ON SOME TYPES OF CONTRACTUAL STABILITY IN A PURE EXCHANGE ECONOMY[1]

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Abstract

In the paper, a game-theoretical analysis of some stable outcomes in pure exchange economies is given. We deal, mostly, with an equilibrium characterization of unimprovable allocations generated by recontracting process close to that introduced by V.Makarov (1980). Rather mild assumptions, providing coincidence of the corresponding contractual core and the set of Walrasian equilibrium allocations, are established, and two examples, demonstrating relevance of the main assumptions, are proposed.

Keywords: M-Contract, Weak quasi-stable contractual system, Weak totally contractual core, Competitive equilibrium.

1 Introduction

This paper contains a game-theoretical analysis of the so-called weak totally contractual allocations, similar to that introduced by V.Makarov in order to describe stable outcomes of some quite natural recontracting processes in pure exchange economies. Detailed presentation of Makarov's original settings can be found in [**?**]. Below, we analyze a slightly strengthened version of contractual blocking, proposed in this paper. Namely, in what follows, no additional restrictions to the stopping rule of the breaking procedure is posed besides the feasibility of the final contractual system (hence, no minimality condition, applied in [**?**, **?**, **?**], and [**?**]). At the same time, like in the initial definition from [**?**],

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any such a final contractual system supposed to be an improvement for each member of blocking coalition (not just at least one of the minimal final system, like it appears to be in [?, ?], and [?]). So, being close to the original definition of contractual blocking, the version exploiting in the paper seems to be quite far from that introduced in [?]. In fact, even for the recontracting process, investigated in the latter paper, the multiplicity of outcomes of contractual breaking procedure is quite typical. Remind [?], that it may often be the case that after by some coalition chosen contracts are broken, the rest (including newly concluded by this coalition) do not constitute a feasible system. It is inherent in the model that in such a situation the breaking process proceeds spontaneously, and stops after feasibility of the contract system is recovered. The only requirement we deal with on this step is minimality: the spontaneous process breaks (nullifies) collection of contracts as small, as possible, provided that not nullified ones constitute a feasible contract system. It is clear that by omitting the minimality condition one enlarges the number of outcomes of the breaking procedure considerably. Therefore, to convince the contractual blocking exploiting in the paper is much more stronger than that applied in [?], we only stress that the former blocking requires blocking coalition to improve upon the initial contract system with any possible breaking outcome (not just with one of them, like in [?]).

Surprisingly, for several rather wide classes of markets, the cores corresponding to the above mentioned types of blocking, are the same. To clarify this phenomenon, we prove one of the main results of the paper, stating that under rather mild assumptions the weak totally contractual core (consisting, by definition, of the allocations that are stable w.r.t the strengthened blocking) is equal to the set of Walrasian equilibrium allocations. To demonstrate the main assumptions in the core equivalence theorem, mentioned above, are relevant, two examples of pure exchange economies having unblocked allocations with no supporting equilibrium prices are given. The most interesting seems to be the last example with no equilibrium allocations and nonempty weak totally contractual core, exhibiting that a weak totally contractual allocation may be chosen as a compromise solution in case the classical market mechanism doesn't work.

2 Weak Totally Contractual Core

Below, we consider a slightly generalized pure exchange model

$$\mathcal{E} = \langle N, \{X_i, w^i, \alpha_i\}_N, \sigma\rangle, \quad (1)$$

where $N := \{1, \ldots, n\}$ is a set of consumers, and $X_i \subseteq \mathbf{R}^l,\ w^i \in \mathbf{R}^l,\ \alpha_i \subseteq X_i \times X_i$ are their consumption sets, initial endowments, and individual preference relations, respectively. As to σ, which is a nonempty subset of 2^N, called a coalitional structure of $\mathcal{E}$, it defines a collection of admissible coalitions $S \subseteq N$, which may join efforts of their participants in order to improve (block) any current allocation of the total initial endowment $\sum_N w^i$.

Remind also (see [?]), that in our notations inclusion $(x, y) \in \alpha_i$ means that y is more preferable than x (w.r.t. the preference relation α_i). For any $x \in X_i$, we denote by $\alpha_i(x)$ a collection of all the bundles $y \in X_i$ that are more preferable than x

$$\alpha_i(x) := \{y \in X_i \mid (x, y) \in \alpha_i\}.$$

As usual, we apply the following shortenings: $x\alpha_i y \Leftrightarrow (x, y) \in \alpha_i$, and

$$\mathcal{P}_i(x) := \{y \in X_i \mid y \in \alpha_i(x),\ x \notin \alpha_i(y)\}$$

(recall, that in the standard interpretation, inclusion $y \in \mathcal{P}_i(x)$ means that y is strictly more preferable than x).

We present first the main notion of the weak contractual domination (blocking) in $\mathcal{E}$. In order to do that, we modify first some relevant definitions from [?], aiming to clarify the main features of the elementary interchange structure we deal with. For each $S \in \sigma$ we fix some subset $M_S \subseteq \mathbf{R}^l$ such that $0 \in M_S$, and put $M_\sigma := \{M_S\}_\sigma$.

Definition 1

A contract (of type M_S) of coalition $S \in \sigma$ is a collection of vectors $\mathrm{v} = \{\mathrm{v}^{ij}\}_{i,j \in S}$ *satisfying conditions:*

(a) $\mathrm{v}^{ii} = 0$ *for any* $i \in N$,

(b) $\mathrm{v}^{ij} \in M_S$ *for any* $i, j \in S$,

(c) $\mathrm{v}^{ij} = -\mathrm{v}^{ji}$ *for any* $i, j \in S$.

Coalition S entering into a contract v will be denoted by $S(\mathrm{v})$, as well.

Components v^{ij}, appearing in definition of the contract $\mathrm{v} = \{\mathrm{v}^{ij}\}_{i,j \in S}$, indicate amount of the corresponding commodities used in the bilateral exchange between the participants $i, j \in S(\mathrm{v})$. Here nonnegative component $\mathrm{v}^{ij}_k \geq 0$ of the vector v^{ij} denotes the amount of commodity k that agent j is obliged to deliver to agent i, and absolute value of negative component $\mathrm{v}_r^{ij} < 0$ measures the amount of commodity r agent i has to deliver to agent j.

As to the subsets M_S, they define admissible types of elementary exchanges within the contract $\mathrm{v} = \{\mathrm{v}^{ij}\}_{i,j\in S}$, e.g., in case $M_S := \{x \in \mathbf{R}^l \mid p^S \cdot x = 0\}$ the only constraint concerning v^{ij}, $i, j \in S$ is that the bilateral exchanges v^{ij} should be equivalent w.r.t. the (fixed) prices p^S.

We call $\mathrm{v} = \{\mathrm{v}^{ij}\}_{i,j\in S}$ *a proper contract*, if either v is a trivial contract (i.e., $\mathrm{v}^{ij} = 0$ for all $i, j \in S(\mathrm{v})$), or for any $i \in S(\mathrm{v})$ there exist $j, k \in S(\mathrm{v})$ such that v^{ij} (v^{ik}) contains strictly positive (strictly negative) components. Note, that the properness of the contract v makes it possible (in principal) to rescind this contract by each member $i \in S(\mathrm{v})$ simply by not delivering the corresponding commodity to the participant $k = k(i)$.

Any finite set $\mathcal{V} = \{\mathrm{v}_r\}_{\mathcal{R}}$, consisting of proper contracts v_r, is called a (proper) *contract system of type* M_σ (*contractual* M_σ*-system*, or c.s. (of type M_σ), for short). Let us stress at once that the elements of $\mathcal{R}$ are supposed to be the "titles" of any type (not necessary natural numbers), naming the contracts belonging to the set $\mathcal{V}$.

Thus, it is supposed that the members of any coalition $S \in \sigma$ may enter several contracts. Moreover, a c.s. $\mathcal{V}$ may contain several samples of the same contract $\mathrm{v}(S)$ differing just by their titles (or, by their names, to be exact). To simplify the terms, we call a c.s. $\tilde{\mathcal{V}} = \{\tilde{\mathrm{v}}_r\}_{\tilde{\mathcal{R}}}$ *a subsystem of c.s.* $\mathcal{V} = \{\mathrm{v}_r\}_{\mathcal{R}}$ if and only if $\tilde{\mathcal{R}} \subseteq \mathcal{R}$, and $\tilde{\mathrm{v}}_r = \mathrm{v}_r$ for any $r \in \tilde{\mathcal{R}}$. Hence, we consider contract systems $\mathcal{V}$ and $\tilde{\mathcal{V}}$ to be identical if and only if $\mathcal{R} = \tilde{\mathcal{R}}$ and $\tilde{\mathrm{v}}_r = \mathrm{v}_r$ for any $r \in \mathcal{R}$.

