \leq s$, and $w_{1i}^s = \rho v + (1-\rho)(v - V_1^i) > \rho v$ as $V_1^i < v$ by the induction and by the above for all $i \leq s$. The conjectured family of equilibria is therefore sustained.

This completes the proof of Proposition 2.■

[26] There is in fact a continuum of equilibria.

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