Toward an Efficiency Rationale for the Public Provision of Private Goods[*]

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Abstract

This paper shows that public provision of private goods may be justified on pure efficiency grounds in an environment where individuals consume both public and private goods. The government's involvement in the provision of private goods provides it with information about individuals' private good purchases that facilitates more efficient revenue extraction for the provision of public goods. We show that public provision of the private good improves economic efficiency under a condition that is always fulfilled under stochastic independence and satisfied for an open set of joint distributions. Our model is an example where there is an efficiency loss from separating revenue and expenditure problems, and is therefore of more general interest for the study of optimal taxation.

Keywords: Publicly Provided Private Goods; Constrained Efficiency; Optimal Taxation.

JEL Classification Number: D61, D82, H42.

1 Introduction

All governments in developed countries not only provide public goods, but also devote considerable resources to the provision of private goods such as health care, housing, day care and education. This paper provides a novel *efficiency rationale* for such public provision of private goods, based on ideas from the literature on commodity bundling.

We model the government as a benevolent social planner who seeks to maximize the expected social surplus in an economy where citizens value a (binary) excludable public good, a (binary) private good, and where utility can be transferred using a perfectly divisible commodity called "money." The key friction is that citizens' valuations for the private and public goods are their private information. Accordingly, we require that the government restrict attention to provision mechanisms that are incentive compatible. In addition, we impose budget balance and individual rationality constraints.

We consider the optimal mechanism under two cases that differ in whether the government can keep track of individuals' purchases of the private good and use that information in constructing fees and access rules for the public good.

As a benchmark, we consider what we call the "separate provision" case, under which the planner designs provision mechanisms for the public and private goods *separately*. It is useful to imagine this as a setup where two government agencies are separately in charge of the public and private good provisions, even though they share a single budget constraint. The key restriction is that the two agencies are not allowed to share information about an individual's purchase in the other agency, otherwise any feasible provision mechanism can be used. In this case, the mechanism design problem simplifies to a problem where the planner sets an access fee for the public good, a provision probability of the public good, and a price for the private good. The solution to this problem has a flavor of *Ramsey pricing*, and the price of the private good is always set above its marginal cost of production. This is because the public good is generally under-provided due to the informational constraints, implying that a sufficiently small "tax" on the private good (which is the wedge between the the marginal cost for the private good and its price) leads to a second-order efficiency loss in the private good market, but generates a first-order efficiency gain due to the increased probability of public good provision. A similar logic also delivers a less obvious result. There should always be a strictly positive access fee for the public good, thus a set of types with strictly positive measure will be excluded from usage. Moreover, we show that whenever the public good is desirable in a first-best world, it will be provided with positive probability in the optimal

separate provision mechanism, with some of its cost covered by the "taxes" collected from the private goods.

The separate provision case is then contrasted with what we refer to as the "joint provision" case, where we allow the social planner to use *all available information* for decisions on both goods. The key difference from the separate provision case is that here a single government agency is in charge of the provision of both the public and private goods, and this agency can therefore condition the pricing and access rules for the public good on information revealed in the private good market. We are primarily interested in the qualitative question of whether joint provision mechanisms can improve expected social welfare over the *optimal* separate provision mechanisms.

The multidimensional nature of the joint provision mechanism design problem prevents us from characterizing the optimal joint provision mechanism. As a result, we follow McAfee *et al.* (1989) and consider "local deviations" from the optimal separate provision mechanism we earlier characterized. We show that, under some conditions, there *always* exists a joint provision mechanism that improves upon the optimal separate provision mechanism in a *subspace* of the feasible joint provision mechanisms. The particular subspace we consider consists of mechanisms fully characterized by three fixed prices, one for the private good, one for the public good, and one for the bundle consisting of both goods. Within this restricted set of feasible joint provision mechanisms, we provide a sufficient conditions for when there exist mechanisms in which *the price of the private good an individual pays depends on whether she also consumes the public good* that generate a higher social welfare than the optimal separate provision mechanism.[1] Importantly, our sufficient condition is fulfilled under stochastic independence between the valuations of the two goods, and thus satisfied by an open set of joint distributions.

The above result can be interpreted as a new efficiency rationale for the public provision of private goods. The justification for such an interpretation is that in order to implement such a pricing policy, it is necessary that the planner keeps track of which individuals purchase the private good and somehow tax them on the basis of whether they also consume the public good. One practical way to achieve this is to allow all agents who opt into (by paying taxes) the "government bundle" access to both the private and the public good, while those who only want to consume the private good pay a separate and different price for the private good alone. This is consistent with

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[1] It is well-known from Adams and Yellen (1976) and McAfee *et al.* (1989) that bundling can increase revenue for a multiproduct monopolist. Fang and Norman (2008) showed that it can also improve social welfare for the case of multiple public good provision. The current paper extends this intuition further to the case of bundling public and private goods.

some important features of local public finance in the U.S.: first, residents who pay property taxes can enjoy the services, many of which has the nature of private goods (e.g., public schools), provided by the local government "for free"; second, most public universities charge different amounts of tuition for in-state or out-of-state residents.[2]

Our paper makes two major contributions to the existing literature on why it may be desirable to provide certain private goods publicly.[3] First, the role played by public provision of private goods in our paper differs substantially from explanations emphasized in the existing literature. In our paper, the government is involved in the provision of private goods because such involvement provides it with information of the private good demands of the individuals, which in turns allows the social planner to more efficiently extract revenues for the provision of public goods. This explanation has more in common with the literature on commodity bundling than the previous literature. In particular, we depart from all existing efficiency based explanations in that we do not rely on preferences for redistribution. This feature of our model is particularly desirable because several important publicly-provided private goods have neutral or regressive distributional effects.

Second, our paper imposes considerably less restrictions than the existing literature on the set of admissible policies in establishing our results. The benchmark outcome we use for comparison is the outcome that can be implemented under the optimal separate provision mechanism; this is in stark contrast to the existing literature where the benchmark outcome is often derived under various implicit restrictions on the market operations. The joint provision mechanism that improves upon the benchmark outcome in our model does require that consumers cannot engage in arbitrage. This restriction, however, seems reasonably descriptive of many publicly-provided private goods that we see in reality. For example, in-state college tuition, public health insurance, day care and public schools are all commodities that are difficult to resell, thus limiting arbitrage. This is important because our model thus also provides an explanation of why some, but not all, private goods are publicly provided.

Our analysis also illustrates a more general point, which is of relevance for the literature on optimal taxation and the traditional division of public economics into taxation and expenditure as separate subfields. Since Ramsey (1927), it has been standard in the optimal taxation literature to ask how to best raise a given target revenue without considering what the tax revenue is intended

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[2]In our setup, the in-state tuition can be interpreted as the price of the private good (college education) for those who also purchase public good; and out-of-state tuition is the price for those who purchase the private good alone.

[3]We review the existing literature on the public provision of private goods in Section 2.

for. Our model is a simple yet plausible example where this standard approach is no longer valid, since the most efficient way to distribute and finance the public good in our model *must* be jointly determined.[4] In other words, the optimal commodity tax to finance the (excludable) public good depends on whether the consumer gets access to the public good. Hence, the analytically convenient dichotomy between government expenditures and revenue comes with an efficiency loss.

The remainder of the paper is structured as follows. Section 2 reviews the related literature on the public provision of private goods; Section 3 presents the model; Section 4 characterizes the optimal separate provision mechanism; Section 5 presents our main results regarding the optimality of public provision of private goods; and finally Section 6 concludes.

2   Related Literature

It is useful to divide the existing literature on public provision of private goods into three strands. The first strand, as represented by Fernandez and Rogerson (1995) and Epple and Romano (1996), is a positive theory of the public provision of private goods driven by political economy considerations.[5] Epple and Romano (1996) consider a political economy model where the level of a private good (say, "health care") and whether private purchases are permitted to supplement the public provision are both determined by majority voting. They show that, under standard assumptions on voter preferences, a regime with positive government provision and no restriction on private supplement is majority preferred to a regime of either only market provision or only government provision. The key intuition is that the median voter, which typically has income below the mean in the population, will receive positive transfers in equilibrium from richer citizens if private goods are publicly provided. Fernandez and Rogerson (1995) use a similar model, with the crucial difference that the private good is only partially subsidized, to show that it is also possible to explain public provision of private goods with regressive distributional consequences. While the political economy considerations in these models are plausible and possibly important for a positive understanding of why government provides private goods, their results do hinge crucially on the restrictions imposed on the policy instruments; that is, what is not explained is why transfers must be in-kind (in the form of publicly provided private goods) instead of pure cash transfers.[6]

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[4]Related points are made in Boadway *et al.* (1998) and Blomquist and Christiansen (2007).

[5]See also Bergstrom and Blomquist (1996) and Gouveia (1997).

[6]However, see Blomquist and Christiansen (1999) for a model where efficiency gains of in-kind transfers are combined with a political economy setup.

The second strand assumes that the government has a preference for redistribution and shows that public provision of private goods may serve as a tool for redistribution from the rich to the poor. As in the first strand of the literature, it also assumes, instead of explains, why in-kind transfers are preferred to cash transfers. For example, Besley and Coate (1991) considered the case where the publicly provided private goods are discrete. In their model, if a household is dissatisfied with the (uniform) quality of publicly provided private goods, it opts out and purchases a higher quality version from the market. If the quality demand increases with income, mainly rich households opt out from the publicly provided private goods, implying that the system of public provision of private goods can serve as a transfer towards less wealthy individuals.[7] As Besley and Coate (1991) acknowledged in their concluding remarks, there may exist other feasible policies that can achieve the same distributional goals more efficiently if one does not restrict attention to in-kind transfers.[8]

The third strand, to which our paper is most closely related, takes a normative perspective and identifies circumstances under which providing a private good in-kind is a more efficient transfer instrument than cash transfers. Blackorby and Donaldson (1988) considered an environment where consumers have private information about their preferences and showed that in-kind transfers can be desirable due to its screening role, which allows better targeted transfers.[9] Coate (1995) considered an environment where the rich has altruistic preferences towards the poor and would like to insure the poor's income risks. If cash assistance is given to the poor, the poor may have incentives not to purchase insurance and exploit the well-know Samaritan's Dilemma. As a result, the rich may instead prefer to directly provide insurance to the poor. His explanation crucially relies on the rich's lack of commitment not to help the poor if the poor does not purchase insurance with the cash assistance.[10] Garratt and Marshall (1994) considers the case of the public financing of college education. They argue that public financing of college education (which is a private good) provides gambles that families desire. Their explanation implicitly relies on the unmodeled notion that private market can not provide such lotteries demanded by the families.

Our paper complements the above literature by analyzing the full mechanism design problem faced by a social planner when individuals have private information about their preferences. Notably, we do not impose any exogenous restrictions on the set of feasible policy instruments other

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[7] Other arguments based on a desire to redistribute income can be found in Blomquist and Christiansen (1995) and Cremer and Gahvari (1997).

[8] See Gahvari and Mattos (2007) for an interesting extension of Besley and Coate (1991).

[9] See also Nichols and Zeckhauser (1982) for similar arguments.

[10] A similar model is considered by Bruce and Waldman (1991).

than the natural participation and incentive compatibility constraints in establishing our comparison benchmark of what can be achieve by the market in the absence of public provision of private goods. The key role played by the public provision of private goods in our paper also substantially differs from targeted transfer (Blackorby and Donaldson 1988), commitment (Coate 1988) and demand for gambles (Garratt and Marshall 1994). In our paper, the government's involvement in the provision of private goods provides it with information about individuals' private good purchases that facilitates more efficient revenue extraction for the provision of public goods. The feature of our model that the efficiency rationale for the public provision of private goods does not rely on redistributive preferences is particularly attractive because several important publicly-provided private goods have neutral or regressive distributional effects.

3 The Environment

Consider an economy populated by a continuum of *ex ante* identical consumers. Consumers have preferences over a binary and excludable public good, a binary private good and a perfectly divisible numeraire good that we will refer to as "money." The public good can be produced at a cost $K > 0$; and the binary private good can be produced at unit cost $c > 0$.

A consumer is characterized by $(\theta_G, \theta_P) \in \Theta_G \times \Theta_P \subseteq R^2$ where θ_G is her valuation for the public good and θ_P is her valuation for the private good. A consumer's valuations (θ_G, θ_P) is her private information. We denote $F : \Theta_G \times \Theta_P \to [0, 1]$ as the cumulative distribution over consumer types (θ_G, θ_P), and write $F_G : \Theta_G \to [0, 1]$ and $F_P : \Theta_P \to [0, 1]$, respectively, as the marginal cumulative distribution of θ_G and θ_P. Consumers are assumed to be risk neutral. The expected payoff of a consumer with type (θ_G, θ_P) who consumes the public good with probability ϕ_G and the private good with probability ϕ_P, and gives up (in expectation) m units of money is given by

$$u(\phi_G, \phi_P, m; \theta_G, \theta_P) = \phi_G \theta_G + \phi_P \theta_P - m. \tag{1}$$

The reader may observe that we have built in the assumption that wealth constraints are always non-binding in (1). If $\overline{\theta_G}$ and $\overline{\theta_P}$ are upper bounds for θ_G and θ_P respectively, this assumption can be justified if every consumer has an endowment of money in excess of $\overline{\theta_G} + \overline{\theta_P}$. We assume that $c < \overline{\theta_P}$.

Remark 1. We have chosen to model the agents in the economy as a continuum in order to obtain a clean characterization of the optimal mechanism. The characterization of the constrained efficient

mechanism is much more complicated in a finite population economy with private information. In particular, the optimal provision rule depends in a non-trivial way on the realized types. However, in a finite economy with sufficiently many agents, the effects of making the provision rule depend on announcements become negligible due to intuition similar to that of the "Paradox of Voting." As a result, the optimal mechanism can be approximated by a mechanism where no agent is pivotal to the decision.[11] Hence, our convenient continuum assumption can be viewed as an approximation of a large finite economy.[12]

Remark 2. However, the continuum specification has a few awkward technical aspects. In particular, we will treat the distribution F as a cross-section of agents, which is necessary to interpret the right hand side of the balanced budget constraint (BB) in Subsection 4.1 as the total revenue from the mechanism; at the same time, the distribution F is also interpreted as the probability distribution over types for any individual agents. As is well-known, this is inconsistent with stochastic independence across agents in standard probability measures (see, however, Al-Najjar 2004 for a solution to the problem).

4 Optimal Separate Provision Mechanisms

In this section we analyze the benchmark case where the mechanism designer cannot condition either the price or the provision probability of one good on a consumer's reported valuation for the other good. The most obvious way to interpret this setup is that the "markets" for the private and public goods are physically separated in space and that the designer lacks the technology to track behavior of individual agents across markets. Formally:

Definition 1 *A **separate provision mechanism** is a quadruple* (ϕ_G, t, ϕ_P, p), *where:*

- $\phi_G : \Theta_G \to [0,1]$ *is the probability of consuming the public good;*

- $t : \Theta_G \to \mathrm{R}$ *is the fee for consuming the public good;*

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[11]See Norman (2004). Schmitz (1997) also makes essentially the same point in the context of monopolistic provision of an excludable public good.