Denote by $\Delta^i(\mathcal{V})$ the net outcome of the agent i resulting after a c.s. $\mathcal{V} = \{\mathrm{v}_r\}_{\mathcal{R}}$ is accepted and realized

$$\Delta^i(\mathcal{V}) := \begin{cases} \sum_{r \mid i\in S(\mathrm{v}_r)} \sum_{j\in S(\mathrm{v}_r)} \mathrm{v}_r^{ij} \quad , & i \in \bigcup_{r\in\mathcal{R}} S(\mathrm{v}_r), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{v}_r^{ij} = -\mathrm{v}_r^{ji}$ for any $i, j \in S(\mathrm{v}_r)$, $r \in \mathcal{R}$, we have that $\sum_N \Delta^i(\mathcal{V}) = 0$ and, hence, $\sum_N x^i(\mathcal{V}) = \sum_N w^i$ with

$$x^i(\mathcal{V}) = x^i(\mathcal{V}, \mathcal{E}) := w^i + \Delta^i(\mathcal{V})$$

to be the resulting outcome of the agent i, obtained by means of the entering into c.s. $\mathcal{V}$. In the sequel we call $x(\mathcal{V}) = x(\mathcal{V}, \mathcal{E}) := (x^i(\mathcal{V}, \mathcal{E}))_N$ *a contractual* M_σ*-allocation* of economy $\mathcal{E}$ (c.a. (of type M_σ), for short).

Definition 2

A contractual system $\mathcal{V}$ is called feasible, if $x^i(\mathcal{V}) \in X_i$ for any $i \in N$.

By definition, feasibility of a c.s. $\mathcal{V}$ guarantees the contractual M_σ-allocation $x(\mathcal{V})$ to belong to the set

$$X(N) = X_{\mathcal{E}}(N) := \{(x^i)_N \in \prod_N X_i \mid \sum_N x^i = \sum_N w^i\}$$

of balanced allocations of the economy $\mathcal{E}$.

Remind, that due to the properness, any contract $\mathrm{v} \in \mathcal{V}$ may, in principal, be broken by any participant $i \in S(\mathrm{v})$. A formal description of the outcomes of rescinding (breaking) contractual subsystems of a feasible c.s. $\mathcal{V} = \{\mathrm{v}_r\}_{r\in\mathcal{R}}$ takes a bit more space. To start with, fix a subsystem $\mathcal{V}' = \{\mathrm{v}_r\}_{r\in\mathcal{R}'}$ of $\mathcal{V}$, consisting of the contracts to be broken, and put

$$\mathcal{U} := \mathcal{V} \setminus \mathcal{V}' = \{\mathrm{v}_r\}_{r\in\mathcal{T}}$$

with $\mathcal{T} := \mathcal{R} \setminus \mathcal{R}'$. Further, let $\mathcal{F}(\mathcal{V}, \mathcal{V}')$ be a collection of all the feasible contractual systems $\tilde{\mathcal{V}}$ of the economy $\mathcal{E}$, satisfying the only requirement

$(*)$ $\tilde{\mathcal{V}} = \{\mathrm{v}_r\}_{r\in\tilde{\mathcal{R}}}$ is a subsystem of $\mathcal{U}$.

Introduce, finally, a version $A_*(\mathcal{V}, \mathcal{R}', \mathcal{E})$ of the set of breaking process outcomes, investigating in this paper:

$$A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}) := \begin{cases} \{\mathcal{U}\} \quad , & \text{if } \mathcal{U} \text{ is a feasible c.s. of } \mathcal{E}, \\ \mathcal{F}(\mathcal{V}, \mathcal{V}') & \text{otherwise.} \end{cases}$$

Thus, $A_*(\mathcal{V}, \mathcal{R}', \mathcal{E})$ contains all the outcomes of the breaking (cancelation) procedure, consisting of no more than two steps. At the first step of this procedure we nullify all the contracts $\mathrm{v}_r,\ r \in \mathcal{R}'$. In case $\mathcal{U} = \{\mathrm{v}_r\}_{\mathcal{R}\setminus\mathcal{R}'}$ is a feasible subsystem, we stop the breaking procedure and put $A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}) := \{\mathcal{U}\}$. Otherwise, at the second step, we continue to nullify contracts $\mathrm{v_r},\ r \in \mathcal{R}''$, with $\mathcal{R}''$ being a subset of $\mathcal{R} \setminus \mathcal{R}'$, in order to guarantee the feasibility of subsystem $\tilde{\mathcal{U}} = \{\mathrm{v}_r\}_{\mathcal{R}\setminus(\mathcal{R}'\cup\mathcal{R}'')}$. So, according to the requirement $(*)$, any subsystem $\tilde{\mathcal{U}}$ of $\mathcal{U}$ can be chosen as a final outcome of the second step, provided that this subsystem is feasible. Certainly, there may be no outcomes at the second step in case $\mathcal{U}$ has no feasible subsystems at all.

Summarizing, we stress once more, that a breaking procedure at the second step (admissible when $\mathcal{U}$ is not feasible, only) supposed to be a spontaneous one. Hence, any feasible subsystem of $\mathcal{U}$ (if exists) may be realized as an outcome of this procedure. Thus, except when U is a feasible system itself ($A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}) = \{\mathcal{U}\}$), we deal with the multiplicity of outcomes, in general. It is clear, even from the first glance, that the multiplicity mentioned causes significant obstacles in the study of contractual stability (for more details see Section 2 below).

To describe which way a coalition $S \in \sigma$ can improve a c.a. $x(\mathcal{V})$, generated by a feasible c.s. $\mathcal{V} = \{\mathrm{v}_r\}_{\mathcal{R}}$ of type M_σ, we suppose first that together with possibility to cancel some contracts v_r with $r \in \mathcal{R}^S = \mathcal{R}^S_{\mathcal{V}} := \{r \in \mathcal{R} \mid S(\mathrm{v}_r) \bigcap S \neq \emptyset\}$ it is allowed for the coalition S to enter some new contract w. Denote by $\mathcal{E}_{\mathrm{w}}$ the modification of $\mathcal{E}$ generated by w:

$$\mathcal{E}_{\mathrm{w}} = \langle N, \{X_i, w^i + \Delta^i(\mathrm{w}), \alpha_i\}_N, \sigma\rangle,$$

where, as before, $\Delta^i(\mathrm{w}) := \Delta^i(\{\mathrm{w}\})$, i.e.,

$$\Delta^i(\mathrm{w}) = \begin{cases} \sum_{j\in\in S(\mathrm{w})} \mathrm{w}^{ij} \quad , & \text{if } i \in S(\mathrm{w}), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3

We say that a coalition $S \in \sigma$ with $|S| \geq 2$ w-improves (w-blocks) a feasible contractual M_σ-system $\mathcal{V} = \{\mathrm{v}_r\}_{\mathcal{R}}$, if there exists a subset $\mathcal{R}' \subseteq \mathcal{R}^S$, and a proper contract w *of type M_S such that $S(\mathrm{w}) = \mathrm{S}$, and for any $\mathcal{W} \in A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}_{\mathrm{w}})$ it holds: $x^i(\mathcal{W}, \mathcal{E}_{\mathrm{w}}) \in \alpha_i(x^i(\mathcal{V}, \mathcal{E}))$ for any $i \in S$, with $x^i(\mathcal{W}, \mathcal{E}_{\mathrm{w}}) \in \mathcal{P}_i(x^i(\mathcal{V}, \mathcal{E}))$ for at least one $i \in S$.*

As to the case of one-element coalition, $S = \{i\} \in \sigma$, it is said that S w-improves a feasible contractual M_σ-system $\mathcal{V} = \{\mathrm{v}_r\}_{\mathcal{R}}$, if there exists a subset $\mathcal{R}' \subseteq \mathcal{R}^S$ such that for any $\mathcal{W} \in A_*(\mathcal{V}, \mathcal{R}', \mathcal{E})$ it holds: $x^i(\mathcal{W}, \mathcal{E})$ belongs to $\mathcal{P}_i(x^i(\mathcal{V}, \mathcal{E}))$.

(Here and below, as before, $\mathcal{P}_i(z) := \{x^i \in \alpha_i(z) \mid (x^i, z) \notin \alpha_i\}$ is the set of those bundles from X_i, which are strictly preferred to z.)

Definition 4

A feasible contractual M_σ-system $\mathcal{V}$ is called a weak quasi-stable, if there is no coalition $S \in \sigma$, which w-improves $\mathcal{V}$.

For any $x \in X(N)$ denote by $\mathcal{C}(x) = \mathcal{C}_{M_\sigma}(x)$ the set of all feasible c.s. $\mathcal{V}$ of type M_σ such that $x = x(\mathcal{V})$. To isolate the allocations $x \in X(N)$ with $\mathcal{C}_{M_\sigma}(x) \neq \emptyset$, whose coalitional stability is independent on the concrete representation via some c.s. $\mathcal{V}$ of type M_σ, we introduce the main notion of the paper.