[12]Note, however, that to get a limit characterization corresponding to the one in this paper, it is necessary to assume that per capita provision costs stay bounded away from zero as the number of participants goes to infinity. Otherwise, a pivot mechanism will work also with a large finite set of agents (see Hellwig 2003).

- $\phi_P : \Theta_P \to [0, 1]$ *is the probability of consuming the private good;*

- $p : \Theta_P \to \mathrm{R}$ *is the fee for consuming the private good.*

Note that both ϕ_G and t are functions of the reported valuation for the public good only; and ϕ_P and p are functions of the reported valuation of the private good only. As will be made clear below, our notion of separate provision mechanism still leaves room for cross-subsidization between the private and public goods, which is the only link allowed between the two design problems.

4.1 The Planning Problem

Given a separate provision mechanism, the expected utility for an agent of type (θ_G, θ_P) is given by $\phi_G(\theta_G)\theta_G - t(\theta_G) + \phi_P(\theta_P)\theta_P - p(\theta_P)$. We assume that the planner seeks to maximize the *ex ante* expected utility of the representative consumer.

The objective function of the planner can be written as

$$
\int_{\Theta_G} \int_{\Theta_P} \left[\phi_G(\theta_G)\theta_G - t(\theta_G) + \phi_P(\theta_P)\theta_P - p(\theta_P)\right] dF(\theta_G, \theta_P) \tag{2}
$$
$$
= \int_{\Theta_G} \left[\phi_G(\theta_G)\theta_G - t(\theta_G)\right] dF_G(\theta_G) + \int_{\Theta_P} \left[\phi_P(\theta_P)\theta_P - p(\theta_P)\right] dF_P(\theta_P).
$$

Since types are assumed to be private information, agents must be willing to disclose their preferences to the planner. That is, it must be *incentive compatible* to report truth-fully,

$$
\phi_G(\theta_G)\theta_G - t(\theta_G) + \phi_P(\theta_P)\theta_P - p(\theta_P) \geq \phi_G(\widehat{\theta_G})\theta_G - t(\widehat{\theta_G}) + \phi_P(\widehat{\theta_P})\theta_P - p(\widehat{\theta_P}), \quad \text{(IC)}
$$
$$
\forall (\theta_G, \theta_P), (\widehat{\theta_G}, \widehat{\theta_P}) \in \Theta_G \times \Theta_P.
$$

We also assume that agents must be willing to participate. Given the continuum-agent formulation, there is no distinction between interim and *ex post* participation constraints, and assuming that the non-participation utility is constant (and normalized to zero), we may write these constraints as

$$
\phi_G(\theta_G)\theta_G - t(\theta_G) + \phi_P(\theta_P)\theta_P - p(\theta_P) \geq 0, \quad \forall (\theta_G, \theta_P) \in \Theta_G \times \Theta_P. \tag{IR}
$$

Finally, we assume that the planner must satisfy the natural feasibility constraint, namely that the total costs for the production of the public and the private goods should not exceed the total revenue collected from the agents. We may write this constraint as

$$
K \left[\sup_{\theta_G \in \Theta_G} \phi_G(\theta_G)\right] + \int_{\Theta_P} c\phi_P(\theta_P) dF_P(\theta_P) \leq \int_{\Theta_G} t(\theta_G) dF_G(\theta_G) + \int_{\Theta_P} p(\theta_P) dF_P(\theta_P). \tag{BB}
$$

8

To understand (BB), note that, since the public good is non-rival, the cost is independent of the number of agents actually consuming the good. That is, it costs K if the good is provided and 0 otherwise. Our formulation allows the mechanism designer to randomize between provision and non-provision. While (BB) says that budget balances in expectation, it is easy to adjust transfers without changing the interim expected payoffs in such a way as to balance the budget for sure (see, e.g., Borgers and Norman 2008).

4.2   Solving the Planning Problem

First, note that an immediate implication of separate provision mechanism assumption is that a mechanism satisfies incentive compatibility constraints (IC) if and only if

$$\phi_G\left(\theta_G\right)\theta_G - t\left(\theta_G\right) \geq \phi_G(\widehat{\theta_G})\theta_G - t(\widehat{\theta_G}) \quad \forall \theta_G, \widehat{\theta_G} \in \Theta_G, \tag{3}$$

$$\phi_P\left(\theta_P\right)\theta_P - p\left(\theta_P\right) \geq \phi_P(\widehat{\theta_P})\theta_P - p(\widehat{\theta_P}) \quad \forall \theta_P, \widehat{\theta_P} \in \Theta_P, \tag{4}$$

where (3) can be viewed as the incentive compatibility constraint for the public good provision and (4) is the incentive compatibility constraint for the private good provision. Hence, the separability restrictions make the model completely single-dimensional, and equivalent to a model where one set of agents care only for the public good and another set of agents care only about the private good.

Further note that, if $(\phi_G^*, t^*, \phi_P^*, p^*)$ is an optimal separate provision mechanism, then (ϕ_G^*, t^*) solves the following problem given (ϕ_P^*, p^*) :

$$\max_{(\phi_G, t):\Theta_G \to [0,1] \times \mathbb{R}} \left\{ \int_\Theta [\phi_G\left(\theta_G\right)\theta_G - t\left(\theta_G\right)] dF_G\left(\theta_G\right) + \int_{\theta_P} [\phi_P^*\left(\theta_P\right)\theta_P - p^*\left(\theta_P\right)] dF_P\left(\theta_P\right) \right\} \tag{5}$$

s.t. $\quad 0 \leq \phi_G\left(\theta_G\right)\theta_G - t\left(\theta_G\right) - \phi_G(\widehat{\theta_G})\theta_G + t(\widehat{\theta_G}), \quad \forall \theta_G, \widehat{\theta_G} \in \Theta_G, \tag{6}$

$$0 \leq \phi_G\left(\theta_G\right)\theta_G - t\left(\theta_G\right) + \phi_P^*\left(\theta_P\right)\theta_P - p^*\left(\theta_P\right), \quad \forall\left(\theta_G, \theta_P\right) \in \Theta_G \times \Theta_P,$$

$$0 \leq \int_{\Theta_G} t\left(\theta_G\right) dF_G\left(\theta_G\right) + \int_{\Theta_P} p^*\left(\theta_P\right) dF_P\left(\theta_P\right) - K \left[\sup_{\theta_G \in \Theta_G} \phi_G\left(\theta_G\right) \right] - \int_{\Theta_P} c\phi_P^*\left(\theta_P\right) dF_P\left(\theta_P\right); \tag{7}$$

and (ϕ_P^*, p^*) solves the following problem given (ϕ_G^*, t^*) :

$$\max_{(\phi_P, p):\Theta_P \to [0,1] \times \mathbb{R}} \left\{ \int_{\Theta_G} [\phi_G^*\left(\theta_G\right)\theta_G - t^*\left(\theta_G\right)] dF_G\left(\theta_G\right) + \int_{\Theta_P} [\phi_P\left(\theta_P\right)\theta_P - p\left(\theta_P\right)] dF_P\left(\theta_P\right) \right\} \tag{8}$$

s.t. $\quad 0 \leq \phi_P\left(\theta_P\right)\theta_P - p\left(\theta_P\right) - \phi_P(\widehat{\theta_P})\theta_P + p(\widehat{\theta_P}) \quad \forall \theta_P, \widehat{\theta_P} \in \Theta_P \tag{9}$

$$0 \leq \phi_G^*\left(\theta_G\right)\theta_G - t^*\left(\theta_G\right) + \phi_P\left(\theta_P\right)\theta_P - p\left(\theta_P\right) \quad \forall\left(\theta_G, \theta_P\right) \in \Theta_G \times \Theta_P$$

$$0 \leq \int_{\Theta_G} t^*\left(\theta_G\right) dF_G\left(\theta_G\right) + \int_{\Theta_P} p\left(\theta_P\right) dF_P\left(\theta_P\right) - K \left[\sup_{\theta_G \in \Theta_G} \phi_G^*\left(\theta_G\right) \right] - \int_{\Theta_P} c\phi_P\left(\theta_P\right) dF_P\left(\theta_P\right).$$

Both (5) and (8) are problems that can be solved using standard techniques going back to Myerson's

(1981) analysis of optimal auction design. Define the "indirect utility functions"

$$
\begin{aligned}
U_G\left(\theta_G\right) &\equiv \theta_G \phi_G\left(\theta_G\right) - t\left(\theta_G\right), & (10) \\
U_P\left(\theta_P\right) &\equiv \theta_P \phi_P\left(\theta_P\right) - p\left(\theta_P\right).
\end{aligned}
$$

A routine argument (see, e.g. Mas-Colell *et al.* 1995, page 888) can be used to establish the following lemmas:

Lemma 1 *Suppose that* $\Theta_G = \left[\underline{\theta_G}, \overline{\theta_G}\right]$ *and that the marginal density* $f_G\left(\theta_G\right)$ *is strictly positive on its support. Then,* (ϕ_G, t) *satisfies the incentive compatibility constraints (6) if and only if* ϕ_G *is weakly increasing in* θ_G *and*

$$
U_G\left(\theta_G\right) = U_G(\widehat{\theta_G}) + \int_{\widehat{\theta_G}}^{\theta_G} \phi_G\left(x\right) dx \quad \forall \theta_G, \widehat{\theta_G} \in \Theta_G.
$$

Lemma 2 *Suppose that* $\Theta_P = \left[\underline{\theta_P}, \overline{\theta_P}\right]$ *and that the marginal density* $f_P\left(\theta_P\right)$ *is strictly positive on its support. Then,* (ϕ_P, p) *satisfies the incentive compatibility constraints (9) if and only if* ϕ_P *is weakly increasing in* θ_P *and*

$$
U_P\left(\theta_P\right) = U(\widehat{\theta_P}) + \int_{\widehat{\theta_P}}^{\theta_P} \phi_P\left(x\right) dx \quad \forall \theta_P, \widehat{\theta_P} \in \Theta_P.
$$

Equally routine procedures (see, e.g. Mas-Colell *et al.* 1995, page 890), using Lemmas 1 and 2, show that the aggregate transfer revenues from the public goods fees and the private goods fees respectively can be determined purely in terms of the utility of the lowest type and the provision rules as

$$
\begin{aligned}
\int_{\Theta_G} t\left(\theta_G\right) dF_G\left(\theta_G\right) &= \int_{\Theta_G} \phi_G\left(\theta_G\right) \left(\theta_G - \frac{1 - F_G\left(\theta_G\right)}{f_G\left(\theta_G\right)}\right) dF_G\left(\theta_G\right) - U_G(\underline{\theta_G}) & (11) \\
\int_{\Theta_P} p\left(\theta_P\right) dF_P\left(\theta_P\right) &= \int_{\Theta_P} \phi_P\left(\theta_P\right) \left(\theta_P - \frac{1 - F_P\left(\theta_P\right)}{f_P\left(\theta_P\right)}\right) dF_P\left(\theta_P\right) - U_P(\underline{\theta_P}). & (12)
\end{aligned}
$$

We also note that, due to the existence of the numeraire good "money," we can without loss of generality assume that the participation constraint of type $\left(\underline{\theta_G}, \underline{\theta_P}\right)$ binds:

Lemma 3 *Suppose that* $(\phi_G^*, t^*, \phi_P^*, p^*)$ *is an optimal separate provision mechanism. Then there exists* $(\hat{t}, \hat{p})$ *such that* $(\phi_G^*, \hat{t}, \phi_P^*, \hat{p})$ *is an optimal separate provision mechanism and*

$$
\phi_G^*\left(\underline{\theta_G}\right) \underline{\theta_G} - \hat{t}\left(\underline{\theta_G}\right) + \phi_P^*\left(\underline{\theta_P}\right) \underline{\theta_P} - \hat{p}\left(\underline{\theta_P}\right) = 0
$$

Since all higher types can mimic $\left(\underline{\theta_G}, \underline{\theta_P}\right)$, incentive compatibility automatically implies that the participation constraints hold for all higher types, provided that it is satisfied for type $\left(\underline{\theta_G}, \underline{\theta_P}\right)$. Using (11) and Lemma 3, we can therefore reformulate (5) as

$$\max_{\phi_G : \Theta_G \to [0,1]} \int_{\Theta_G} [\phi_G\left(\theta_G\right)\theta_G] \, dF_G\left(\theta_P\right) - K\phi_G\left(\overline{\theta_G}\right) \tag{13}$$

$$\text{s.t. } 0 \leq \int_{\Theta_G} \phi_G\left(\theta_G\right)\left(\theta_G - \frac{1 - F_G\left(\theta_G\right)}{f_G\left(\theta_G\right)}\right) dF_G\left(\theta_G\right) - K\phi_G\left(\overline{\theta_G}\right)$$

$$+ \int_{\Theta_P} [p^*\left(\theta_P\right) - c\phi_P^*\left(\theta_P\right)] \, dF_P\left(\theta_P\right) \tag{14}$$

$$0 \leq \phi_G\left(\theta_G\right) \leq 1 \text{ for all } \theta_G \tag{15}$$

$$\phi_G\left(\cdot\right) \text{ is weakly increasing.} \tag{16}$$

To understand the objective function, observe that the social feasibility constraint (7) must bind. The objective function (13) is thus simply obtained by substitution of (7) into the objective function of the problem, eliminating the constants, and noting, by the fact that ϕ_G is monotonic, that $\sup_{\theta_G \in \Theta_G} \phi_G\left(\theta_G\right) = \phi_G\left(\overline{\theta_G}\right)$. The integral constraint (14), together with the monotonicity requirement (16), combines all incentive and participation constraints; and the constraints in (15) are the boundary constraints for the provision probabilities.

To facilitate further interpretations, it may be useful to observe that the problem for a profit maximizing monopolist for the public good would be to maximize

$$\int_{\Theta_G} \phi_G\left(\theta_G\right)\left(\theta_G - \frac{1 - F_G\left(\theta_G\right)}{f_G\left(\theta_G\right)}\right) dF_G\left(\theta_G\right) - K\phi_G\left(\overline{\theta_G}\right) \tag{17}$$

subject only to the constraints (15) and (16). For this problem, the "no-haggling" logic of Stokey (1979), Myerson (1981) and Riley and Zeckhauser (1983) immediately implies that the profit-maximizing mechanism is, without loss of generality, one where the monopolist charges a single price.[13] However, this result does not extend to our problem where profits appear as a constraint in (14). In general, the solution to the problem (13) may very well be a randomized mechanism.[14] However, randomizations can be ruled out by making an additional restriction on the distribution of types. Define

$$x_G\left(\theta_G\right) \equiv \theta_G - \frac{1 - F_G\left(\theta_G\right)}{f_G\left(\theta_G\right)}, \tag{18}$$

---

[13] If there is a profit-maximizing random mechanism, then a single price mechanism that charges any price in the support of the randomized mechanism is also optimal.

[14] This is easy to realize by considering the case with two types, θ_G^l and θ_G^h. Assuming that charging a flat fee equal to θ_G^l would violate the budget constraint, whereas charging θ_G^h would give a strict surplus, it is obvious that the surplus can be made higher by letting the low type agents consume with some probability. The example can easily be extended to continuous densities.

which is often referred to as the "virtual surplus." We can then show the following result:

Lemma 4 *Suppose that $x_G(\theta_G)$ as defined in (18) is weakly increasing in θ_G and that ϕ_G^* is a solution to (13). Then there exists some t^* such that*

$$\phi_G^*(\theta_G) = \begin{cases} \phi_G(\overline{\theta_G}) & \text{for } \theta_G \geq \frac{t^*}{\phi_G(\overline{\theta_G})} \\ 0 & \text{for } \theta_G < \frac{t^*}{\phi_G(\overline{\theta_G})} \end{cases} \tag{19}$$

Lemma 4 follows almost immediately from the fact that $\phi_G(\theta_G)$ appears linearly in both the objective function and the constraint. Thus, characterizing the solution to (13) is reduced to determining two variables: (1) the probability of public good provision $\phi_G(\overline{\theta_G})$; and (2) a user fee (or equivalently, a threshold valuation for being allowed to consume the good when it is produced).[15]

In the same spirit, the private goods problem (8) may be reformulated as

$$\max_{(\phi_P):\Theta_P \to [0,1]} \int_{\Theta_P} [\phi_P(\theta_P)(\theta_P - c)] \, dF_P(\theta_P) \tag{20}$$

$$\text{s.t. } 0 \leq \int_{\Theta_P} \phi_P(\theta_P) \left[\theta_P - \frac{1 - F_P(\theta_P)}{f_P(\theta_P)} - c \right] dF_P(\theta_G)$$

$$+ \int_{\Theta_G} t^*(\theta_G) \, dF_G(\theta_G) - K\phi_G^*(\overline{\theta_G}) \tag{21}$$

$$0 \leq \phi_P(\theta_P) \leq 1 \text{ for all } \theta_P \tag{22}$$

$$\theta_P(\cdot) \text{ is weakly increasing.} \tag{23}$$

Again the objective function is derived by substituting the (binding) budget constraint into the objective function of Problem (8) and eliminating constants. A similar argument allows us to conclude that all we need to do is to find a price to charge for the private good. Again, the fact that (20) is not a profit-maximization problem makes it necessary to make regularity assumptions on the virtual surplus in order to rule out a randomized optimal mechanism. Define

$$x_P(\theta_P) = \theta_P - \frac{1 - F_P(\theta_P)}{f_P(\theta_P)}. \tag{24}$$

We have the following result:

---

[15] The continuum-consumer assumption in itself trivializes the provision decision in the sense that this can no longer be made contingent on the realized distribution of types. However, we still need to make a (standard) regularity assumption in order to obtain the fixed price characterization. See Norman (2004) for an assumption that justifies this particular continuum model as a limit of finite economies.

Lemma 5 *Suppose that $x_P(\theta_P)$ as defined in (24) is weakly increasing and that ϕ_P^* is a solution to (20). Then, there exists some p^* such that*

$$\phi_P^*(\theta_P) = \begin{cases} 0 & \text{if } \theta_P < p^* \\ 1 & \text{if } \theta_P \geq p^*. \end{cases} \tag{25}$$

Lemmas 4 and 5 show that, if the marginal distributions are such that the virtual surplus for each good is monotonic in type, the maximization of (2) subject to (IC), (IR) and (BB) reduces to a simple optimization problem in three variables, with some slight abuse of notations, (1) a flat user fee, t^*, for the public good; (2) a probability that the public good will be provided, ϕ_G^* (which may also be reinterpreted as the quantity of the public good); and (3) a fixed price, p^*, for the private good.

Hence, using Lemmas 4 and 5, we obtain the following simplified planning problem:

$$\max_{\{t,\phi_G,p\}} \phi_G \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} \theta_G dF_G(\theta_G) - K\phi_G + \int_p^{\overline{\theta_P}} (\theta_P - c) \, dF_P(\theta_P) \tag{26}$$

$$\text{s.t } 0 \leq t\left[1 - F_G\left(\frac{t}{\phi_G}\right)\right] + (p-c)\left[1 - F_P(p)\right] - K\phi_G, \tag{27}$$

$$0 \leq \phi_G \leq 1. \tag{28}$$

Proposition 1 *Suppose that $\mathrm{E}\theta_G > K$ and that $\underline{\theta_G} < K$. Then, in any optimal solution (t^*, ϕ_G^*, p^*) to (26), the following must be true:*

1. $p^* > c$;

2. $\phi_G^* > 0$;

3. $t^* > 0$.

Proof. Let λ, μ and γ respectively be the Lagrangian multiplier for the constraint (27) and the boundary constraints $\phi_G \geq 0$ and $\phi_G \leq 1$. The Kuhn-Tucker necessary conditions for an optimum are:

$$0 = \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} \theta_G dF_G(\theta_G) + f_G\left(\frac{t}{\phi_G}\right)\left(\frac{t}{\phi_G}\right)^2 - K + \lambda f_G\left(\frac{t}{\phi_G}\right)\left(\frac{t}{\phi_G}\right)^2 - \lambda K + \mu - \gamma \tag{29}$$

$$\mu\phi_G = 0, \gamma(1 - \phi_G) = 0, \mu \geq 0, \gamma \geq 0 \tag{30}$$

$$0 = -\frac{t}{\phi_G}f_G\left(\frac{t}{\phi_G}\right) + \lambda\left[1 - F_G\left(\frac{t}{\phi_G}\right) - f_G\left(\frac{t}{\phi_G}\right)\frac{t}{\phi_G}\right] \tag{31}$$

$$0 = -(p-c)f_P(p) + \lambda\left[(1 - F_P(p)) - (p-c)f_P(p)\right] \tag{32}$$

$$0 = \lambda\left\{t\left[1 - F_G\left(\frac{t}{\phi_G}\right)\right] + (p-c)\left[1 - F_P(p)\right] - K\phi_G\right\}, \lambda \geq 0 \tag{33}$$

PART 1: If $p^* < c$, then the first term on the right hand side in (32) is strictly positive and the second is weakly positive, implying that the condition cannot hold. Suppose that $p^* = c$. Then, from (32) either $\lambda = 0$ or $1 - F_P(c) = 0$. Since the second condition is ruled out by the assumption that $c < \overline{\theta_P}$, the only possibility that remains is that $\lambda = 0$. But if $\lambda = 0$ at the optimal solution, then constraint (27) is not binding, which implies that t^*, ϕ_G^* must solve the following problem:

$$\max_{\{t, \phi_G\}} \phi_G \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} \theta_G dF_G(\theta_G) - K\phi_G$$

$$\text{s.t.} \qquad 0 \leq \phi_G \leq 1.$$

We note that if $\phi_G^* > 0$ in the solution, then the objective is monotonically decreasing in t over $\left[\underline{\theta_G}, \overline{\theta_G}\right]$, thus if $\phi_G^* > 0$, it must be that $t^* = \phi_G^* \underline{\theta_G}$. Thus, if $\phi_G^* > 0$ it must maximize

$$\phi_G \left[\int_{\underline{\theta_G}}^{\overline{\theta_G}} \theta_G dF_G(\theta_G) - K \right] = \phi_G \left[E\theta_G - K \right].$$

By assumption the bracketed expression $[E\theta_G - K]$ is strictly positive, so the solution must be $\phi_G^* = 1$; and thus $t^* = \underline{\theta_G}$. But, substituting $p^* = c, t^* = \underline{\theta_G}$ and $\phi_G^* = 1$ into the constraint (27) we see that

$$t^* \left[1 - F_G \left(\frac{t^*}{\phi_G^*} \right) \right] + (p^* - c)\left[1 - F_P(p^*) \right] - K\phi_G^* = \underline{\theta_G} - K < 0.$$

Hence, the resource constraint (27) is violated, which contradicts the assertion that $\lambda = 0$. Thus we conclude that $p^* > c$ in any solution to (26).

PART 2: Part 1 establishes that in the optimum $p^* > c$, thus if we have $\phi_G^* = 0$, there must be a strict budget surplus (the tax collected from the private goods due to $p^* > c$ is unspent). It is easy to show that the social surplus can be improved by spending such budget surplus. To see this, fix $p^*, t^* = 0, \phi_G^* = 0$. But consider $\phi_G^{*\prime}$ given by

$$\phi_G^{*\prime} = \frac{(p^* - c)\left[1 - F_P(p^*) \right]}{K} > 0$$

By construction, constraint (27) is satisfied by the alternative simple mechanism $\left(t^* = 0, \phi_G^{*\prime}, p^* \right)$. But the surplus under this alternative is

$$\phi_G^{*\prime} \int_0^{\overline{\theta_G}} \theta_G dF_G(\theta_G) + \int_{p^*}^{\overline{\theta_P}} (\theta_P - c) dF_P(\theta_P) > \int_{p^*}^{\overline{\theta_P}} (\theta_P - c) dF_P(\theta_P).$$

Hence, $\left(t^* = 0, \phi_G^{*\prime}, p^* \right)$ results in a strict increase in surplus relative to $\left(t^* = 0, \phi_G^* = 0, p^* \right)$. A contradiction.

14

PART 3: This is obvious if $\underline{\theta}_G > 0$, since $t^* = \phi_G^* \underline{\theta}_G$ would be non-distortionary. Suppose $t^* = 0$ and $\underline{\theta}_G \leq 0$. Since the necessary condition (31) must be satisfied, we have $\lambda \left[1 - F_G (0) \right] = 0$. This condition can only hold if $\lambda = 0$. But we have shown in the proof of Part 1 that if $\lambda = 0$, then $p^* = c$, which contradicts our conclusion in Part 1. ∎

Thus, under some standard regularity conditions, the optimal separate provision mechanism is characterized by two prices – one for the public and one for the private good – and a probability to provide the public good. In particular, it shows that the only government intervention on the private goods under the optimal separate provision mechanism can be interpreted as a "unit tax" in the amount of $p^* - c$ and leave its provision solely to the "private sector." Hence, under the optimal separate provision mechanism, the private good can be provided in a completely decentralized manner via a competitive market (subject to a tax); and the unit tax on the private goods is used as a cross subsidy from the private good to the production of the public good. The logic is simply that the welfare cost of a small tax on the private good is second order, since the consumers who stop consuming the private good have valuations just marginally above the cost of production. What we find more interesting is that a public project with positive expected social benefits should always be undertaken with some positive probability. That is, there is a role for randomizations in the model. It is also interesting to observe that there should always be a strictly positive user fee for the public good. The logic for this result is similar to the argument for why there should be a positive tax on the private good, but runs counter the idea that excluding consumers when the marginal cost is zero is always bad.

The characterization of (t^*, ϕ_G^*, p^*) in Proposition 1 will be used as the comparison benchmark in the section in which we examine whether joint provision mechanisms can improve social welfare. In obtaining the above characterization, it is worth emphasizing again that we imposed only the restriction neither the access probabilities nor the fees for both the private and the public goods depends on the consumer's reported valuation of the other good, but did not impose any other restrictions were made on the set of admissible mechanisms. This is in contrast to the existing literature we reviewed in Section 2 where the desirability of public provision of private goods is typically established with some restriction on the available policy instruments or some kind of market imperfection when the government is not involved in the provision of the private goods.

5 Public Provision of Both Goods

We have argued that the optimal separate provision mechanism characterized in Section 4 is consistent with an economy where the private good is traded on a competitive market (subject to a tax), and where the public good is provided by a government entity with resources coming from public good user fees and private good tax revenue. We will now consider a setup where the government is able to condition the provision probability and price for each of the two goods on the reported valuations of *both goods*. We interpret this as *public provision of both goods* since this is inconsistent with a world where the trading of the private good is done anonymously in the "private sector."

In general, a direct revelation mechanism can be represented as a quadruple $\left(\widetilde{\phi_G}, \widetilde{t}, \widetilde{\phi_P}, \widetilde{p} \right)$, where the difference with Section 4 is that all these functions are over the domain $\Theta_G \times \Theta_P$, whereas the corresponding objects in the separable case are functions of either Θ_G or Θ_P. This leads to a multidimensional mechanism design problem, and there is no known methodology to characterize incentive compatibility in an analytically tractable way.

To get a tractable problem, we will proceed along the lines of McAfee *et al.* [25] and consider a class of simple mechanisms. Specifically, we will add a single instrument to the separate provision case, so that instead of considering mechanisms of the form (t, ϕ_G, p) as characterized in Proposition 1, we will consider mechanisms of the form (t, ϕ_G, p, τ), where (t, ϕ_G, p) have the same interpretations as before, respectively as the user fee for the public good, the provision probability for the public good, and the price of the private good; but τ now is the fee charged for an agent who consumes both the public and the private good. If $\tau \neq t + p$, this requires that the government be actively involved in provision of the private good because such a scheme is feasible for the government only if it could monitor the consumers' purchases of the private goods.

While it is obviously a limitation that we are not able to characterize the constrained efficient mechanism for the full-blown mechanism design problem, the reader may note that, if we find that $\tau \neq t + p$ in the solution to our simplified problem, then it must be that the constrained efficient mechanism is also one in which the government takes an active part in the provision of the private good. Therefore, this suffices to answer the qualitative question we are interested in, namely whether public provision of a private good can be efficiency enhancing.

5.1 Some Preliminaries

Consider a (simple pricing) mechanism on the form (t, ϕ_G, p, τ). It is easy to see that a consumer will demand:

Only the Public Good if:

$$
\begin{aligned}
\phi_G \theta_G - t &\geq 0 \\
\phi_G \theta_G - t &\geq \theta_P - p \\
\phi_G \theta_G - t &\geq \phi_G \theta_G + \theta_P - \tau
\end{aligned}
\tag{34}
$$

Only the Private Good if:

$$
\begin{aligned}
\theta_P - p &\geq 0 \\
\theta_P - p &\geq \phi_G \theta_G - t \\
\theta_P - p &\geq \phi_G \theta_G + \theta_P - \tau
\end{aligned}
\tag{35}
$$

The Bundle if:

$$
\begin{aligned}
\phi_G \theta_G + \theta_P - \tau &\geq 0 \\
\phi_G \theta_G + \theta_P - \tau &\geq \phi_G \theta_G - t \\
\phi_G \theta_G + \theta_P - \tau &\geq \theta_P - p
\end{aligned}
\tag{36}
$$

Which of the inequalities in (34), (35) and (36) are relevant depends on whether the bundle is cheaper or more expensive than the components. Figures 1 and 2 respectively represent the consumer's demand for the private and public goods as a function of her valuation types (θ_G, θ_P) where for simplicity we multiplied her public valuation type by the provision probability ϕ_G. The following two claims are obvious from the inspection of Figures 1 and 2.