Definition 5

An allocation $x \in X(N)$ is called a weak totally contractual M_σ-allocation (w.t.c.a. *(of type M_σ), for short), if $\mathcal{C}_{M_\sigma}(x)$ is nonempty set, and it contains only weak quasi-stable contractual systems (i.e., for every $\mathcal{V} \in \mathcal{C}_{M_\sigma}(x)$ it holds: $\mathcal{V}$ is a weak quasi-stable allocation).*

A set $D_*^{M_\sigma}(\mathcal{E})$ of all the weak totally contractual M_σ-allocations of $\mathcal{E}$ is called *a weak totally contractual core (of type M_σ) of economy $\mathcal{E}$.*

Remark 1. Note, that like in this paper, the main point at issue in [?] is the fact that after by some coalition $S \in \sigma$ chosen contracts with numbers $r \in \mathcal{R}'$ are broken, the rest, with numbers $r \in \mathcal{R} \setminus \mathcal{R}'$ may not constitute a feasible contractual system in $\mathcal{E}_{\mathrm{w}}$. In both papers, it is supposed that the breaking process proceeds spontaneously, and stops after feasibility of the contractual system $\mathcal{U} = \{\mathrm{v}_r\}_{r \in \mathcal{R} \setminus \mathcal{R}'}$ is recovered. One of the distinctive feature of the stopping rule applied in the paper mentioned is that the set of outcomes (denoted there by $A(\mathcal{V}, \mathcal{R}', \mathcal{E}_{\mathrm{w}})$) supposed to consist of only the maximal feasible subsystems of $\mathcal{U}$ (remind, that here, in this paper, we deal with $A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}_{\mathrm{w}})$ that consists of all the feasible subsystem of $\mathcal{U}$). Consequently, we have

$$A(\mathcal{V}, \mathcal{R}', \mathcal{E}_{\mathrm{w}}) \subseteq A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}_{\mathrm{w}}).$$

Second (and the last) distinction concerns the blocking rule: to improve a feasible contractual M-system $\mathcal{V}$ it is sufficient (according to [?]) to find at least one maximal feasible subsystem of $\mathcal{U}$ that improves $\mathcal{V}$ (unlike the w-blocking in this paper, which requires any feasible subsystem of $\mathcal{U}$ to be able to improve $\mathcal{V}$).
Summarizing, we have that the sets $D^M(\mathcal{E}),\ M \in \mathcal{M}$, of totally contractual M-allocations (unblocked M-allocation in the sense of blocking from [?]) satisfy inclusions

$$D^M(\mathcal{E}) \subseteq D_*^M(\mathcal{E}), \quad M \in \mathcal{M}.$$

Surprisingly, for several rather wide classes of pure exchange economies it holds: $D^M(\mathcal{E}) = D_*^M(\mathcal{E})$ (for more details, see Sect.4 below).

Note, that any coalitional structure σ admits the (unique) representation as the union of (locally) irreducible coalitional substructures, inscribed into the corresponding components of N. Therefore, in what follows, we suppose σ to be irreducible itself. For the sake of completeness, remind corresponding definition from ([**?**]) (as usual, below we call a partition $\{N_1, N_2\}$ nontrivial, if N_1 and N_2 are nonempty):

- *Irreducibility-assumption*: for any nontrivial partition $\{N_1, N_2\}$ of N there is $S \in \sigma$ such that $S \bigcap N_1$ and $S \bigcap N_2$ are nonempty sets.

Besides, everywhere below it is assumed that all the subsets M_S are identical and equal to some linear subspace $M \subseteq \mathbf{R}^l$ satisfying the following *Sign-assumption*:

- *Sign-assumption*: every nonzero vector $z \in M$ contains both strictly positive and strictly negative components.

Remark 2. Observe, that from the Bipolar Theorem and Minkovski Separation Theorem it follows immediately that a linear subspace $M \neq \mathbf{R}^l$ satisfies the above mentioned Sign-assumption if and only if its polar subspace $M^0 := \{p \in \mathbf{R}^l \mid p \cdot z = 0,\ z \in M\}$ meets the requirement

$$M^0 \bigcap \text{int}\mathbf{R}^l_+ \neq \emptyset. \tag{2}$$

It is clear, that in case relation (2) is satisfied there exists a nonempty finite subset $P_M \subseteq \mathbf{R}^l$ containing at least one strictly positive vector, such that $M = \{z \in \mathbf{R}^l \mid p \cdot z = 0,\ p \in P_M\}$. Hence, the economical meaning of the constraint imposed on the type of the contracts we consider below is as follows: elementary exchanges bundles v^{ij} should be compatible with all the fixed-price vectors p, belonging to some nonempty finite subset $P_M \subseteq \mathbf{R}^l$, containing at least one strictly positive price vector $\bar{p}$.
Under Sign-assumption it can easily be shown also , that for any $S \subseteq N$ with $|S| \geq 3$ there exists a proper "zero-contract" w of type M (i.e., a proper contract $\mathrm{w} \neq 0$ of type M such that (i) $S(\mathrm{w}) = S$, and (ii) $\Delta^i(\mathrm{w}) = 0$ for any $i \in S$). It means that the properness of any contract v of type $M \neq \{0\}$ with $|S(\mathrm{v})| \geq 3$ is guaranteed "automatically": $\mathrm{u} := \mathrm{v} + \lambda \mathrm{w}$ is a proper contract of type M with $S(\mathrm{u}) = \mathrm{S(v)}$ and $\Delta^i(\mathrm{u}) = \Delta^i(\mathrm{v})$, $i \in N$, provided that $\mathrm{w} \neq 0$ is a proper "zero-contract" of type M with $S(\mathrm{w}) = \mathrm{S(v)}$, and $\lambda > 0$ is large enough. Hence, everywhere below we may (and will) deal with some contracts and contract systems of type M, not necessarily satisfying properness-assumption for not-two-person coalitions (at least, in those situations, where only properness "modulo zero-contract" matters).

Denote by $\mathcal{C}_M$ a collection of all c.s. of type M_σ with $M_S = M$ for any $S \in \sigma$. In what follows, the contracts v of type $M_S = M$, as well as the systems $\mathcal{V} \in \mathcal{C}_M$, and allocations $x = x(\mathcal{V})$, will be called *M-contracts, M-systems and M-allocations*, respectively. To present more detailed description of some unblocked (in the sense of Definition **??**) M-allocations $x(\mathcal{V})$, we characterize first all feasible allocations, attainable by entering into the contract systems of type M. Put

$$X^M_{\mathcal{E}} := \{x \in \prod_N X_i \mid \exists \mathcal{V} \in \mathcal{C}_M : x = x(\mathcal{V})\}.$$

Proposition **??** below was established in ([**?**]); for its crucial role in the further considerations and for the sake of completeness we reproduce it here together with the proof that seems to be rather instructive.

Proposition 1

If σ is irreducible coalition structure, then

$$X^M_{\mathcal{E}} = \{x \in X(N) \mid \exists \Delta \in \Delta_M(N) : x = w + \Delta\},$$

where $\Delta_M(N) := \{\Delta = (\Delta^i)_N \in M^N \mid \sum_N \Delta^i = 0\}$, $w := (w^i)_N$.

Proof. If $x \in X^M_{\mathcal{E}}$, then $x = x(\mathcal{V}) = w + \Delta(\mathcal{V})$ for some $\mathcal{V} \in \mathcal{C}_M$, where $\Delta(\mathcal{V}) := (\Delta^i(\mathcal{V}))_N$. By definition of M-contract it follows that $\Delta^i(\mathcal{V}) \in M$, $\sum_N \Delta^i(\mathcal{V}) = 0$ and, consequently, we get desired: $x = w + \Delta$ for some $\Delta \in \Delta_M(N)$.

Let now $x \in X(N)$ with $x = w + \Delta$ for some $\Delta \in \Delta_M(N)$. To prove that there is $\mathcal{V} \in \mathcal{C}_M$, such that $\Delta = \Delta(\mathcal{V})$, we apply induction on the number $|N|$ of the economical agents of $\mathcal{E}$. It is clear, that in case $|N| = 2$ we have : $\mathcal{V} = \{\mathrm{v}\}$, with $\mathrm{v}^{12} := \Delta^1$, and $\mathrm{v}^{21} := \Delta^2$. Suppose that our assertion is valid for all economies of type (1) with $|N| \leq m$. Fix some economy $\mathcal{E}$ with $|N| = |N_{\mathcal{E}}| = m + 1$. If $N \in \sigma$, then, modulo "zero-contract", in accord with Remark 2, a c.s. required consists of the only M-contract v, defined by the formula: $\mathrm{v}^{ii+1} := \sum_{k=1}^{i} \Delta^k$, $i = 1, \ldots, m$; $\mathrm{v}^{ij} = 0$, $j - i > 1$ (note, that due to the equalities $\mathrm{v}^{ij} = -\mathrm{v}^{ji}$ to give a complete description of the contract v it is sufficient to indicate all the vectors v^{ij} with $i, j \in S(\mathrm{v})$ satisfying inequalities $i < j$).

Consider the case $N \notin \sigma$. Entering into the trivial contracts for some coalitions $S \in \sigma$, if necessary, we may assume without loss of generality, that σ is minimal (w.r.t. inclusion in 2^N) irreducible coalitional structure. Fix some $S^0 \in \sigma$. It is not hard to prove that there exists a partition $\eta^0 = \{N^0_1, N^0_2\}$ of N such that (i)coalitions $S^0_k := S^0 \bigcap N^0_k$, $k = 1, 2$, are nonempty; (ii)for any $T \in \sigma \setminus \{S^0\}$ either $T \subseteq N_1^0$, or $T \subseteq N_2^0$; and (iii)coalitional structures $\sigma_k^0 := \{T \in \sigma \mid T \subseteq N_k^0\} \bigcup \{S_k^0\}$, $k = 1, 2$, are irreducible in N_1^0, N_2^0, respectively. Indeed, existence of a partition η^0, satisfying conditions (i) and (ii), follows immediately from the minimality of the irreducible coalitional structure σ (since in case $\sigma \setminus \{T\}$ is irreducible for some $T \in \sigma$ we get a contradiction). As to the irreducibility of σ_1^0, for example, it can easily be shown by consideration of three cases for any nontrivial partition $\{N_1^1, N_1^2\}$ of N_1^0:

1) $S^0 \bigcap N_1^k \neq \emptyset$, $k = 1, 2$;

2) $S^0 \bigcap N_1^1 \neq \emptyset, S^0 \bigcap N_1^2 = \emptyset$;

3) $S^0 \bigcap N_1^1 = \emptyset, S^0 \bigcap N_1^2 \neq \emptyset$.

Specifically, considering partitions $\{N_1^2, N_1^1 \bigcup N_2\}$ and $\{N_1^1, N_1^2 \bigcup N_2\}$ in cases 2) and 3), respectively, we get required coalitions from σ_1^0 on the basis of condition (ii). In fact, since in both these cases coalition S^0 doesn't intersect one of the elements of the corresponding nontrivial partition of N, by (ii) and irreducibility of σ, in each of these cases there exists a coalition $S \in \sigma_1^0$ such that $S \bigcap N_1^k \neq \emptyset$ for any $k = 1, 2$.

Now, by making use of the partition η^0, we divide initial economy $\mathcal{E}$ into two appropriate "smaller" economies

$$\mathcal{E}_{(k)} := \langle N_{(k)}, X_i^{(k)}, w_{(k)}^i, \alpha_i^{(k)}\}_{N_{(k)}}, \sigma_{(k)} \rangle, \quad k = 1, 2,$$

satisfying inductive hypothesis and "implementing" the allocation $x = w + \Delta$ with $\Delta = (\Delta^1, \ldots, \Delta^n) \in \Delta_M(N)$ by means of some contractual M-system (as a result of a suitable exchange, both within and between these economies). To this end we fix some economic agents $i_k \in S_k^0$, $k = 1, 2$, and put

$$N_{(k)} := N_k^0, \quad k = 1, 2;$$

$$w_{(k)}^i := w^i, \quad i \in N_{(k)}, \ k = 1, 2;$$

$$X_i^{(k)} := X_i \quad i \in N_{(k)} \setminus \{i_k\}, \ k = 1, 2.$$

As to the consumption sets $X_{i_1}^{(1)}$ and $X_{i_2}^{(2)}$, we put $X_{i_1}^{(1)} := \{w^{i_1} - \sum_{i \in N_{(1)} \setminus \{i_1\}} \Delta^i\}$ and $X_{i_2}^{(2)} := \{w^{i_2} + \sum_{i \in N_{(1)} \cup \{i_2\}} \Delta^i\}$. Finally, we take $\sigma_{(k)}$ to be equal to σ_k^0, $k = 1, 2$, and put $\alpha_i^{(k)} := \alpha_i$ for any $i \in N_{(k)}$ and $k = 1, 2$. To apply inductive hypothesis, take $\Delta_{(k)} \in \Delta_M(N_{(k)}),\ k = 1, 2,$ with

$$\Delta_{(1)}^{i_1} := -\sum_{i \in N_{(1)} \setminus \{i_1\}} \Delta^i, \ \ \Delta_{(2)}^{i_2} := \sum_{i \in N_{(1)} \cup \{i_2\}} \Delta^i,$$

and $\Delta_{(k)}^i := \Delta^i$ for any $i \in N_{(k)} \setminus \{i_k\},\ \ k = 1, 2$. Further, for each $k = 1, 2$, put $w_{(k)} := (w^i)_{N_{(k)}}$ and $\Delta_{(k)} := (\Delta_{(k)}^i)_{N_{(k)}}$. It is clear, that $x_{(1)} := w_{(1)} + \Delta_{(1)}$ and $x_{(2)} := w_{(2)} + \Delta_{(2)}$ are balanced allocations of the economies $\mathcal{E}_{(1)},\ \mathcal{E}_{(2)},$ respectively. Hence, considerations, given above, implies: economies $\mathcal{E}_{(1)}, \mathcal{E}_{(2)}$ and corresponding allocations $x_{(1)},\ x_{(2)}$ satisfy our inductive hypothesis, and, consequently, there exist M-systems $\mathcal{V}_{(1)}$ and $\mathcal{V}_{(2)}$ such that $x_{(k)} = x(\mathcal{V}_{(k)}, \mathcal{E}_{(k)})$, each $k = 1, 2$.

Now we design a contract $\mathrm{v}_{(S^0)}$, which coalition S^0 should enter to in order to organize (together with the M-systems $\mathcal{V}_{(1)}$ and $\mathcal{V}_{(2)}$) an M-system $\mathcal{V}$, satisfying equality $x = x(\mathcal{V}) = x(\mathcal{V}, \mathcal{E})$. To do so, we put

$$\mathrm{v}_{(S^0)}^{i_1 i_2} := \sum_{i \in N_{(1)}} \Delta^i, \quad \mathrm{v}_{(S^0)}^{i_2 i_1} := -\sum_{i \in N_{(1)}} \Delta^i,$$

and $\mathrm{v}_{(S^0)}^{ij} := 0$ for any $i, j \in S^0$ such that $\{i, j\} \neq \{i_1, i_2\}$. It is clear, that the system $\mathcal{V} := \mathcal{V}_{(1)} \bigcup \mathcal{V}_{(2)} \bigcup \{\mathrm{v}_{(S^0)}\}$, with $\mathrm{v}_{(S^0)}$ thus defined (and modified in the spirit of of Remark 2, if necessary), meets our requirement, which proves the proposition. $\Box$

3 Weak Totally Contractual Set and Equilibrium Allocations

Below, we restrict ourselves to the case, when M is a hypersubspace of $\mathbf{R}^l$ (i.e., we assume, that $\dim M = l - 1$). In this situation it turns out that any w.t.c. allocation is an equilibrium allocation under fairly natural regularity assumptions. Vice versa, under essentially weaker assumptions any (competitive) equilibrium allocation turns out to be a w.t.c. allocation, provided that a subspace M is properly chosen.

Remind (see, e.g., [**?**], [**?**]), that $\bar{x} \in X(N)$ is called a *competitive equilibrium allocation* (equilibrium allocation, for short), if there exists a nonzero vector $\bar{p} \in \mathbf{R}^l$ such that

$$\mathcal{P}_i(\bar{x}^i) \bigcap B_i(\bar{p}) = \emptyset, \quad i \in N, \tag{3}$$

where, as usual, $B_i(\bar{p})$ is the budget set of an agent i at prices $\bar{p}$:

$$B_i(\bar{p}) := \{x^i \in X_i \mid \bar{p} \cdot x^i \leq \bar{p} \cdot w^i\}.$$

Denote by $W(\mathcal{E})$ the set of all equilibrium allocations of economy $\mathcal{E}$. Remind, that any vector $\bar{p}$, satisfying (3), is called an equilibrium price vector (equilibrium price, for short), supporting the allocation $\bar{x}$.

Theorem 1

Let $\bar{x}$ be an equilibrium allocation of the economy $\mathcal{E}$. If there is a strictly positive equilibrium price vector $\bar{p}$, supporting $\bar{x}$, then $\bar{x}$ belongs to $D_*^M(\mathcal{E})$ with $M := \{z \in \mathbf{R}^l \mid \bar{p} \cdot z = 0\}$.