Claim 1 *If $\tau \leq t + p$, the second inequality in (34) and (35) is implied by the other two inequalities. And,*

- *the proportion of agents purchasing the public good only is*

$$
\int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} \int_{\underline{\theta_P}}^{\tau - t} f(\theta_G, \theta_P) \, d\theta_P d\theta_G
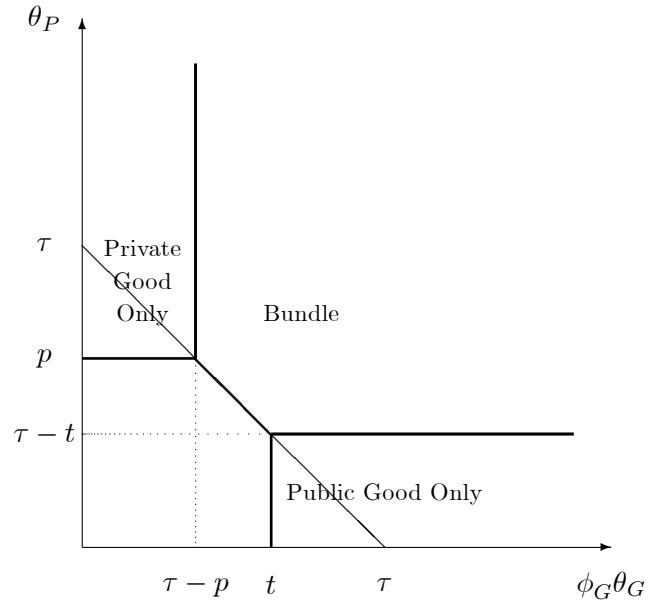$$

17

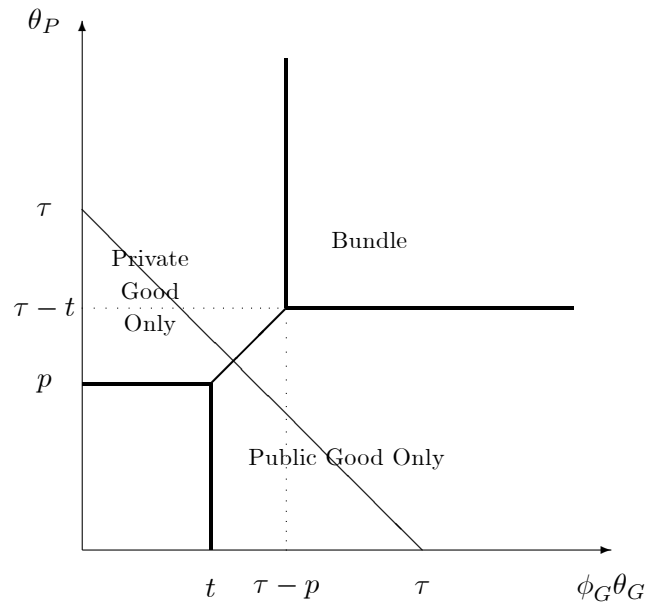Figure 1: Consumers' Demand for the Case $\tau \leq t + p$.

Figure 2: Consumers' Demand for the Case $\tau > t + p$.

18

- *the proportion of agents purchasing the private good only is*

$$\int_p^{\overline{\theta_P}} \int_{\underline{\theta_G}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G, \theta_P\right) d\theta_G d\theta_P$$

- *the proportion of agents purchasing the bundle is*

$$\int_{\frac{\tau-p}{\phi_G}}^{\frac{t}{\phi_G}} \int_{\tau-\phi_G\theta_G}^{\overline{\theta_P}} f\left(\theta_G, \theta_P\right) d\theta_P d\theta_G + \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} \int_{\tau-t}^{\overline{\theta_P}} f\left(\theta_G, \theta_P\right) d\theta_P d\theta_G$$

Claim 2 *If $\tau \geq t + p$, the first inequality in (36) is implied by the other two inequalities. And,*

- *the proportion of agents purchasing the public good only is*

$$\int_{\frac{t}{\phi_G}}^{\frac{\tau-p}{\phi_G}} \int_{\underline{\theta_P}}^{\phi_G\theta_G+p-t} f\left(\theta_G, \theta_P\right) d\theta_P d\theta_G + \int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}} \int_{\underline{\theta_P}}^{\tau-t} f\left(\theta_G, \theta_P\right) d\theta_P d\theta_G$$

- *the proportion of agents purchasing the private good only is*

$$\int_p^{\tau-t} \int_{\underline{\theta}}^{\frac{\theta_P+t-p}{\phi_G}} f\left(\theta_G, \theta_P\right) d\theta_G d\theta_P + \int_{\tau-t}^{\overline{\theta_P}} \int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G, \theta_P\right) d\theta_G d\theta_P$$

- *the proportion of agents purchasing the bundle is*

$$\int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}} \int_{\tau-t}^{\overline{\theta_P}} f\left(\theta_G, \theta_P\right) d\theta_P d\theta_G$$

Define by $G_1\left(t, p, \tau; \phi_G\right)$ the budget surplus (when positive) given a mechanism (t, ϕ_G, p, τ) under case one when $\tau \leq t + p$. Using Claim 1, it can be written as:

$$G_1\left(t, p, \tau; \phi_G\right) = t\left[\int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} \int_{\underline{\theta_P}}^{\tau-t} f\left(\theta_G, \theta_P\right) d\theta_P d\theta_G\right] + (p-c)\left[\int_p^{\overline{\theta_P}} \int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G, \theta_P\right) d\theta_G d\theta_P\right] \tag{37}$$

$$+ (\tau - c)\left[\int_{\frac{\tau-p}{\phi_G}}^{\frac{t}{\phi_G}} \int_{\tau-\phi_G\theta_G}^{\overline{\theta_P}} f\left(\theta_G, \theta_P\right) d\theta_P d\theta_G + \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} \int_{\tau-t}^{\overline{\theta_P}} f\left(\theta_G, \theta_P\right) d\theta_P d\theta_G\right] - K\phi_G,$$

Symmetrically, we let $G_2\left(t, p, \tau; \phi_G\right)$ denote the budget surplus given a mechanism (t, ϕ_G, p, τ) where $\tau \geq t + p$, which using Claim 2 can be written as:

$$G_2\left(t, p, \tau; \phi_G\right) = t\left[\int_{\frac{t}{\phi_G}}^{\frac{\tau-p}{\phi_G}} \int_{\underline{\theta_P}}^{\phi_G\theta_G+p-t} f\left(\theta_G, \theta_P\right) d\theta_P d\theta_G + \int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}} \int_{\underline{\theta_P}}^{p} f\left(\theta_G, \theta_P\right) d\theta_P d\theta_G\right] \tag{38}$$

$$+ (p-c)\left[\int_p^{\tau-t} \int_{\underline{\theta}}^{\frac{\theta_P+t-p}{\phi_G}} f\left(\theta_G, \theta_P\right) d\theta_G d\theta_P + \int_{\tau-t}^{\overline{\theta_P}} \int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G, \theta_P\right) d\theta_G d\theta_P\right]$$

$$+ (\tau - c)\left[\int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}} \int_{\tau-t}^{\overline{\theta_P}} f\left(\theta_G, \theta_P\right) d\theta_P d\theta_G\right] - K\phi_G.$$

19

Next, let $S_1(t, p, \tau; \phi_G)$ denote the social surplus associated with (t, ϕ_G, p, τ) in the case when $\tau \leq t + p$,

$$S_1(t, p, \tau; \phi_G) = \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} \int_{\underline{\theta_P}}^{\tau - t} \phi_G \theta_G f(\theta_G, \theta_P) \, d\theta_P d\theta_G + \int_p^{\overline{\theta_P}} \int_{\underline{\theta_G}}^{\frac{\tau - p}{\phi_G}} (\theta_P - c) f(\theta_G, \theta_P) \, d\theta_G d\theta_P \qquad (39)$$
$$+ \int_{\frac{\tau - p}{\phi_G}}^{\frac{t}{\phi_G}} \int_{\tau - \phi_G \theta_G}^{\overline{\theta_P}} (\phi_G \theta_G + \theta_P - c) f(\theta_G, \theta_P) \, d\theta_P d\theta_G$$
$$+ \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} \int_{\tau - t}^{\overline{\theta_P}} (\phi_G \theta_G + \theta_P - c) f(\theta_G, \theta_P) \, d\theta_P d\theta_G - K\phi_G.$$

and let $S_2(t, p, \tau; \phi_G)$ be the social surplus associated with (t, ϕ_G, p, τ) in the case when $\tau \geq t + p$,

$$S_2(t, p, \tau; \phi_G) = \int_{\frac{t}{\phi_G}}^{\frac{\tau - p}{\phi_G}} \int_{\underline{\theta_P}}^{\phi_G \theta_G + p - t} \phi_G \theta_G f(\theta_G, \theta_P) \, d\theta_P d\theta_G + \int_{\frac{\tau - p}{\phi_G}}^{\overline{\theta_G}} \int_{\underline{\theta_P}}^{\tau - t} \phi_G \theta_G f(\theta_G, \theta_P) \, d\theta_P d\theta_G \qquad (40)$$
$$+ \int_p^{\tau - t} \int_{\underline{\theta}}^{\frac{\theta_P + t - p}{\phi_G}} (\theta_P - c) f(\theta_G, \theta_P) \, d\theta_G d\theta_P + \int_{\tau - t}^{\overline{\theta_P}} \int_{\underline{\theta}}^{\frac{\tau - p}{\phi_G}} (\theta_P - c) f(\theta_G, \theta_P) \, d\theta_G d\theta_P$$
$$+ \int_{\frac{\tau - p}{\phi_G}}^{\overline{\theta_G}} \int_{\tau - t}^{\overline{\theta_P}} (\phi_G \theta_G + \theta_P - c) f(\theta_G, \theta_P) \, d\theta_P d\theta_G - K\phi_G.$$

For notational brevity we will let $z = (t, p, \tau)$ and $z^* = (t^*, p^*, t^* + p^*)$, where (t^*, ϕ_G^*, p^*) is an optimal solution to the separate provision problem (26) in Section 4. Now we define two auxiliary problems. First consider the problem of:

$$\max_{(t,p,\tau)} S_1(t, p, \tau; \phi_G^*) \qquad (41)$$
$$\text{s.t.} \qquad G_1(t, p, \tau; \phi_G^*) \geq 0$$
$$t + p - \tau \geq 0.$$

Problem (41) gives the best simple pricing policy in the form of (t, ϕ_G, p, τ) under the restriction that $\phi_G = \phi_G^*$ and $\tau \leq t + p$; that is, the public good provision probability is fixed at the level as in the optimal separate provision mechanism characterized in Proposition 1, and the bundle is restricted to be no more expensive than separate purchase of its components. Problem (41) will tell us the maximal attainable social surplus when one deviates from the the optimal separable provision mechanism among a particular set of perturbations (namely, charging the bundle of public and private goods a lower price than the sum of its components), while respecting the budget balance constraint. Notice that the separate provision mechanism prices $z^* = (t^*, p^*, t^* + p^*)$ is in the constraint set of Problem (41).

20

Symmetrically, consider the problem of:

$$\max_{(t,p,\tau)} S_2\left(t,p,\tau;\phi_G^*\right) \tag{42}$$
$$\text{s.t.} \quad G_2\left(t,p,\tau;\phi_G^*\right) \geq 0$$
$$\tau - t - p \geq 0.$$

Problem (42) gives the best simple pricing policy in the form of (t,ϕ_G,p,τ) under the restriction that $\phi_G = \phi_G^*$ and $\tau \geq t+p$. That is, the public good provision probability is fixed at the level as in the optimal separate provision mechanism characterized in Proposition 1, and the bundle is restricted to be no cheaper than separate purchase of its components. Problem (42) will tell us the maximal attainable social surplus when one deviates from the the optimal separable provision mechanism among another particular set of perturbations (namely, charging the bundle of public and private goods a higher price than the sum of its components), while respecting the budget balance constraint. Notice again that the separate provision mechanism prices $z^* = (t^*,p^*,t^*+p^*)$ is in the constraint set of Problem (42).

If there is no potential improvement of joint provision mechanism over the optimal separate provision mechanism we characterized in Proposition 1, a necessary condition is that $(t^*,p^*,\tau = t^*+p^*)$ must solve both Problems (41) and (42).

5.2 The Main Result

Now we show our main result, which is a sufficient condition under which $(t^*,p^*,\tau = t^*+p^*)$ cannot simultaneously solve both Problems (41) and (42). As a result, when the identified condition is satisfied, there must exist a simple pricing joint provision mechanism that improves social welfare upon the optimal separate provision mechanisms.

To this end, we first introduce a few notations. Differentiating (37) and evaluating at $z = z^* =$

$\left(t^{*}, p^{*}, t^{*}+p^{*}\right)$, we find that:[16]

$$
\begin{aligned}
\frac{\partial G_{1}\left(z^{*}; \phi_{G}^{*}\right)}{\partial t} &= \int_{\frac{t^{*}}{\phi_{G}^{*}}}^{\overline{\theta_{G}}} \left\{F_{P}\left(p^{*} | \theta_{G}\right) + \left(p^{*} - c\right) f_{P}\left(p^{*} | \theta_{G}\right)\right\} f_{G}\left(\theta_{G}\right) d\theta_{G} \\
&\quad - \frac{t^{*}}{\phi_{G}^{*}} F_{P}\left(p^{*} | \frac{t^{*}}{\phi_{G}^{*}}\right) f_{G}\left(\frac{t^{*}}{\phi_{G}^{*}}\right)
\end{aligned} \tag{43a}
$$

$$
\begin{aligned}
\frac{\partial G_{1}\left(z^{*}; \phi_{G}^{*}\right)}{\partial p} &= \int_{\underline{\theta_{G}}}^{\frac{t^{*}}{\phi_{G}^{*}}} \left[\left\{\left(1 - F_{P}\left(p^{*} | \theta_{G}\right)\right) - \left(p^{*} - c\right) f_{P}\left(p^{*} | \theta_{G}\right)\right\}\right] f_{G}\left(\theta_{G}\right) d\theta_{G} \\
&\quad + \frac{t^{*}}{\phi_{G}^{*}} \left[1 - F_{P}\left(p^{*} | \frac{t^{*}}{\phi_{G}^{*}}\right)\right] f_{G}\left(\frac{t^{*}}{\phi_{G}^{*}}\right)
\end{aligned} \tag{43b}
$$

$$
\begin{aligned}
\frac{\partial G_{1}\left(z^{*}; \phi_{G}^{*}\right)}{\partial \tau} &= \int_{\frac{t^{*}}{\phi_{G}^{*}}}^{\overline{\theta_{G}}} \left\{\left(1 - F_{P}\left(p^{*} | \theta_{G}\right)\right) - \left(p^{*} - c\right) f_{P}\left(p^{*} | \theta_{G}\right)\right\} f_{G}\left(\theta_{G}\right) d\theta_{G} \\
&\quad - \frac{t^{*}}{\phi_{G}^{*}} \left[1 - F_{P}\left(p^{*} | \frac{t^{*}}{\phi_{G}^{*}}\right)\right] f_{G}\left(\frac{t^{*}}{\phi_{G}^{*}}\right)
\end{aligned} \tag{43c}
$$

These expressions inform us about the effect on the budget when one slightly perturbs the relevant prices t, p and τ.

Likewise, if we differentiate $S_{1}\left(\cdot\right)$ in (39) and evaluate at $z = z^{*}$, we obtain:

$$
\frac{\partial S_{1}\left(z^{*}; \phi_{G}^{*}\right)}{\partial t} = \int_{\frac{t^{*}}{\phi_{G}^{*}}}^{\overline{\theta_{G}}} \left(p^{*} - c\right) f_{P}\left(p^{*} | \theta_{G}\right) f_{G}\left(\theta_{G}\right) d\theta_{G} - \frac{t^{*}}{\phi_{G}^{*}} F_{P}\left(p^{*} | \frac{t^{*}}{\phi_{G}^{*}}\right) f_{G}\left(\frac{t^{*}}{\phi_{G}^{*}}\right) \tag{44a}
$$

$$
\frac{\partial S_{1}\left(z^{*}; \phi_{G}^{*}\right)}{\partial p} = -\int_{\underline{\theta_{G}}}^{\frac{t^{*}}{\phi_{G}^{*}}} \left(p^{*} - c\right) f_{P}\left(p^{*} | \theta_{G}\right) f_{G}\left(\theta_{G}\right) d\theta_{G} + \frac{t^{*}}{\phi_{G}^{*}} \left[1 - F_{P}\left(p^{*} | \frac{t^{*}}{\phi_{G}^{*}}\right)\right] f_{G}\left(\frac{t^{*}}{\phi_{G}^{*}}\right) \tag{44b}
$$

$$
\frac{\partial S_{1}\left(z^{*}; \phi_{G}^{*}\right)}{\partial \tau} = -\int_{\frac{t^{*}}{\phi_{G}^{*}}}^{\overline{\theta_{G}}} \left(p^{*} - c\right) f_{P}\left(p^{*} | \theta_{G}\right) f_{G}\left(\theta_{G}\right) d\theta_{G} - \frac{t^{*}}{\phi_{G}^{*}} \left[1 - F_{P}\left(p^{*} | \frac{t^{*}}{\phi_{G}^{*}}\right)\right] f_{G}\left(\frac{t^{*}}{\phi_{G}^{*}}\right). \tag{44c}
$$

We can also write out explicitly the gradients for S_{2} and G_{2}, which we omit here.

Remark. The reader may notice that expression $\partial G_{1}\left(z^{*}; \phi_{G}^{*}\right) / \partial p$ in (43b) corresponds directly to the expression in Proposition 1 of McAfee *et al.* (1989). Expression (43a) can also be written in that form by reversing the roles of θ_{G} and θ_{P}. This close correspondence with McAfee *et al.* (1989) is not a coincidence. The derivatives reported above can be thought of as the effect on profits given a marginal increase in t, p and τ respectively, which is exactly what McAfee *et al.* (1989) was analyzing. In their case, going from (43b) to their main result is relatively straightforward since they asked for a direction where the mixed bundling mechanism increases *profits* relative to separate pricing. In particular, $\left(t^{*}, p^{*}\right)$ in their case were chosen to solve a monopolist profit maximization problem under separate pricing, thus p^{*}, for example, must satisfy the first order optimality condition $\left[1 - F_{P}\left(p^{*}\right)\right] - \left(p^{*} - c\right) f_{P}\left(p^{*}\right) = 0$. Hence, the first term in (43b), if θ_{G} and

---

[16] The details of the derivations for (43) and (44) are available in an Appendix from the authors' websites.

θ_P are stochastically independent, becomes:

$$\int_{\underline{\theta_G}}^{\frac{t^*}{\phi_G^*}} \left\{ [1 - F_P\left(p^*|\theta_G\right)] - \left(p^* - c\right) f_P\left(p^*|\theta_G\right) \right\} f_G\left(\theta_G\right) d\theta_G \tag{45}$$

$$= \int_{\underline{\theta_G}}^{\frac{t^*}{\phi_G^*}} \left\{ [1 - F_P\left(p^*\right)] - \left(p^* - c\right) f_P\left(p^*\right) \right\} f_G\left(\theta_G\right),$$

which is equal to zero from the optimality condition of p^* in their problem. Thus it follows immediately from (43b) that a small increase in the price of the private good (or a small decrease in the price from the bundle) would increase the profits in the case of stochastic independence. It can also be verified that, by rewriting $\partial G_1\left(z^*; \phi_G^*\right)/\partial t$ (to be explicit about the optimality condition for t^* for the monopolist profit-maximization problem), that a small increase in the price of the public good also increases profits if θ_G and θ_P are stochastically independent.

Our problem differs from McAfee *et al.* (1989) in two respects. First of all, our goal is to demonstrate that bundling can increase *social welfare* rather than profits. Secondly, because (t^*, p^*) in our problem, as characterized in Proposition 1, are *not chosen to maximize profits*, we cannot use the first order conditions from the optimal separable provision mechanism in the same way as McAfee *et al.* (1989).

Our first preliminary result is that in fact, evaluated at an optimal separate provision mechanism prices $(t^*, p^*, \tau = t^* + p^*)$, the partial derivatives of G_1 and G_2 are the same, and the partial derivatives of S_1 and S_2 also coincide. Letting $DG_i\left(z;; \phi_G\right)$ and $DS_i\left(z; \phi_G\right)$ denote the gradient vectors for $i = 1, 2$, we thus have that;

Lemma 6 $DG_1\left(z^*; \phi_G^*\right) = DG_2\left(z^*; \phi_G^*\right)$ *and* $DS_1\left(z^*; \phi_G^*\right) = DS_2\left(z^*; \phi_G^*\right)$

Now we establish a useful lemma:

Lemma 7 *Let λ^* be the multiplier on constraint (27) corresponding to the solution (ϕ_G^*, t^*, p^*) of problem (26). Also, let λ_i be the multiplier on the resource constraint $G_i\left(t, p, \tau; \phi_G^*\right)$ for $i = 1, 2$ in problem (41) and (42). Then,*

1. $\lambda_1 = \lambda^*$ if z^* solves problem (41);

2. $\lambda_2 = \lambda^*$ if z^* solves problem (42).

Proof. First consider (41). If z^* solves the problem, the Kuhn-Tucker necessary conditions for a solution must be fulfilled at z^*. Hence, there must exist $\lambda_1 > 0$ and $\mu_1 \geq 0$ such that

$$\frac{\partial S_1\left(z^*; \phi_G^*\right)}{\partial t} + \lambda_1 \frac{\partial G_1\left(z^*; \phi_G^*\right)}{\partial t} + \mu_1 = 0 \tag{46a}$$

$$\frac{\partial S_1\left(z^*; \phi_G^*\right)}{\partial p} + \lambda_1 \frac{\partial G_1\left(z^*; \phi_G^*\right)}{\partial p} + \mu_1 = 0 \tag{46b}$$

$$\frac{\partial S_1\left(z^*; \phi_G^*\right)}{\partial \tau} + \lambda_1 \frac{\partial G_1\left(z^*; \phi_G^*\right)}{\partial \tau} - \mu_1 = 0 \tag{46c}$$

$$\mu_1\left(t + p - \tau\right) = 0, \quad \mu_1 \geq 0 \tag{46d}$$

Using the expressions for the partial derivatives in (43) and (44), it is easy to check that:

$$\frac{\partial S_1\left(z^*; \phi_G^*\right)}{\partial t} + \frac{\partial S_1\left(z^*; \phi_G^*\right)}{\partial \tau} = -\frac{t^*}{\phi_G^*} f_G\left(\frac{t^*}{\phi_G^*}\right) \tag{47a}$$

$$\frac{\partial G_1\left(z^*; \phi_G^*\right)}{\partial t} + \frac{\partial G_1\left(z^*; \phi_G^*\right)}{\partial \tau} = \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f_G\left(\theta_G\right) d\theta_G - \frac{t^*}{\phi_G^*} f_G\left(\frac{t^*}{\phi_G^*}\right)$$

$$= \left[1 - F_G\left(\frac{t^*}{\phi_G^*}\right)\right] - \frac{t^*}{\phi_G^*} f_G\left(\frac{t^*}{\phi_G^*}\right). \tag{47b}$$

Combining (46a) and (46c), and using (47), we have that

$$-\frac{t^*}{\phi_G^*} f_G\left(\frac{t^*}{\phi_G^*}\right) + \lambda_1 \left\{\left[1 - F_G\left(\frac{t^*}{\phi_G^*}\right)\right] - \frac{t^*}{\phi_G^*} f_G\left(\frac{t^*}{\phi_G^*}\right)\right\} = 0. \tag{48}$$

This condition is the same as (31), the first order condition to the problem when the goods are sold separately. It follows that $\lambda_1 = \lambda^*$, since otherwise (48) will be violated. This proves the first part.

For the second part, we note that the Kuhn-Tucker conditions for Problem (42) are

$$\frac{\partial S_2\left(z^*; \phi_G^*\right)}{\partial t} + \lambda_2 \frac{\partial G_2\left(z^*; \phi_G^*\right)}{\partial t} - \mu_2 = \frac{\partial S_1\left(z^*; \phi_G^*\right)}{\partial t} + \lambda_2 \frac{\partial G_1\left(z^*; \phi_G^*\right)}{\partial t} - \mu_2 = 0$$

$$\frac{\partial S_2\left(z^*; \phi_G^*\right)}{\partial p} + \lambda_2 \frac{\partial G_2\left(z^*; \phi_G^*\right)}{\partial p} - \mu_2 = \frac{\partial S_1\left(z^*; \phi_G^*\right)}{\partial p} + \lambda_2 \frac{\partial G_1\left(z^*; \phi_G^*\right)}{\partial p} - \mu_2 = 0$$

$$\frac{\partial S_2\left(z^*; \phi_G^*\right)}{\partial \tau} + \lambda_2 \frac{\partial G_2\left(z^*; \phi_G^*\right)}{\partial \tau} + \mu_2 = \frac{\partial S_1\left(z^*; \phi_G^*\right)}{\partial \tau} + \lambda_2 \frac{\partial G_1\left(z^*; \phi_G^*\right)}{\partial \tau} + \mu_2 = 0$$

$$\mu_2\left(t + p - \tau\right) = 0, \mu_2 \geq 0$$

where the first equality in each line follows from Lemma 6. The same argument applies. ∎

Together, Lemmas 6 and 7 makes the Kuhn-Tucker conditions for problem (41) comparable with those of problem (42). Now our main result about the sufficient condition for the welfare improvement potential of public provision of private goods over the optimal separate provision mechanism follows:

Proposition 2 *Let λ^* be the multiplier on constraint (27) corresponding to the solution (ϕ_G^*, t^*, p^*) of problem (26). Then, there exists a feasible simple pricing policy (t, p, τ) that generates a higher social surplus than the optimal separate provision mechanism whenever*

$$DS_1(z^*; \phi_G^*) + \lambda^* DG_1(z^*; \phi_G^*) \neq 0. \tag{49}$$

Proof. From Lemma 7, we know that if z^* solves both problems (41) and (42), the multiplier in each problem must be given by λ^*. Thus if z^* is the best simple pricing policy for problem (41), then

$$
\begin{aligned}
\frac{\partial S_1(z^*; \phi_G^*)}{\partial t} + \lambda^* \frac{\partial G_1(z^*; \phi_G^*)}{\partial t} + \mu_1 &= 0 \\
\frac{\partial S_1(z^*; \phi_G^*)}{\partial p} + \lambda^* \frac{\partial G_1(z^*; \phi_G^*)}{\partial p} + \mu_1 &= 0 \\
\frac{\partial S_1(z^*; \phi_G^*)}{\partial \tau} + \lambda^* \frac{\partial G_1(z^*; \phi_G^*)}{\partial \tau} - \mu_1 &= 0 \\
\mu_1(t + p - \tau) = 0, \qquad \mu_1 &\geq 0.
\end{aligned}
\tag{50}
$$

Similarly if z^* is the best simple pricing policy for problem (42), then by using Lemma 6, we have

$$
\begin{aligned}
\frac{\partial S_1(z^*; \phi_G^*)}{\partial t} + \lambda^* \frac{\partial G_1(z^*; \phi_G^*)}{\partial t} - \mu_2 &= 0 \\
\frac{\partial S_1(z^*; \phi_G^*)}{\partial p} + \lambda^* \frac{\partial G_1(z^*; \phi_G^*)}{\partial p} - \mu_2 &= 0 \\
\frac{\partial S_1(z^*; \phi_G^*)}{\partial \tau} + \lambda^* \frac{\partial G_1(z^*; \phi_G^*)}{\partial \tau} + \mu_2 &= 0 \\
\mu_2(t + p - \tau) = 0, \qquad \mu_2 &\geq 0
\end{aligned}
\tag{51}
$$

Assume that $\mu_1 > 0$. Then, (50) implies that $\frac{\partial S_1(z^*; \phi_G^*)}{\partial t} + \lambda^* \frac{\partial G_1(z^*; \phi_G^*)}{\partial t} < 0$, which makes it impossible to find $\mu_2 \geq 0$ such that (51) holds. Symmetrically, if $\mu_2 > 0$, then $\frac{\partial S_1(z^*; \phi_G^*)}{\partial t} + \lambda^* \frac{\partial G_1(z^*; \phi_G^*)}{\partial t} > 0$, which makes it impossible to find $\mu_1 \geq 0$ such that (50) holds. Since z^* must solve both (41) and (42) for there to be no improvement we conclude that $\mu_1 = \mu_2 = 0$, or else there is some z better than z^*. The claim follows. ∎

Because the simple pricing joint provision mechanism is a subset of all feasible joint provision mechanisms, an immediate implication of Proposition 2 is that whenever condition (49) is satisfied, the optimal joint provision mechanism must generate higher social welfare than the optimal separate provision mechanism.

5.3 Stochastic Independence

Now we use Proposition 2 above to examine the case where θ_G and θ_P are stochastically independent. In this case we have that there is indeed always an improvement over the optimal separate provision policy.

Proposition 3 *Suppose that θ_G and θ_P are stochastically independent. Then*

$$\frac{\partial S_1\left(z^*; \phi_G^*\right)}{\partial p} + \lambda^* \frac{\partial G_1\left(z^*; \phi_G^*\right)}{\partial p} > 0.$$

Proof. When $f_P\left(\theta_P | \theta_G\right) = f_P\left(\theta_P\right)$ for all θ_P we have that

$$
\begin{aligned}
\frac{\partial G_1\left(z^*; \phi_G^*\right)}{\partial p} &= \left[1 - F_P\left(p^*\right) - \left(p^* - c\right) f_P\left(p^*\right)\right] F_G\left(\frac{t^*}{\phi_G^*}\right) + \frac{t^*}{\phi_G^*}\left[1 - F_P\left(p^*\right)\right] f_G\left(\frac{t^*}{\phi_G^*}\right) \\
&= \frac{\left(p^* - c\right) f_P\left(p^*\right)}{\lambda^*} F_G\left(\frac{t^*}{\phi_G^*}\right) + \frac{t^*}{\phi_G^*}\left[1 - F_P\left(p^*\right)\right] f_G\left(\frac{t^*}{\phi_G^*}\right)
\end{aligned}
$$

where the second equality uses (32), the first order condition for p^* in the separable case. Next,

$$\frac{\partial S_1\left(z^*; \phi_G^*\right)}{\partial p} = -\left(p^* - c\right) f_P\left(p^*\right) F_G\left(\frac{t^*}{\phi_G^*}\right) + \frac{t^*}{\phi_G^*}\left[1 - F_P\left(p^*\right)\right] f_G\left(\frac{t^*}{\phi_G^*}\right)$$

Hence,

$$
\begin{aligned}
\frac{\partial S_1\left(z^*; \phi_G^*\right)}{\partial p} + \lambda^* \frac{\partial G_1\left(z^*; \phi_G^*\right)}{\partial p} &= -\left(p^* - c\right) f_P\left(p^*\right) F_G\left(\frac{t^*}{\phi_G^*}\right) + \frac{t^*}{\phi_G^*}\left[1 - F_P\left(p^*\right)\right] f_G\left(\frac{t^*}{\phi_G^*}\right) \\
&\quad + \lambda^* \left\{ \frac{\left(p^* - c\right) f_P\left(p^*\right)}{\lambda^*} F_G\left(\frac{t^*}{\phi_G^*}\right) + \frac{t^*}{\phi_G^*}\left[1 - F_P\left(p^*\right)\right] f_G\left(\frac{t^*}{\phi_G^*}\right) \right\} \\
&= \left(1 + \lambda^*\right) \frac{t^*}{\phi_G^*}\left[1 - F_P\left(p^*\right)\right] f_G\left(\frac{t^*}{\phi_G^*}\right) > 0. \qquad \blacksquare
\end{aligned}
$$

Proposition 3 establishes that when the valuations for the private and public goods are stochastically independent, there is a *welfare improving* role for the public provision of private goods over the optimal separate provision mechanism we characterized in Proposition 1. It also implies, by continuity, that there exists an *open set* of joint distribution functions for θ_G and θ_P for which public provision of private goods can be preferred from an efficiency viewpoint.[17]

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[17]We conjecture is that condition (49) is satisfied *generically* in the sense that it holds for almost all joint distribution functions.