Proof. Since $\bar{x} \in W(\mathcal{E})$ implies inclusion $\bar{x} \in X(N) \bigcap \prod_N B_i(\bar{p})$, we have: $\Delta^i := \bar{x}^i - w^i$ belongs to M for any $i \in N$. It is clear, that irreducibility of σ and Proposition**??** imply: $\bar{x} = w + \Delta \in X_{\mathcal{E}}^M$, where, as before, $w = (w^i)_N$, and $\Delta = (\Delta^i)_N$. Suppose, the allocation $\bar{x}$ can be w-improved by some coalition $S \in \sigma$. Then, by Definition **??**, there exist at least one contractual system $\mathcal{V} \in \mathcal{C}_M(\bar{x})$ and one participant $i \in S$ such that $x^i(\mathcal{V}) \in \mathcal{P}_i(\bar{x}^i)$ and $x^j(\mathcal{V}) \in \alpha_j(\bar{x}^j)$ for any $j \in S \setminus \{i\}$. By definition of equilibrium allocation, the former inclusion $x^i(\mathcal{V}) \in \mathcal{P}_i(\bar{x}^i)$ implies $\mathcal{P}_i(\bar{x}^i) \bigcap B_i(\bar{p}) = \emptyset$. But then $\bar{p} \cdot x^i(\mathcal{V}) > \bar{p} \cdot w^i$, which contradicts to the inclusion $x^i(\mathcal{V}) \in X_i^M$ yielding $\bar{p} \cdot x^i(\mathcal{V}) = \bar{p} \cdot w^i$.

Thus, there are no coalitions $S \in \sigma$ that can w-improve $\bar{x}$ and, hence $\bar{x}$ belongs to the weak totally contractual core (of type M) of the economy $\mathcal{E}$. □

To prove the reverse inclusion $D_*^M(\mathcal{E}) \subseteq W(\mathcal{E})$ we have to add some compatibility assumptions concerning the coalitional structure σ (as well, as traditional convexity and monotonicity assumptions, imposed on the consumption sets X_i and individual preference relations α_i). Namely, from now on it is supposed that for any $i \in N$ it holds:

(a) X_i is a convex set;

(b) α_i is a reflexive binary relation;

(c) for any $x^i \in \widehat{X_i} := \mathrm{Pr}_{X_i} X(N)$ there exists $z \in \mathrm{int}\mathbf{R}_+^l$ such that $x^i + z \in \mathcal{P}_i(x^i)$;

(d) for any $x^i \in X_i$ the set $\mathcal{P}_i(x^i)$ is convex and, besides, $(x^i, y^i] \subseteq \mathcal{P}_i(x^i)$ for any $y^i \in \mathcal{P}_i(x^i)$, where $(x^i, y^i] := \{(1-t)x^i + ty^i \mid t \in (0,1]\}$.

To propose a compatibility assumption concerning σ, remind first, that for any $i \in N$ we denote by σ_i a set of those coalitions $S \in \sigma$ that contain i :

$$\sigma_i := \{S \in \sigma \mid i \in S\}.$$

Second, we introduce a useful strengthening of the notion of σ-divisibility, proposed earlier in ([**?**]).

Definition 6

A coalition $T \subseteq N$ is said to be strongly σ-divisible, if for any $i \in N$ there are two coalitions $R, S \in \sigma_i$ such that

$$R \cap (T \setminus S) \neq \emptyset. \tag{4}$$

Further, as in [**?**], for any $x \in X_{\mathcal{E}}^M$ put

$$N_x := \{i \in N \mid x^i \in \mathrm{int}_M X_i\},$$

where $\mathrm{int}_M X_i$ is the (relative) interior of $X_i^M := (w^i + M) \cap X_i$ in the affine subspace $w^i + M$.

The main result of this section is as follows.

Theorem 2

Suppose $\bar{x}$ belongs to $D_*^M(\mathcal{E})$ for some M satisfying Sign-assumption. If $N_{\bar{x}}$ is a strongly σ-divisible subset of N, then $\bar{x}$ is an equilibrium allocation of $\mathcal{E}$.

Proof. Let $\bar{x} \in D_*^M(\mathcal{E})$, and i be any agent of the economy $\mathcal{E}$. In order to show that $\bar{x}^i$ is a locally maximal element on X_i^M w.r.t. the binary relation α_i, we fix first some coalitions $R, S \in \sigma_i$ and a participant $k \in R$ such that $k \in N_{\bar{x}}$ and $k \notin S$ (remind, that the existence of such coalitions R, S and a participant k follows directly from the fact that $N_{\bar{x}}$ is a strongly σ-divisible set). Since $\bar{x}^k \in \mathrm{int}_M X_k$, there is some $U \subseteq M$, being a symmetric neighbourhood of zero in M, such that $\bar{x}^k + U \subseteq X_k^M$. Assuming that $\bar{x}^i$ is not a locally maximal element on X_i^M w.r.t. the binary relation α_i, we have that there exists $z \in U$ satisfying inclusion $\bar{x}^i + z \in \mathcal{P}_i(\bar{x}^i)$. To get a contradiction, we construct now a contractual M-system $\bar{\mathcal{V}} \in \mathcal{C}(\bar{x})$, which can be w-improved by coalition S (chosen in the very beginning of the proof). To do so, fix zero M-contract w with $S(\mathrm{w}) = \mathrm{S}$ (and $\mathrm{v}^{ij} = 0$ for any $\{i, j\} \subseteq S$), and consider an arbitrary $\mathcal{V} \in \mathcal{C}(\bar{x})$. Further, denote by $\bar{\mathrm{v}}$ an M-contract satisfying requirements: $S(\bar{\mathrm{v}}) = R$, and

$$\bar{\mathrm{v}}^{jr} = \begin{cases} z & , \quad j = i,\, r = k, \\ 0 & , \quad \{j, r\} \neq \{i, k\} \end{cases}$$

(it follows directly from Remark 2 that it always possible to construct a proper "almost zero-contract" v of type M with $|S(\mathrm{v})| \geq 3$ and $\mathrm{v}^{ij} = 0$ for all but one two-element subsets $\{i,j\} \subseteq S$). Put $\bar{\mathcal{V}} := \mathcal{V} \cup \{\mathrm{v}_1, \mathrm{v}_2\}$ with $\mathrm{v}_1 = \bar{\mathrm{v}}$, $\mathrm{v}_2 = -\bar{\mathrm{v}}$, where, as usual, $(-\mathrm{v})^{ij} = -\mathrm{v}^{ij}$ for any $\{i,j\} \subseteq S(\mathrm{v})$. Further, consider $\mathcal{R}' \subseteq \mathcal{R}^S$ to be equal to $\{2\}$. Note first, that by definition of the contracts v, w and contractual system $\mathcal{V}$ it follows immediately that $A_*(\bar{\mathcal{V}}, \mathcal{R}', \mathcal{E}_{\mathrm{w}})$ contains only one system, namely, $A_*(\bar{\mathcal{V}}, \mathcal{R}', \mathcal{E}_{\mathrm{w}}) = \{\mathcal{U}\}$ with $\mathcal{U} := \mathcal{V} \cup \{\mathrm{v}_1\}$ (no multiplicity of outcomes of cancelation process due to a proper chosen breaking system $\mathcal{R}'$). Second, due to the fact that preference relations α_j, $j \in N$, are reflexive, and by definition of z and v_1 we get: $x^j(\mathcal{U}, \mathcal{E}_{\mathrm{w}}) \in \alpha_j(x^j(\bar{\mathcal{V}}, \mathcal{E}))$ for any $j \in S$, with $x^i(\mathcal{U}, \mathcal{E}_{\mathrm{w}}) \in \mathcal{P}_i(x^i(\bar{\mathcal{V}}, \mathcal{E}))$. Hence, according to the Definition **??**, we conclude, that coalition S w-improves a feasible contractual M-system $\bar{\mathcal{V}}$ belonging to $\mathcal{C}(\bar{x})$.

Thus, we have obtained desired contradiction with assumption $\bar{x} \in D_*^M(\mathcal{E})$ and, consequently, finished the proof of the fact that for any $i \in N$, the bundle $\bar{x}^i$ is a local maximum on X_i^M w.r.t. his preference relation α_i.

More precisely, we have established that for any $i \in N$, there exists a symmetric neighbourhood of zero in M, say V, such that $\mathcal{P}_i(\bar{x}^i) \cap (\bar{x}^i + V) \cap X_i = \emptyset$. To continue the proof of the inclusion $\bar{x} \in W(\mathcal{E})$, check first that in fact the intersection $\mathcal{P}_i(\bar{x}^i) \cap X_i^M$ is empty, as well. To verify the latter assertion, just mention, that for any $y^i \in \mathcal{P}_i(\bar{x}^i) \bigcap X_i^M$ (if such y^i exists) the bundle $\bar{z}(t) := ty^i + (1-t)\bar{x}^i$ belongs to the neighbourhood $\bar{x}^i + V$ of $\bar{x}^i$ for any $t \in (0,1)$ small enough. But this fact (together with the convexity assumption (d)) contradicts to the local maximality of $\bar{x}^i$, which had already been proven above.