6   Conclusion

This paper shows that public provision of private goods may be justified on pure efficiency grounds in an environment where individuals consume both public and private goods. The government's involvement in the provision of private goods provides it with information about individuals' private good purchases that facilitates more efficient revenue extraction for the provision of public goods. We show that public provision of the private good improves economic efficiency under a condition that is always fulfilled under stochastic independence and satisfied for an open set of joint distributions.

The efficiency rationale for the public provision of private goods we advanced in this paper differs substantially from explanations emphasized in the existing literature, which typically rely on preferences for redistribution. Our result is also derived under considerably less stringent restrictions on the set of admissible policies than the existing literature. We do not impose any restrictions on the benchmark outcome that can be implemented under the optimal separate provision mechanism, which is in stark contrast to the existing literature where the benchmark outcome is often derived under various implicit restrictions on the market operations. The no-arbitrage restriction we require for the joint provision mechanism to improve upon the benchmark outcome in our model also accords with many publicly-provided private goods we see in reality because in-state college tuition, public health insurance, and public schools are all commodities that are difficult to resell. Thus our paper also provides an explanation of why some, but not all, private goods are publicly provided.

Finally, our analysis also exemplifies a more general point about the optimal taxation literature. Our design problem combines a simple commodity taxation problem with an excludable public good provision problem. We found that the marginal price for access to the public good for consumers that purchase the private good should be different from those who do not. Hence, our model is a stylized example where provision and user fees for the public good, and taxes on the private good must be jointly determined in order to achieve economic efficiency. Our paper thus illustrates that the standard practice of separating the question of how a given budget should be spent from the question of how a given tax revenue should be raised generates efficiency losses.

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A   Appendix: Omitted Details of Some Derivations

A.1   Derivation of Derivatives in (43):

For simplicity of notation, define:

$$
A_1\left(z;\phi_G\right) \equiv t\left[\int_{\frac{t}{\phi_G}}^{\overline{\theta_G}}\int_{\underline{\theta_P}}^{\tau-t} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G\right],
$$

$$
B_1\left(z;\phi_G\right) \equiv (p-c)\left[\int_p^{\overline{\theta_P}}\int_{\underline{\theta_G}}^{\frac{\tau-p}{\phi_G^*}} f\left(\theta_G,\theta_P\right)d\theta_G d\theta_P\right],
$$

$$
C_1\left(z;\phi_G\right) \equiv (\tau-c)\left[\int_{\frac{\tau-p}{\phi_G}}^{\frac{t}{\phi_G}}\int_{\tau-\phi_G\theta_G}^{\overline{\theta_P}} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G + \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}}\int_{\tau-t}^{\overline{\theta_P}} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G\right],
$$

so that

$$
G_1\left(z;\phi_G\right) = A_1\left(z;\phi_G\right) + B_1\left(z;\phi_G\right) + C_1\left(z;\phi_G\right) - K\phi_G.
$$

Differentiating with respect to t, we have:

$$
\frac{\partial A_1\left(z;\phi_G\right)}{\partial t} = \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}}\int_{\underline{\theta_P}}^{\tau-t} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G - t\left[\int_{\underline{\theta_P}}^{\tau-t} f\left(\frac{t}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P + \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} f\left(\theta_G,\tau-t\right)d\theta_G\right],
$$

$$
\frac{\partial B_1\left(z;\phi_G\right)}{\partial t} = 0,
$$

$$
\frac{\partial C_1\left(z;\phi_G\right)}{\partial t} = (\tau-c)\left[\int_{\tau-t}^{\overline{\theta_P}} f\left(\frac{t}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P - \int_{\tau-t}^{\overline{\theta_P}} f\left(\frac{t}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P + \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} f\left(\theta_G,\tau-t\right)d\theta_G\right]
$$

$$
= (\tau-c)\left[\int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} f\left(\theta_G,\tau-t\right)d\theta_G\right].
$$

Thus,

$$
\frac{\partial A_1\left(z^*;\phi_G^*\right)}{\partial t} = \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\int_{\underline{\theta_P}}^{p^*} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G - t^*\left[\int_{\underline{\theta_P}}^{p^*} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P + \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f\left(\theta_G,p^*\right)d\theta_G\right],
$$

$$
\frac{\partial B_1\left(z^*;\phi_G^*\right)}{\partial t} = 0,
$$

$$
\frac{\partial C_1\left(z^*;\phi_G^*\right)}{\partial t} = (t^*+p^*-c)\left[\int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f\left(\theta_G,p^*\right)d\theta_G\right].
$$

So,

$$
\begin{aligned}
\frac{\partial G_1(z^*)}{\partial t} &= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} \left\{ \int_{\underline{\theta_P}}^{p^*} f(\theta_G,\theta_P)\, d\theta_P + (p^*-c) f(\theta_G,p^*) \right\} d\theta_G - t^* \int_{\underline{\theta_P}}^{p^*} f\left(\frac{t^*}{\phi_G^*},\theta_P\right) \frac{1}{\phi_G^*} d\theta_P \qquad \text{(A1)} \\[2ex]
&= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} \left\{ \int_{\underline{\theta_P}}^{p^*} \underbrace{\frac{f(\theta_G,\theta_P)}{\int_{\underline{\theta_P}}^{\overline{\theta_P}} f(\theta_G,\theta_P)\, d\theta_P}}_{f_P(\theta_P|\theta_G)} d\theta_P + (p^*-c) \underbrace{\frac{f(\theta_G,p^*)}{\int_{\underline{\theta_P}}^{\overline{\theta_P}} f(\theta_G,\theta_P)\, d\theta_P}}_{f_P(p^*|\theta_G)} \right\} \underbrace{\left(\int_{\underline{\theta_P}}^{\overline{\theta_P}} f(\theta_G,\theta_P)\, d\theta_P \right)}_{f_G(\theta_G)} d\theta_G d\theta_G \\[2ex]
&\quad - t^* \int_{\underline{\theta_P}}^{p^*} \underbrace{\frac{f\left(\frac{t^*}{\phi_G^*},\theta_P\right)}{\int_{\underline{\theta_P}}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right) d\theta_P}}_{f_P\left(\theta_P|\frac{t^*}{\phi_G^*}\right)} \frac{1}{\phi_G^*} d\theta_P \underbrace{\int_{\underline{\theta_P}}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right) d\theta_P}_{f_G\left(\frac{t^*}{\phi_G^*}\right)} \\[2ex]
&= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} \left\{ \int_{\underline{\theta_P}}^{p^*} f_P(\theta_P|\theta_G)\, d\theta_P + (p^*-c) f_P(p^*|\theta_G) \right\} f_G(\theta_G)\, d\theta_G - \frac{t^*}{\phi_G^*} \int_{\underline{\theta_P}}^{p^*} f_P\left(\theta_P|\frac{t^*}{\phi_G^*}\right) d\theta_P\, f_G\left(\frac{t^*}{\phi_G^*}\right) \\[2ex]
&= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} \left\{ F_P(p^*|\theta_G) + (p^*-c) f_P(p^*|\theta_G) \right\} f_G(\theta_G)\, d\theta_G - \frac{t^*}{\phi_G^*} F_P\left(p^*|\frac{t^*}{\phi_G^*}\right) f_G\left(\frac{t^*}{\phi_G^*}\right).
\end{aligned}
$$

Differentiating with respect to τ yields:

$$
\begin{aligned}
\frac{dA_1(z;\phi_G)}{d\tau} &= t\left[\int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} f(\theta_G,\tau-t)\, d\theta_G \right] \\[2ex]
\frac{\partial B_1(z;\phi_G)}{\partial \tau} &= (p-c)\left[\int_{p}^{\overline{\theta_P}} f\left(\frac{\tau-p}{\phi_G^*},\theta_P\right) \frac{1}{\phi_G^*} d\theta_P \right] \\[2ex]
\frac{\partial C_1(z;\phi_G)}{\partial \tau} &= \int_{\frac{\tau-p}{\phi_G}}^{\frac{t}{\phi_G}} \int_{\tau-\phi_G\theta_G}^{\overline{\theta_P}} f(\theta_G,\theta_P)\, d\theta_P d\theta_G + \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} \int_{\tau-t}^{\overline{\theta_P}} f(\theta_G,\theta_P)\, d\theta_P d\theta_G \\[2ex]
&\quad - (\tau-c)\left[\int_{p}^{\overline{\theta_P}} f\left(\frac{\tau-p}{\phi_G},\theta_P\right) \frac{1}{\phi_G} d\theta_P + \int_{\frac{\tau-p}{\phi_G}}^{\frac{t}{\phi_G}} f(\theta_G,\tau-\phi_G\theta_G)\, d\theta_G \right] \\[2ex]
&\quad - (\tau-c)\left[\int_{\frac{t}{\phi_G}}^{\overline{\theta_G}} f(\theta_G,\tau-t)\, d\theta_G \right]
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{dA_1(z^*;\phi_G^*)}{d\tau} &= t^*\left[\int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f(\theta_G,\tau-t^*)\, d\theta_G \right] \\[2ex]
\frac{\partial B_1(z^*;\phi_G^*)}{\partial \tau} &= (p^*-c)\left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right) \frac{1}{\phi_G^*} d\theta_P \right] \\[2ex]
\frac{\partial C_1(z^*;\phi_G^*)}{\partial \tau} &= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} \int_{p^*}^{\overline{\theta_P}} f(\theta_G,\theta_P)\, d\theta_P d\theta_G - (p^*+t^*-c)\left\{ \left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right) \frac{1}{\phi_G^*} d\theta_P \right] + \left[\int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f(\theta_G,p^*)\, d\theta_G \right] \right\}.
\end{aligned}
$$

It follows that

$$\frac{\partial G_1\left(z^*;\phi_G^*\right)}{\partial \tau} = \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\int_{p^*}^{\overline{\theta_P}} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G - t^*\left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P\right] - (p^*-c)\left[\int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f\left(\theta_G,p^*\right)d\theta_P d\theta_G\right]$$

$$= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left\{\int_{p^*}^{\overline{\theta_P}} f\left(\theta_G,\theta_P\right)d\theta_P - (p^*-c)f\left(\theta_G,p^*\right)\right\}d\theta_G - t^*\left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P\right]$$

$$\text{/same steps as in (A1)/} = \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left\{\left[1-F_P\left(p^*|\theta_G\right)\right]-(p^*-c)f_P\left(p^*|\theta_G\right)\right\}f_G\left(\theta_G\right)d\theta_G - \frac{t^*}{\phi_G^*}\left[1-F_P\left(p^*|\frac{t^*}{\phi_G^*}\right)\right]f_G\left(\frac{t^*}{\phi_G^*}\right).$$

Differentiating with respect to p yields:

$$\frac{\partial A_1\left(z;\phi_G\right)}{\partial p} = 0,$$

$$\frac{\partial B_1\left(z;\phi_G\right)}{\partial p} = \int_{p}^{\overline{\theta_P}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G,\theta_P\right)d\theta_G d\theta_P - (p-c)\left[\int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G,p\right)d\theta_G + \int_{p}^{\overline{\theta_P}} f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P\right],$$

$$\frac{\partial C_1\left(z;\phi_G\right)}{\partial p} = (\tau-c)\left[\int_{p}^{\overline{\theta_P}} f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P\right].$$

Hence,

$$\frac{\partial A_1\left(z^*;\phi_G^*\right)}{\partial p} = 0,$$

$$\frac{\partial B_1\left(z^*;\phi_G^*\right)}{\partial p} = \int_{p^*}^{\overline{\theta_P}}\int_{\underline{\theta}}^{\frac{t^*}{\phi_G^*}} f\left(\theta_G,\theta_P\right)d\theta_G d\theta_P - (p^*-c)\left[\int_{\underline{\theta}}^{\frac{t^*}{\phi_G^*}} f\left(\theta_G,p^*\right)d\theta_G + \int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P\right],$$

$$\frac{\partial C_1\left(z^*;\phi_G^*\right)}{\partial p} = (t^*+p^*-c)\left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P\right],$$

and

$$\frac{\partial G_1\left(z^*;\phi_G^*\right)}{\partial p} = \int_{\underline{\theta}}^{\frac{t^*}{\phi_G^*}}\left[\int_{p^*}^{\overline{\theta_P}} f\left(\theta_G,\theta_P\right)d\theta_P - (p^*-c)f\left(\theta_G,p^*\right)\right]d\theta_G + t^*\left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P\right]$$

$$\text{/same steps as in (A1)/} = \int_{\underline{\theta}}^{\frac{t^*}{\phi_G^*}}\left[\left\{\left(1-F_P\left(p^*|\theta_G\right)\right)-(p^*-c)f_P\left(p^*|\theta_G\right)\right\}\right]f_G\left(\theta_G\right)d\theta_G$$

$$+ \frac{t^*}{\phi_G^*}\left(1-F_P\left(p^*|\frac{t^*}{\phi_G^*}\right)\right)f_G\left(\frac{t^*}{\phi_G^*}\right).$$

A.2 Derivation of Derivatives in (44):

Let

$$D_1\left(z;\phi_G\right) = \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}}\int_{\underline{\theta_P}}^{\tau-t} \phi_G\theta_G f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G,$$

$$E_1\left(z;\phi_G\right) = \int_{p}^{\overline{\theta_P}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}} (\theta_P-c)f\left(\theta_G,\theta_P\right)d\theta_G d\theta_P,$$

$$F_1\left(z;\phi_G\right) = \int_{\frac{\tau-p}{\phi_G}}^{\frac{t}{\phi_G}}\int_{\tau-\phi_G\theta_G}^{\overline{\theta_P}} (\phi_G\theta_G+\theta_P-c)f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G + \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}}\int_{\tau-t}^{\overline{\theta_P}} (\phi_G\theta_G+\theta_P-c)f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G.$$

3

Hence,

$$
\frac{\partial D_1\left(z;\phi_G\right)}{\partial t} = -\int_{\underline{\theta_P}}^{\tau-t} t f\left(\frac{t}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P - \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}}\phi_G\theta_G f\left(\theta_G,\tau-t\right)d\theta_G
$$

$$
\frac{\partial D_1\left(z;\phi_G\right)}{\partial p} = 0
$$

$$
\frac{\partial D_1\left(z;\phi_G\right)}{\partial \tau} = \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}}\phi_G\theta_G f\left(\theta_G,\tau-t\right)d\theta_G
$$

and thus,

$$
\frac{\partial D_1\left(z^*;\phi_G^*\right)}{\partial t} = -\int_{\underline{\theta_P}}^{p^*} t^* f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P - \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\phi_G^*\theta_G f\left(\theta_G,p^*\right)d\theta_G \tag{A2}
$$

$$
\frac{\partial D_1\left(z^*;\phi_G^*\right)}{\partial p} = 0
$$

$$
\frac{\partial D_1\left(z^*;\phi_G^*\right)}{\partial \tau} = \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\phi_G^*\theta_G f\left(\theta_G,p^*\right)d\theta_P d\theta_G.
$$

Similarly,

$$
\frac{\partial E_1\left(z;\phi_G\right)}{\partial t} = 0,
$$

$$
\frac{\partial E_1\left(z;\phi_G\right)}{\partial p} = -\int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}}(p-c)f\left(\theta_G,p\right)d\theta_G - \int_p^{\overline{\theta_P}}(\theta_P-c)f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P,
$$

$$
\frac{\partial E_1\left(z;\phi_G\right)}{\partial \tau} = \int_p^{\overline{\theta_P}}(\theta_P-c)f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P,
$$

thus,

$$
\frac{\partial E_1\left(z^*;\phi_G^*\right)}{\partial t} = 0 \tag{A3}
$$

$$
\frac{\partial E_1\left(z^*;\phi_G^*\right)}{\partial p} = -\int_{\underline{\theta}}^{\frac{t^*}{\phi_G^*}}(p^*-c)f\left(\theta_G,p^*\right)d\theta_G - \int_{p^*}^{\overline{\theta_P}}(\theta_P-c)f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P
$$

$$
\frac{\partial E_1\left(z^*;\phi_G^*\right)}{\partial \tau} = \int_{p^*}^{\overline{\theta_P}}(\theta_P-c)f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P.
$$

4

Finally,

$$\frac{\partial F_1\left(z;\phi_G\right)}{\partial t} = \int_{\tau-t}^{\overline{\theta_P}}\left(t+\theta_P-c\right)f\left(\frac{t}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P - \int_{\tau-t}^{\overline{\theta_P}}\left(t+\theta_P-c\right)f\left(\frac{t}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P$$

$$+ \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}}\left(\phi_G\theta_G+\tau-t-c\right)f\left(\theta_G,\tau-t\right)d\theta_G$$

$$= \int_{\frac{t}{\phi_G}}^{\overline{\theta_G}}\left(\phi_G\theta_G+\tau-t-c\right)f\left(\theta_G,\tau-t\right)$$

$$\frac{\partial F_1\left(z;\phi_G\right)}{\partial p} = \int_{p}^{\overline{\theta_P}}\left(\tau-p+\theta_P-c\right)f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P$$

$$\frac{\partial F_1\left(z;\phi_G\right)}{\partial \tau} = -\int_{p}^{\overline{\theta_P}}\left(\tau-p+\theta_P-c\right)f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P - \int_{\frac{\tau-p}{\phi_G}}^{\frac{t}{\phi_G}}\left(\tau-c\right)f\left(\theta_G,\tau-\phi_G\theta_G\right)d\theta_G$$

$$-\int_{\frac{t}{\phi_G}}^{\overline{\theta_G}}\left(\phi_G\theta_G+\tau-t-c\right)f\left(\theta_G,\tau-t\right)d\theta_G$$

thus,

$$\frac{\partial F_1\left(z^*;\phi_G^*\right)}{\partial t} = \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left(\phi_G^*\theta_G+p^*-c\right)f\left(\theta_G,p^*\right)d\theta_G \tag{A4}$$

$$\frac{\partial F_1\left(z^*;\phi_G^*\right)}{\partial p} = \int_{p^*}^{\overline{\theta_P}}\left(t^*+\theta_P-c\right)f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P$$

$$\frac{\partial F_1\left(z^*;\phi_G^*\right)}{\partial \tau} = -\int_{p^*}^{\overline{\theta_P}}\left(t^*+\theta_P-c\right)f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P - \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left(\phi_G^*\theta_G+p^*-c\right)f\left(\theta_G,p^*\right)d\theta_G.$$

Hence,

$$\frac{\partial S_1\left(z^*;\phi_G^*\right)}{\partial t} = \frac{\partial D_1\left(z^*;\phi_G^*\right)}{\partial t} + \frac{\partial E_1\left(z^*;\phi_G^*\right)}{\partial t} + \frac{\partial F_1\left(z^*;\phi_G^*\right)}{\partial t}$$

$$= -\int_{\underline{\theta_P}}^{p^*}t^*f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P - \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\phi_G^*\theta_G f\left(\theta_G,p^*\right)d\theta_G + \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left(\phi_G^*\theta_G+p^*-c\right)f\left(\theta_G,p^*\right)d\theta_G$$

$$= -\int_{\underline{\theta_P}}^{p^*}\frac{t^*}{\phi_G^*}f\left(\frac{t^*}{\phi_G^*},\theta_P\right)d\theta_P + \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left(p^*-c\right)f\left(\theta_G,p^*\right)d\theta_G$$

$$= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left(p^*-c\right)f_P\left(p^*|\theta_G\right)f_G\left(\theta_G\right)d\theta_G - \frac{t^*}{\phi_G^*}F_P\left(p^*|\frac{t^*}{\phi_G^*}\right)f_G\left(\frac{t^*}{\phi_G^*}\right)$$

and,

$$\frac{\partial S_1\left(z^*;\phi_G^*\right)}{\partial p} = \frac{\partial D_1\left(z^*;\phi_G^*\right)}{\partial p} + \frac{\partial E_1\left(z^*;\phi_G^*\right)}{\partial p} + \frac{\partial F_1\left(z^*;\phi_G^*\right)}{\partial p}$$

$$= -\int_{\underline{\theta}}^{\frac{t^*}{\phi_G}}\left(p^*-c\right)f\left(\theta_G,p^*\right)d\theta_G - \int_{p^*}^{\overline{\theta_P}}\left(\theta_P-c\right)f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P$$

$$+ \int_{p^*}^{\overline{\theta_P}}\left(t^*+\theta_P-c\right)f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P$$

$$= -\int_{\underline{\theta}}^{\frac{t^*}{\phi_G}}\left(p^*-c\right)f\left(\theta_G,p^*\right)d\theta_G + \int_{p^*}^{\overline{\theta_P}}\frac{t^*}{\phi_G^*}f\left(\frac{t^*}{\phi_G^*},\theta_P\right)d\theta_P$$

$$= -\int_{\underline{\theta}}^{\frac{t^*}{\phi_G}}\left(p^*-c\right)f_P\left(p^*|\theta_G\right)f_G\left(\theta_G\right)d\theta_G + \frac{t^*}{\phi_G^*}\left[1-F_P\left(p^*|\frac{t^*}{\phi_G^*}\right)\right]f_G\left(\frac{t^*}{\phi_G^*}\right),$$

and,

$$\frac{\partial S_1\left(z^*;\phi_G^*\right)}{\partial \tau} = \frac{\partial D_1\left(z^*;\phi_G^*\right)}{\partial \tau} + \frac{\partial E_1\left(z^*;\phi_G^*\right)}{\partial \tau} + \frac{\partial F_1\left(z^*;\phi_G^*\right)}{\partial \tau}$$

$$= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\phi_G^*\theta_G f\left(\theta_G,p^*\right)d\theta_P d\theta_G + \int_{p^*}^{\overline{\theta_P}}\left(\theta_P-c\right)f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P$$

$$- \int_{p^*}^{\overline{\theta_P}}\left(t^*+\theta_P-c\right)f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P - \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left(\phi_G^*\theta_G+p^*-c\right)f\left(\theta_G,p^*\right)d\theta_G$$

$$= -\int_{p^*}^{\overline{\theta_P}}\frac{t^*}{\phi_G^*}f\left(\frac{t^*}{\phi_G^*},\theta_P\right)d\theta_P - \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left(p^*-c\right)f\left(\theta_G,p^*\right)d\theta_G$$

$$= -\int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left(p^*-c\right)f_P\left(p^*|\theta_G\right)f_G\left(\theta_G\right)d\theta_G - \frac{t^*}{\phi_G^*}\left[1-F_P\left(p^*|\frac{t^*}{\phi_G^*}\right)\right]f_G\left(\frac{t^*}{\phi_G^*}\right).$$

A.3 Calculations in Proving Lemma 6

- **Part 1:** $DG_1\left(z^*;\phi_G^*\right) = DG_2\left(z^*;\phi_G^*\right)$:

Write $G_2\left(z;\phi_G\right) = A_2\left(z;\phi_G\right) + B_2\left(z;\phi_G\right) + C_2\left(z;\phi_G\right) - K\phi_G$ where:

$$A_2\left(z;\phi_G\right) \equiv t\left[\int_{\frac{t}{\phi_G}}^{\frac{\tau-p}{\phi_G}}\int_{\underline{\theta_P}}^{\phi_G\theta_G+p-t}f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G + \int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}}\int_{\underline{\theta_P}}^{\tau-t}f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G\right]$$

$$B_2\left(z;\phi_G\right) \equiv \left(p-c\right)\left[\int_{p}^{\tau-t}\int_{\underline{\theta}}^{\frac{\theta_P+t-p}{\phi_G}}f\left(\theta_G,\theta_P\right)d\theta_G d\theta_P + \int_{\tau-t}^{\overline{\theta_P}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}}f\left(\theta_G,\theta_P\right)d\theta_G d\theta_P\right]$$

$$C_2\left(z;\phi_G\right) \equiv \left(\tau-c\right)\left[\int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}}\int_{\tau-t}^{\overline{\theta_P}}f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G\right].$$

6

Differentiating A_2 we get:

$$
\frac{\partial A_2\left(z;\phi_G\right)}{\partial t} = \left[\int_{\frac{t}{\phi_G}}^{\frac{\tau-p}{\phi_G}}\int_{\underline{\theta_P}}^{\phi_G\theta_G+p-t} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G + \int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}}\int_{\underline{\theta_P}}^{\tau-t} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G\right]
$$

$$
-t\left[\int_{\underline{\theta_P}}^{p} f\left(\frac{t}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P + \int_{\frac{t}{\phi_G}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G,\phi_G\theta_G+p-t\right)d\theta_G + \int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}} f\left(\theta_G,\tau-t\right)d\theta_G\right]
$$

$$
\frac{\partial A_2\left(z;\phi_G\right)}{\partial p} = -t\int_{\underline{\theta_P}}^{\tau-t} f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P + t\int_{\frac{t}{\phi_G}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G,\phi_G\theta_G+p-t\right)d\theta_G + t\int_{\underline{\theta_P}}^{\tau-t} f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P
$$

$$
= t\int_{\frac{t}{\phi_G}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G,\phi_G\theta_G+p-t\right)d\theta_G
$$

$$
\frac{\partial A_2\left(z;\phi_G\right)}{\partial\tau} = t\int_{\underline{\theta_P}}^{\tau-t} f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P - t\int_{\underline{\theta_P}}^{\tau-t} f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P + t\int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}} f\left(\theta_G,\tau-t\right)d\theta_G
$$

$$
= t\int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}} f\left(\theta_G,\tau-t\right)d
$$

Hence,

$$
\frac{\partial A_2\left(z^*;\phi_G^*\right)}{\partial t} = \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\int_{\underline{\theta_P}}^{p^*} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G - t^*\left[\int_{\underline{\theta_P}}^{p^*} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P + \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f\left(\theta_G,p^*\right)d\theta_G\right]
$$

$$
\frac{\partial A_2\left(z^*;\phi_G^*\right)}{\partial p} = 0
$$

$$
\frac{\partial A_2\left(z^*;\phi_G^*\right)}{\partial\tau} = t^*\int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f\left(\theta_G,p^*\right)d\theta_G.
$$

Similarly,

$$
\frac{\partial B_2\left(z;\phi_G\right)}{\partial t} = (p-c)\left[-\int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G,\tau-t\right)d\theta_G + \int_{p}^{\tau-t} f\left(\frac{\theta_P+t-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P + \int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G,\tau-t\right)d\theta_G\right]
$$

$$
= (p-c)\left[\int_{p}^{\tau-t} f\left(\frac{\theta_P+t-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P\right]
$$

$$
\frac{\partial B_2\left(z;\phi_G\right)}{\partial p} = \int_{p}^{\tau-t}\int_{\underline{\theta}}^{\frac{\theta_P+t-p}{\phi_G}} f\left(\theta_G,\theta_P\right)d\theta_G d\theta_P + \int_{\tau-t}^{\overline{\theta_P}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G
$$

$$
-(p-c)\left[\int_{\underline{\theta}}^{\frac{t}{\phi_G}} f\left(\theta_G,p\right)d\theta_G + \int_{p}^{\tau-t} f\left(\frac{\theta_P+t-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P + \int_{\tau-t}^{\overline{\theta_P}} f\left(\frac{\tau-p}{\phi_G},\theta_P\right)d\theta_P\right]
$$

$$
\frac{\partial B_2\left(z;\phi_G\right)}{\partial\tau} = (p-c)\left[\int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G,\tau-t\right)d\theta_G d\theta_P - \int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}} f\left(\theta_G,\tau-t\right)d\theta_G + \int_{\tau-t}^{\overline{\theta_P}} f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_G\right]
$$

$$
= (p-c)\left[\int_{\tau-t}^{\overline{\theta_P}} f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_G\right].
$$

Hence,

$$\frac{\partial B_2\left(z^*;\phi_G^*\right)}{\partial t} = 0$$

$$\frac{\partial B_2\left(z^*;\phi_G^*\right)}{\partial p} = \int_{p^*}^{\overline{\theta_P}}\int_{\underline{\theta}}^{\frac{t^*}{\phi_G^*}} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G - \left(p^*-c\right)\left[\int_{\underline{\theta}}^{\frac{t^*}{\phi_G^*}} f\left(\theta_G,p^*\right)d\theta_G + \int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)d\theta_P\right]$$

$$\frac{\partial B_2\left(z^*;\phi_G^*\right)}{\partial \tau} = \left(p^*-c\right)\left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_G\right].$$

Similarly,

$$\frac{\partial C_2\left(z;\phi_G\right)}{\partial t} = \left(\tau-c\right)\left[\int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}} f\left(\theta_G,\tau-t\right)d\theta_G\right]$$

$$\frac{\partial C_2\left(z;\phi_G\right)}{\partial p} = \left(\tau-c\right)\left[\int_{\tau-t}^{\overline{\theta_P}} f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P\right]$$

$$\frac{\partial C_2\left(z;\phi_G\right)}{\partial \tau} = \int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}}\int_{\tau-t}^{\overline{\theta_P}} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G - \left(\tau-c\right)\left[\int_{\tau-t}^{\overline{\theta_P}} f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P + \int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}} f\left(\theta_G,\tau-t\right)d\theta_G\right].$$