Summarizing, we have that for every $i \in N$ the consumption bundle $\bar{x}^i$ is maximal on X_i^M w.r.t. α_i. Proceeding now like in [**?**] (Theorem 4.13), pick some $\bar{p} \in M^0 \bigcap \mathrm{int}\mathbf{R}_+^l$ and prove that $\mathcal{P}_i(\bar{x}^i) \bigcap B_i(\bar{p}) = \emptyset$ for any $i \in N$. Let $y^i \in \mathcal{P}_i(\bar{x}^i)$ for some $i \in N$. Since $\bar{p} \cdot x^i = \bar{p} \cdot w^i$ for any $x^i \in X_i^M$ (in particular, $\bar{p} \cdot \bar{x}^i = \bar{p} \cdot w^i$, each $i \in N$), from the maximality of $\bar{x}^i$ on X_i^M it follows that $\bar{p} \cdot y^i \neq \bar{p} \cdot w^i$. Suppose, that $\bar{p} \cdot y^i < \bar{p} \cdot w^i$. Because of the inclusion $\bar{x} \in X(N)$ we have, due to the monotonicity assumption (c): there is $z \in \mathrm{int}\mathbf{R}_+^l$ such that $\bar{y}^i := \bar{x}^i + z \in \mathcal{P}_i(\bar{x}^i)$. It is clear, that $\bar{p} \cdot \bar{y}^i > \bar{p} \cdot w^i$. But then there exists $\bar{t} \in (0,1)$ such that the bundle $z(\bar{t}) := \bar{t}y^i + (1-\bar{t})\bar{y}^i$ belongs to X_i^M. On the other hand, due to the convexity of $\mathcal{P}_i(\bar{x}^i)$ we have: $z(\bar{t}) \in \mathcal{P}_i(\bar{x}^i)$, which contradicts to the maximality of $\bar{x}^i$ on X_i^M w.r.t. α_i.

Contradiction obtained proves the inequality $\bar{p} \cdot y^i > \bar{p} \cdot w^i$, which implies the relation $y^i \notin B_i(\bar{p})$. Due to the arbitrariness of element y^i taken from $\mathcal{P}_i(\bar{x}^i)$, we have desired: $\mathcal{P}_i(\bar{x}^i) \cap B_i(\bar{p}) = \emptyset$. $\square$

Note, that the cardinality of $N_{\bar{x}}$ may be very small. It is clear, however, that σ-divisibility condition takes the simplest form in case $N_{\bar{x}} = N$.

To easify formulations, from now on we assume that hypersubspace M under consideration always satisfies Sign-assumption.

Corollary 1

If $|\sigma_i| \geq 2$ for any $i \in N$, then $D^M_{**}(\mathcal{E}) \subseteq W(\mathcal{E})$, where

$$D^M_{**}(\mathcal{E}) := D^M_{*}(\mathcal{E}) \bigcap X^M_0, \quad X^M_0 := \prod_N \mathit{int}_M X_i.$$

Proof. Let $\bar{x} \in D^M_{**}(\mathcal{E})$ be given. To prove that $\bar{x}$ belongs to $W(\mathcal{E})$, we mention first that due to the inclusion $\bar{x} \in X^M_0$ we have $N_{\bar{x}} = N$. To show that $\bar{x}$ satisfies all the requirements of Theorem**??**, it is sufficient to prove that under assumptions of our corollary the grand coalition N is strongly σ-divisible. Fix an arbitrary $i \in N$. By applying assumption $|\sigma_i| \geq 2$, select two different coalitions R and S, belonging to σ_i. Due to $R \neq S$ we have that either $R \backslash S \neq \emptyset$, or $S \backslash R \neq \emptyset$ (otherwise we get equality $R = S$ contradicting the choice of R, S). It is clear, that in both cases the relation (**??**) is satisfied (with $T = N$ and renaming the coalitions selected, if necessary). Hence, $N_{\bar{x}}$ is a strongly σ-divisible subset of N and, consequently, by Theorem **??** we get inclusion required: $\bar{x} \in W(\mathcal{E})$. □

The results obtained allow us to find some rather large classes of exchange models, admitting an equilibrium characterization for any w.t.c. allocation $\bar{x}$, without direct analysis of the structure of $N_{\bar{x}}$. To give some examples, introduce first additional characteristics of the exchange model $\mathcal{E}$. Denote by $\mathcal{M}$ the set of all hypersubspaces $M \subseteq \mathbf{R}^l$ satisfying Sign-assumption, and for any $M \in \mathcal{M}$ put

$$S^M_{\mathcal{E}} := \{i \in N \mid \alpha_i \text{ is complete, transitive, } w^i \in X_i, \text{ and } \alpha_i(w^i) \bigcap X^M_i \subseteq \mathrm{int}_M X_i\}.$$

Definition 7

An exchange model $\mathcal{E}$ is called C_M-regular, if $S^M_{\mathcal{E}} \neq \emptyset$.

Definition 8

An exchange model $\mathcal{E}$ is called $C_{\mathcal{M}}$-regular, if it is C_M-regular for any $M \in \mathcal{M}$.

Remark 3. Note, that by definition of the sets $S^M_{\mathcal{E}}$ we have that for any $M \in \mathcal{M}$ it holds: $w^i \in \mathrm{int}_M X_i$ for each $i \in S^M_{\mathcal{E}}$. Consequently, for any $C_{\mathcal{M}}$-regular economy $\mathcal{E}$ we have: $w^i \in \mathrm{int}_M X_i$ for any $M \in \mathcal{M}$ and $i \in S^M_{\mathcal{E}}$.

By applying, mutatis mutandis, argumentation, used in the proof of Theorem **??**, we obtain the following results.

Proposition 2
Let $\mathcal{E}$ be a C_M-regular pure exchange model with w^i belonging to X_i for any $i \in N \setminus S^M_{\mathcal{E}}$. If $S^M_{\mathcal{E}}$ is a strongly σ-divisible subset of N, and one-element coalition $\{i\}$ belongs to σ for any $i \in S^M_{\mathcal{E}}$, then $D^M_*(\mathcal{E}) \subseteq W(\mathcal{E})$.

Proof. Observe first, that by definition of the set $S^M_{\mathcal{E}}$ and due to the assumption $w^i \in X_i,\ i \in N \setminus S^M_{\mathcal{E}}$, of our proposition, we have that $w^i \in X_i$ for any $i \in N$ (initial endowments of the agents belong to their consumption sets). Let $\bar{x}$ be an arbitrary element of $D^M_*(\mathcal{E})$. It is not very hard to verify that $S^M_{\mathcal{E}}$ is contained in $N_{\bar{x}}$. To start with the proof of this assertion, note that due to the completeness of α_i and inclusion $\{i\} \in \sigma$, fulfilled for any $i \in S^M_{\mathcal{E}}$, we get: $\bar{x}^i \in \alpha_i(w^i)$ for any $i \in S^M_{\mathcal{E}}$. Indeed, suppose $\bar{x}^i \notin \alpha_i(w^i)$ for some $i \in S^M_{\mathcal{E}}$. Then, by completeness of α_i we get $w^i \in \mathcal{P}_i(\bar{x}^i)$. Applying the latter inclusion, we prove that the coalition $\{i\}$ can w-improve the allocation $\bar{x}$. Doing so, fix some $\mathcal{V} \in \mathcal{C}(\bar{x})$ and put $S = \{i\}$, $\mathrm{w} = 0$ and $\mathcal{R}' = \mathcal{R}^S$. Note, that due to the inclusions $w^i \in X_i$, $i \in N$, we have that $A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}_{\mathrm{w}}) \neq \emptyset$ (to argue, just mention, that in case the contractual system $\mathcal{U} := \{\mathrm{v_r}\}_{\mathrm{r} \in \mathcal{R} \setminus \mathcal{R}^{\mathrm{S}}}$ happens not to be feasible, we can just break all the rest, nullifying all the contracts from $\mathcal{V} \setminus \mathcal{U}$; the resulting empty system gives us initial endowment allocation $w = (w^1, \ldots, w^n)$, which belongs to $X(N)$ due to the inclusions $w^i \in X_i$ mentioned). Moreover, it is easy to check that $x^i(\mathcal{V}', \mathcal{E}_{\mathrm{w}}) = w^i$ for any $\mathcal{V}' \in A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}_{\mathrm{w}})$ and, hence, according to the Definition **??** and our supposition $w^i \in \mathcal{P}_i(\bar{x}^i)$ we have: coalition $S = \{i\}$ w-improves the allocation $\bar{x}$. But the latter contradicts to the assumption $\bar{x} \in D^M_*(\mathcal{E})$. Due to the contradiction obtained, we get: $\bar{x}^i$ belongs to $\alpha_i(w^i)$ for any $i \in S^M_{\mathcal{E}}$. Since, evidently, $\bar{x}^i$ belongs to X^M_i for any $i \in N$, by inclusions $\bar{x}^i \in \alpha_i(w^i)$ and $\alpha_i(w^i) \cap X^M_i \subseteq \mathrm{int}_M X_i$, $i \in S^M_{\mathcal{E}}$, we get: $\bar{x}^i \in \mathrm{int}_M X_i$ for any $i \in S^M_{\mathcal{E}}$. Thus, the inclusion

$$S^M_{\mathcal{E}} \subseteq N_{\bar{x}}, \tag{5}$$

mentioned in the beginning of our proof, is established.