Hence,

$$\frac{\partial C_2\left(z^*;\phi_G^*\right)}{\partial t} = \left(t^*+p^*-c\right)\left[\int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f\left(\theta_G,p^*\right)d\theta_G\right]$$

$$\frac{\partial C_2\left(z^*;\phi_G^*\right)}{\partial p} = \left(t^*+p^*-c\right)\left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P\right]$$

$$\frac{\partial C_2\left(z^*;\phi_G^*\right)}{\partial \tau} = \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\int_{p^*}^{\overline{\theta_P}} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G - \left(t^*+p^*-c\right)\left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P + \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f\left(\theta_G,p^*\right)d\theta_G\right]$$

Combining terms we get that

$$\frac{\partial G_2\left(z^*;\phi_G^*\right)}{\partial t} = \frac{\partial A_2\left(z^*\right)}{\partial t} + \frac{\partial B_2\left(z^*\right)}{\partial t} + \frac{\partial C_2\left(z^*\right)}{\partial t}$$

$$= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\int_{\underline{\theta_P}}^{p^*} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G - t^*\left[\int_{\underline{\theta_P}}^{p^*} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P + \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f\left(\theta_G,p^*\right)d\theta_G\right]$$

$$+ \left(t^*+p^*-c\right)\left[\int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f\left(\theta_G,p^*\right)d\theta_G\right]$$

$$= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left\{\int_{\underline{\theta_P}}^{p^*} f\left(\theta_G,\theta_P\right)d\theta_P + \left(p^*-c\right)f\left(\theta_G,p^*\right)\right\}d\theta_G - t^*\int_{\underline{\theta_P}}^{p^*} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P$$

$$= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left\{\int_{\underline{\theta_P}}^{p^*} f_P\left(\theta_P|\theta_G\right)d\theta_P + \left(p^*-c\right)f_P\left(p^*|\theta_G\right)\right\}f_G\left(\theta_G\right)d\theta_G - \frac{t^*}{\phi_G^*}\int_{\underline{\theta_P}}^{p^*} f_P\left(\theta_P|\frac{t^*}{\phi_G^*}\right)d\theta_P f_G\left(\frac{t^*}{\phi_G^*}\right)$$

$$= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left\{F_P\left(p^*|\theta_G\right) + \left(p^*-c\right)f_P\left(p^*|\theta_G\right)\right\}f_G\left(\theta_G\right)d\theta_G - \frac{t^*}{\phi_G^*}F_P\left(p^*|\frac{t^*}{\phi_G^*}\right)f_G\left(\frac{t^*}{\phi_G^*}\right)$$

$$= \frac{\partial G_1\left(z^*\right)}{\partial t},$$

8

and,

$$
\begin{aligned}
\frac{\partial G_2\left(z^*;\phi_G^*\right)}{\partial p} &= \frac{\partial A_2\left(z^*;\phi_G^*\right)}{\partial p} + \frac{\partial B_2\left(z^*;\phi_G^*\right)}{\partial p} + \frac{\partial C_2\left(z^*;\phi_G^*\right)}{\partial p} \\
&= \int_{p^*}^{\overline{\theta_P}}\int_{\underline{\theta}}^{\frac{t^*}{\phi_G^*}} f\left(\theta_G,\theta_P\right)d\theta_G d\theta_P - (p^*-c)\left[\int_{\underline{\theta}}^{\frac{t^*}{\phi_G^*}} f\left(\theta_G,p^*\right)d\theta_G + \int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)d\theta_P\right] \\
&\quad + (t^*+p^*-c)\left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P\right] \\
&= \int_{\underline{\theta}}^{\frac{t^*}{\phi_G^*}}\left\{\int_{p^*}^{\overline{\theta_P}} f\left(\theta_G,\theta_P\right)d\theta_P - (p^*-c)f\left(\theta_G,p^*\right)\right\}d\theta_G + t^*\left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P\right] \\
&= \int_{\underline{\theta}}^{\frac{t^*}{\phi_G^*}}\left\{1 - F_P\left(p^*|\theta_G\right) - (p^*-c)f_P\left(p^*|\theta_G\right)\right\}f_G\left(\theta_G\right)d\theta_G + t^*\left[1 - F_P\left(p^*|\frac{t^*}{\phi_G^*}\right)\right]f_G\left(\frac{t^*}{\phi_G^*}\right) \\
&= \frac{\partial G_1\left(z^*\right)}{\partial p}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial G_2\left(z^*;\phi_G^*\right)}{\partial \tau} &= \frac{\partial A_2\left(z^*;\phi_G^*\right)}{\partial \tau} + \frac{\partial B_2\left(z^*;\phi_G^*\right)}{\partial \tau} + \frac{\partial C_2\left(z^*;\phi_G^*\right)}{\partial \tau} \\
&= t^*\int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f\left(\theta_G,p^*\right)d\theta_G + (p^*-c)\left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_G\right] \\
&\quad + \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\int_{p^*}^{\overline{\theta_P}} f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G - (t^*+p^*-c)\left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P + \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} f\left(\theta_G,p^*\right)d\theta_G\right] \\
&= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left\{\int_{p^*}^{\overline{\theta_P}} f\left(\theta_G,\theta_P\right)d\theta_P - (p^*-c)f\left(\theta_G,p^*\right)\right\}d\theta_G - t^*\left[\int_{p^*}^{\overline{\theta_P}} f\left(\frac{t^*}{\phi_G^*},\theta_P\right)\frac{1}{\phi_G^*}d\theta_P\right] \\
&= \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\left\{1 - F_P\left(p^*|\theta_G\right) - (p^*-c)f_P\left(p^*|\theta_G\right)\right\}f_G\left(\theta_G\right)d\theta_G - \frac{t^*}{\phi_G^*}\left[1 - F_P\left(p^*|\frac{t^*}{\phi_G^*}\right)f_G\left(\frac{t^*}{\phi_G^*}\right)\right] \\
&= \frac{\partial G_1\left(z^*\right)}{\partial \tau}
\end{aligned}
$$

- **Part 2:** $DS_1\left(z^*;\phi_G^*\right) = DS_2\left(z^*;\phi_G^*\right)$:

Write $S_2\left(z;\phi_G\right) = D_2\left(z;\phi_G\right) + E_2\left(z;\phi_G\right) + F_2\left(z;\phi_G\right) - K\phi_G$ where

$$
\begin{aligned}
D_2\left(z;\phi_G\right) &= \int_{\frac{t}{\phi_G}}^{\frac{\tau-p}{\phi_G}}\int_{\underline{\theta_P}}^{\phi_G\theta_G+p-t} \phi_G\theta_G f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G + \int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}}\int_{\underline{\theta_P}}^{\tau-t} \phi_G\theta_G f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G \\
E_2\left(z;\phi_G\right) &= \int_{p}^{\tau-t}\int_{\underline{\theta}}^{\frac{\theta_P+t-p}{\phi_G}} \left(\theta_P-c\right)f\left(\theta_G,\theta_P\right)d\theta_G d\theta_P + \int_{\tau-t}^{\overline{\theta_P}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}} \left(\theta_P-c\right)f\left(\theta_G,\theta_P\right)d\theta_G d\theta_P \\
F_2\left(z;\phi_G\right) &= \int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}}\int_{\tau-t}^{\overline{\theta_P}} \left(\phi_G\theta_G+\theta_P-c\right)f\left(\theta_G,\theta_P\right)d\theta_P d\theta_G.
\end{aligned}
$$

9

Thus,

$$\frac{\partial D_2\left(z;\phi_G\right)}{\partial t} = -\int_{\underline{\theta_P}}^{p}\frac{t}{\phi_G}f\left(\frac{t}{\phi_G},\theta_P\right)d\theta_P - \int_{\frac{t}{\phi_G}}^{\frac{\tau-p}{\phi_G}}\phi_G\theta_G f\left(\theta_G,\phi_G\theta_G+p-t\right)d\theta_G - \int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}}\phi_G\theta_G f\left(\theta_G,\tau-t\right)d\theta_G$$

$$\frac{\partial D_2\left(z;\phi_G\right)}{\partial p} = -\int_{\underline{\theta_P}}^{\tau-t}\frac{\tau-p}{\phi_G}f\left(\frac{\tau-p}{\phi_G},\theta_P\right)d\theta_P + \int_{\frac{t}{\phi_G}}^{\frac{\tau-p}{\phi_G}}\phi_G\theta_G f\left(\theta_G,\phi_G\theta_G+p-t\right)d\theta_P d\theta_G + \int_{\underline{\theta_P}}^{\tau-t}\frac{\tau-p}{\phi_G}f\left(\frac{\tau-p}{\phi_G},\theta_P\right)$$

$$= \int_{\frac{t}{\phi_G}}^{\frac{\tau-p}{\phi_G}}\phi_G\theta_G f\left(\theta_G,\phi_G\theta_G+p-t\right)d\theta_P d\theta_G$$

$$\frac{\partial D_2\left(z;\phi_G\right)}{\partial \tau} = \int_{\underline{\theta_P}}^{\tau-t}\frac{\tau-p}{\phi_G}f\left(\frac{\tau-p}{\phi_G},\theta_P\right)d\theta_P - \int_{\underline{\theta_P}}^{\tau-t}\frac{\tau-p}{\phi_G}f\left(\frac{\tau-p}{\phi_G},\theta_P\right)d\theta_P + \int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}}\phi_G\theta_G f\left(\theta_G,\tau-t\right)d\theta_G$$

$$= \int_{\frac{\tau-p}{\phi_G}}^{\overline{\theta_G}}\phi_G\theta_G f\left(\theta_G,\tau-t\right)d\theta_G,$$

so,

$$\frac{\partial D_2\left(z^*;\phi_G^*\right)}{\partial t} = -\int_{\underline{\theta_P}}^{p^*}\frac{t^*}{\phi_G^*}f\left(\frac{t^*}{\phi_G^*},\theta_P\right)d\theta_P - \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\phi_G^*\theta_G f\left(\theta_G,p^*\right)d\theta_G$$

$$\frac{\partial D_2\left(z^*;\phi_G^*\right)}{\partial p} = 0$$

$$\frac{\partial D_2\left(z^*;\phi_G^*\right)}{\partial \tau} = \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}}\phi_G^*\theta_G f\left(\theta_G,p^*\right)d\theta_G,$$

which is the same as in (A2).

Similarly,

$$\frac{\partial E_2\left(z;\phi_G\right)}{\partial t} = -\int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}}\left(\tau-t-c\right)f\left(\theta_G,\tau-t\right)d\theta_G + \int_{p}^{\tau-t}\left(\theta_P-c\right)f\left(\frac{\theta_P+t-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P$$

$$+ \int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}}\left(\tau-t-c\right)f\left(\theta_G,\tau-t\right)d\theta_G$$

$$= \int_{p}^{\tau-t}\left(\theta_P-c\right)f\left(\frac{\theta_P+t-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P$$

$$\frac{\partial E_2\left(z;\phi_G\right)}{\partial p} = -\int_{\underline{\theta}}^{\frac{t}{\phi_G}}\left(p-c\right)f\left(\theta_G,p\right)d\theta_G - \int_{p}^{\tau-t}\left(\theta_P-c\right)f\left(\frac{\theta_P+t-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P$$

$$- \int_{\tau-t}^{\overline{\theta_P}}\left(\theta_P-c\right)f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P$$

$$\frac{\partial E_2\left(z;\phi_G\right)}{\partial \tau} = \int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}}\left(\tau-t-c\right)f\left(\theta_G,\tau-t\right)d\theta_G - \int_{\underline{\theta}}^{\frac{\tau-p}{\phi_G}}\left(\tau-t-c\right)f\left(\theta_G,\tau-t\right)d\theta_G$$

$$+ \int_{\tau-t}^{\overline{\theta_P}}\left(\theta_P-c\right)f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P$$

$$= \int_{\tau-t}^{\overline{\theta_P}}\left(\theta_P-c\right)f\left(\frac{\tau-p}{\phi_G},\theta_P\right)\frac{1}{\phi_G}d\theta_P,$$

10

so,

$$\frac{\partial E_2\left(z^*; \phi_G^*\right)}{\partial t} = 0$$

$$\frac{\partial E_2\left(z^*; \phi_G^*\right)}{\partial p} = -\int_{\underline{\theta}}^{\frac{t^*}{\phi_G^*}} \left(p^* - c\right) f\left(\theta_G, p^*\right) d\theta_G - \int_{p^*}^{\overline{\theta_P}} \left(\theta_P - c\right) f\left(\frac{t^*}{\phi_G^*}, \theta_P\right) \frac{1}{\phi_G^*} d\theta_P$$

$$\frac{\partial E_2\left(z^*; \phi_G^*\right)}{\partial \tau} = \int_{p^*}^{\overline{\theta_P}} \left(\theta_P - c\right) f\left(\frac{t^*}{\phi_G^*}, \theta_P\right) \frac{1}{\phi_G^*} d\theta_P,$$

which is the same as the expressions in (A3).

Finally,

$$\frac{\partial F_2\left(z; \phi_G\right)}{\partial t} = \int_{\frac{\tau - p}{\phi_G}}^{\overline{\theta_G}} \left(\phi_G \theta_G + \tau - t - c\right) f\left(\theta_G, \tau - t\right) d\theta_G$$

$$\frac{\partial F_2\left(z; \phi_G\right)}{\partial p} = \int_{\tau - t}^{\overline{\theta_P}} \left(\tau - p + \theta_P - c\right) f\left(\frac{\tau - p}{\phi_G}, \theta_P\right) \frac{1}{\phi_G} d\theta_P$$

$$\frac{\partial F_2\left(z; \phi_G\right)}{\partial \tau} = -\int_{\tau - t}^{\overline{\theta_P}} \left(\tau - p + \theta_P - c\right) f\left(\frac{\tau - p}{\phi_G}, \theta_P\right) \frac{1}{\phi_G} d\theta_P - \int_{\frac{\tau - p}{\phi_G}}^{\overline{\theta_G}} \left(\phi_G \theta_G + \tau - t - c\right) f\left(\theta_G, \tau - t\right) d\theta_G,$$

so,

$$\frac{\partial F_2\left(z^*; \phi_G^*\right)}{\partial t} = \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} \left(\phi_G^* \theta_G + p^* - c\right) f\left(\theta_G, p^*\right) d\theta_G$$

$$\frac{\partial F_2\left(z^*; \phi_G^*\right)}{\partial p} = \int_{p^*}^{\overline{\theta_P}} \left(t^* + \theta_P - c\right) f\left(\frac{t^*}{\phi_G^*}, \theta_P\right) \frac{1}{\phi_G^*} d\theta_P$$

$$\frac{\partial F_2\left(z^*; \phi_G^*\right)}{\partial \tau} = -\int_{p^*}^{\overline{\theta_P}} \left(t^* + \theta_P - c\right) f\left(\frac{t^*}{\phi_G^*}, \theta_P\right) \frac{1}{\phi_G^*} d\theta_P - \int_{\frac{t^*}{\phi_G^*}}^{\overline{\theta_G}} \left[\phi_G^* \theta_G + p^* - c\right] f\left(\theta_G, p^*\right) d\theta_G$$

which is the same as the expressions in (A4). Since all the components are identical the result follows.