It is clear, that any superset of a strongly σ-divisible set is a strongly σ-divisible set, as well. Hence, by inclusion (5) we have that $N_{\bar{x}}$ is a strongly σ-divisible set. Consequently, by applying Theorem **??** we get required: $\bar{x} \in W(\mathcal{E})$. □

Below, we demonstrate one of the quite clear implications of Proposition **??**.

Corollary 2

If $\mathcal{E}$ is C_M-regular exchange economy with $w^i \in X_i$ for any $i \in N \setminus S^M_{\mathcal{E}}$, one-element coalition $\{i\}$ belongs to σ for any $i \in N$, and, moreover, for any $i \in N$ there exists a coalition $R \in \sigma_i$ such that

$$(R \setminus \{i\}) \cap S^M_{\mathcal{E}} \neq \emptyset, \tag{6}$$

then every weak totally contractual M-allocation is an equilibrium allocation.

Proof. The main lines of the proof are the same, as in the proof of Proposition **??**. First, we establish that $\bar{x}^i$ belongs to $\alpha_i(w^i)$ for any $\bar{x} \in D^M_*(\mathcal{E})$ and $i \in S^M_{\mathcal{E}}$. Here we apply argumentation from the proof of Proposition **??**, stating that assumption $w^i \in \mathcal{P}_i(\bar{x}^i)$ with $i \in S^M_{\mathcal{E}}$ implies that the one-element coalition $\{i\}$ w-improves allocation $\bar{x}$. Second, as a consequence of inclusions $\bar{x}^i \in \alpha_i(w^i)$, $i \in S^M_{\mathcal{E}}$, we get the insertion $S^M_{\mathcal{E}} \subseteq N_{\bar{x}}$. Finally, by taking for any $i \in N$ coalitions $S = \{i\}$ and R to be equal to that appearing in (6), we have that $S^M_{\mathcal{E}}$ is a strongly σ-divisible subset of N. Referring to Proposition **??**, we complete the proof. □

To present a "global" version of the results, obtained in term of the C_M-regularity, we introduce one more important notion. Put

$$D_*(\mathcal{E}) := \bigcup_{M \in \mathcal{M}} D^M_*(\mathcal{E}), \tag{7}$$

$$S_{\mathcal{E}} := \bigcup_{M \in \mathcal{M}} S^M_{\mathcal{E}}. \tag{8}$$

Definition 9

We call $D_*(\mathcal{E})$, defined by the formula (7), a weak totally contractual core of an economy $\mathcal{E}$.

Theorem 3

If $\mathcal{E}$ is $C_{\mathcal{M}}$-regular, $S^M_{\mathcal{E}}$ is strongly σ-divisible for every $M \in \mathcal{M}$, $\{i\} \in \sigma$ for every $i \in S_{\mathcal{E}}$ (with $S_{\mathcal{E}}$ to be defined by (8)), and $w^i \in X_i$ for any $i \in N \setminus S_{\mathcal{E}}$, then the weak totally contractual core $D_*(\mathcal{E})$ of economy $\mathcal{E}$ is contained in the set $W(\mathcal{E})$ of competitive equilibria of $\mathcal{E}$.

In order to provide an easier verifiable regularity condition than that, given in Definition **??**, we introduce another characteristic of the exchange model under consideration:

$$T_{\mathcal{E}} := \{i \in N \mid \alpha_i \text{ is complete, transitive, } w^i \in X_i, \text{ and } \alpha_i(w^i) \bigcap \check{X}_i \subseteq \text{int}X_i\},$$

where $\check{X}_i := X_i \setminus (w^i + \text{int}\mathbf{R}^l_+), \;\; i \in N$.

Definition 10
An exchange model $\mathcal{E}$ is called C-regular, if $T_{\mathcal{E}} \neq \emptyset$.

It is clear, that a slight modification of the proof of Proposition **??** gives the following analogs of Theorem 3.

Theorem 4
If $T_{\mathcal{E}}$ is strongly σ-divisible, $w^i \in X_i$ for any $i \in N \setminus T_{\mathcal{E}}$, and $\{i\} \in \sigma$ for every $i \in T_{\mathcal{E}}$, then $D_*(\mathcal{E}) \subseteq W(\mathcal{E})$.

Proof. Observe, that in the theorem under consideration C-regularity of $\mathcal{E}$ follows from the fact that it is assumed $T_{\mathcal{E}}$ to be a strongly σ-divisible subset of N and, hence, $T_{\mathcal{E}} \neq \emptyset$. Since, in some sense, the role of $T_{\mathcal{E}}$ in our theorem is quite similar to that, played by $S^M_{\mathcal{E}}$ in Proposition **??**, we just adapt here the main lines of the proof of this proposition.

Let $\bar{x} \in D^M_*(\mathcal{E})$ for some $M \in \mathcal{M}$. Due to the assumption that $T_{\mathcal{E}}$ is strongly σ-divisible, the only thing we need to directly apply Theorem **??** and get the inclusion required $\bar{x} \in W(\mathcal{E})$, is to prove the insertion $T_{\mathcal{E}} \subseteq N_{\bar{x}}$. To this end, according to the assumptions $\alpha_i(w^i) \cap \check{X}_i \subseteq \text{int}X_i$, $i \in T_{\mathcal{E}}$, given, it would be enough to establish inclusions: $\bar{x}^i \in \check{X}_i$, and $\bar{x}^i \in \sigma_i(w^i)$ for any $i \in T_{\mathcal{E}}$. Note, that the first inclusions, $\bar{x}^i \in \check{X}_i$, follows directly from the relations $\bar{x}^i - w^i \in M$ and $M \in \mathcal{M}$, fulfilled for any $i \in N$ (remind, that by Sign-assumption, for any non-zero $x \in M$ with $M \in \mathcal{M}$ we have that $x_k < 0$ for some k). As to the second one, we apply the same argumentation, as in the proof of Proposition **??**, which shows that supposition $w^i \in \mathcal{P}_i(\bar{x}^i)$ for some $i \in T_{\mathcal{E}}$ implies that coalition $\{i\}$ can w-improve the allocation $\bar{x}$. Applying the contradiction obtained (remind, it was supposed that $\bar{x}$ belongs to $D^M_{\mathcal{E}}$) and completeness of the preference relations α_i, $i \in T_{\mathcal{E}}$, we get required: $\bar{x}^i \in \alpha_i(w^i)$ for any $i \in T_{\mathcal{E}}$.

According to the remarks, given above, the proof of our theorem is completed. □

Corollary 3

If $\mathcal{E}$ is C-regular with $\{i\} \in \sigma$, $i \in T_{\mathcal{E}}$, $w^i \in X_i$ for any $i \in N \setminus T_{\mathcal{E}}$, and, besides, for any $i \in N$ there exists a coalition $R \in \sigma_i$ such that

$$(R \setminus \{i\}) \cap T_{\mathcal{E}} \neq \emptyset,$$

then $D_*(\mathcal{E}) \subseteq W(\mathcal{E})$.

We omit the proof of Corollary **??** because it almost literally reproduces the proof of Corollary **??** (with $S_{\mathcal{E}}$ replaced by $T_{\mathcal{E}}$).

As usual, we say that α_i is locally monotonic, if for any $x^i \in \mathrm{Pr}_{X_i} X(N)$ there exists $\delta(x^i) > 0$ such that $x^i + z \in \mathcal{P}_i(x^i)$, whenever $z \in \mathbf{R}^l_+$ and $0 < \|z\| < \delta(x^i)$. Observe, that α_i is always locally monotonic under the following standard assumptions: α_i is strongly monotonic, and $X_i = \mathbf{R}^l_+$.

Taking account that local monotonicity guarantees equilibrium prices to be strictly positive, and summarizing the results, which has been proven above, we obtain the following core equivalence result.

Theorem 5

Suppose a pure exchange economy $\mathcal{E}$ satisfies the assumptions of either one of Theorems **??**, **??**, *or Corollary* **??**. *If α_i is locally monotonic for at least one agent of the economy $\mathcal{E}$, then $D_*(\mathcal{E}) = W(\mathcal{E})$.*

4 Conclusion

Two remarks should be given in the conclusion. First, let us mention once more that w-blocking introduced in the paper is not that much stronger than blocking considered in [?]. Although we have that directly by definition it follows that coalition S can block a feasible contractual M-system $\mathcal{V}$ whenever $\mathcal{V}$ can be w-blocked by S (i.e., w-blocking implies blocking, which means in our terms that the former is stronger than the latter), there are several rather wide classes of pure exchange economies with totally contractual set $D(\mathcal{E})$ and weak totally contractual core $D_*(\mathcal{E})$ to be equal. In fact, due to the inclusion $W^{\mathcal{M}}(\mathcal{E}) \subseteq D(\mathcal{E})$[3] established in [?], one can easily see that any condition, which provides

[3]Here $W^{\mathcal{M}}(\mathcal{E})$ consists of those allocations from $W(\mathcal{E})$ that can be supported by a strictly positive prices.

insertion $D_*(\mathcal{E}) \subseteq W(\mathcal{E})$, guarantees the coincidence of $D(\mathcal{E})$ and $D_*(\mathcal{E})$. In particular, due to Theorem 5 we have the following assertion.

Proposition 3
*Let $\mathcal{E}$ satisfies the assumptions of either one of Theorems **??**, **??**, or Corollary **??**. If α_i is locally monotonic for at least one agent of the economy $\mathcal{E}$, then $D(\mathcal{E}) = D_*(\mathcal{E})$.*

Almost the same argumentation, based on Corollary **??** and Proposition **??**, yields the "individual" version of Proposition **??** (the only alteration we need in the corresponding proof is the replacement of $W(\mathcal{E})$, $D(\mathcal{E})$, and $D_*(\mathcal{E})$ by $W^M(\mathcal{E})$, $D^M(\mathcal{E})$, and $D_*^M(\mathcal{E})$ with $W^M(\mathcal{E}) := W(\mathcal{E}) \bigcap X_{\mathcal{E}}^M,\ M \in \mathcal{M}$).

Proposition 4
If $\mathcal{E}$ is C_M-regular exchange economy with $w^i \in X_i$ for any $i \in N \setminus S_{\mathcal{E}}^M$, one-element coalition $\{i\}$ belongs to σ for any $i \in N$, and, moreover, for any $i \in N$ there exists a coalition $R \in \sigma_i$ such that

$$(R \setminus \{i\}) \cap S_{\mathcal{E}}^M \neq \emptyset,$$

then $D^M(\mathcal{E}) = D_*^M(\mathcal{E})$.

Proposition 5
Let $\mathcal{E}$ be a C_M-regular pure exchange model with w^i belonging to X_i for any $i \in N \setminus S_{\mathcal{E}}^M$. If $S_{\mathcal{E}}^M$ is a strongly σ-divisible subset of N, and one-element coalition $\{i\}$ belongs to σ for any $i \in S_{\mathcal{E}}^M$, then $D^M(\mathcal{E}) = D_*^M(\mathcal{E})$.

Second (and the final) remark concerns the importance of the strong σ-divisibility in Theorem **??**. To demonstrate the main assumption in the core equivalence theorem is relevant, two examples of pure exchange economies having unblocked (in the sense of Definition **??**) allocations with no supporting equilibrium prices are given. The most interesting is the second example with no equilibrium allocations and nonempty weak totally contractual core, exhibiting that the weak totally contractual allocation may be chosen as a compromise solution in case the classical market mechanism doesn't work.

Example 1 [$W(\mathcal{E}_1) \neq \emptyset,\ D_*(\mathcal{E}_1) \setminus W(\mathcal{E}_1) \neq \emptyset$].

Let $\mathcal{E}_1$ be pure exchange economy defined by the following parameters:

$$N = \{1, 2, 3, 4, 5\}, \quad \sigma = \{\{1, 2\}, \{2, 3\}, \{3, 4, 5\}\};$$

$$X_i = \mathbf{R}^2_+,\ i \in N, \quad w^1 = (3,0),\ w^2 = (6,0),\ w^3 = (6,1),\ w^4 = (9,1),\ w^5 = (6,3);$$

$$u_i(x_1, x_2) = \begin{cases} x_1 + 4x_2 & , \quad i = 1, 2, \\ 2x_1 + 3x_2 & , \quad i = 3, 4, 5. \end{cases}$$

Put

$$M = \{x \in \mathbf{R}^2 \mid x_1 + 3x_2 = 0\}$$

and consider the allocation $\bar{x} = (\bar{x}^i)_{i \in N} \in X^M_{\mathcal{E}_1}$, defined as follows:

$$\bar{x}^1 = (0,1),\ \bar{x}^2 = (0,2),\ \bar{x}^3 = (9,0),\ \bar{x}^4 = (12,0),\ \bar{x}^5 = (9,2).$$

Note, that directly from Definition **??** it follows that any strongly σ-divisible coalition contains at least two participants. Hence, due to $N_{\bar{x}} = \{5\}$ we get that in our case $N_{\bar{x}}$ is not a strongly σ-divisible subset of N. Further, by analyzing restrictions of utility functions u_i to the corresponding intervals X^M_i, one can easily check that for any $i \in N \setminus \{5\}$ the bundle $\bar{x}^i$ is a maximal element on X^M_i w.r.t. the individual preference relation, generated by u_i. Consequently, there is only one coalition $S = \{3,4,5\} \in \sigma$ that may be able to w-improve some contractual system from $\mathcal{C}(\bar{x})$. Suppose, it really w-improves $\bar{x}$. Since the functions $u_i,\ i = 3,4,5$, are strictly increasing w.r.t x_1 on the corresponding sets X^M_i, the definition of w-blocking implies that there exists $x \in X^M_{\mathcal{E}}$ such that $x^3_1 \geq \bar{x}^3_1,\ x^4_1 \geq \bar{x}^4_1$ and $x^5_1 > \bar{x}^5_1$. Consequently, $\sum_{i \in S} x^i_1 > 30$, which contradicts to the relations $\sum_{i \in S} x^i_1 \leq \sum_{i \in N} x^i_1 = \sum_{i \in N} w^i_1 = 30$. Therefore, we get $\bar{x} \in D^M_*(\mathcal{E})$.

At the same time, one can easily check, that the set $W(\mathcal{E}_1)$ of competitive equilibria of the economy $\mathcal{E}_1$ consists of the only allocation $\widehat{x} = (\widehat{x}^i)_{i \in N} \in X(N)$, given by the data:

$$\widehat{x}^1 = (0, 5/3),\ \widehat{x}^2 = (0, 10/3),\ \widehat{x}^3 = (39/5, 0),\ \widehat{x}^4 = (54/5, 0),\ \widehat{x}^5 = (57/5, 0),$$

with the (normed) supporting equilibrium price vector $\widehat{p}$, given by the equality: $\widehat{p} = (5/14, 9/14)$.

Hence, we have got the required: $W(\mathcal{E}_1) \neq \emptyset$ and $D_*(\mathcal{E}_1) \setminus W(\mathcal{E}_1) \neq \emptyset$.

Example 2 [$W(\mathcal{E}_2) = \emptyset,\ D_*(\mathcal{E}_2) \neq \emptyset$].

Consider one more pure exchange economy $\mathcal{E}_2$, given by the data:

$$N = \{1,2,3\}, \quad \sigma = \{\{1,2\},\{2,3\}\};$$

$$X_i = \mathbf{R}^2_+,\ i \in N; \quad w^1 = (0,1),\ w^2 = (6,1),\ w^3 = (6,3);$$

$$u_i(x_1, x_2) = \begin{cases} x_1 + 4x_2 & , \quad i = 1, \\ x_1 & , \quad i = 2, 3. \end{cases}$$

Put, as in the previous example, $M = \{x \in \mathbf{R}^2 \mid x_1 + 3x_2 = 0\}$, and pick the balanced allocation $\bar{x} = (\bar{x}^i)_{i\in N} \in M_{\mathcal{E}_2}(N)$ of the economy $\mathcal{E}_2$, defined as follows:

$$\bar{x}^1 = (0,1),\ \bar{x}^2 = (9,0),\ \bar{x}^3 = (3,4).$$

Due to the same reason, as in the first example, $N_{\bar{x}} = \{3\}$ is not a strongly σ-divisible set in $\mathcal{E}_2$. Further, suppose, $\bar{x} \notin D_*^M(\mathcal{E}_2)$. Doing, like in the previous example, one can easily check that the bundles $\bar{x}^1$ and $\bar{x}^2$ are maximal elements in X_1^M and X_2^M, respectively (w.r.t. the individual preferences, defined by the corresponding utility functions $u_1$and u_2). Consequently, the only coalition $S \in \sigma$, which might be w-improving for some contractual system from $\mathcal{C}(\bar{x})$, is $S = \{2,3\}$. Suppose, that $S = \{2,3\}$ really w-improves some $\mathcal{V} \in \mathcal{C}(\bar{x})$. By definition of w- improvement and construction of $\bar{x}$ and utility functions u_2, u_3, it means that there exists $x = (x^i)_{i\in N} \in X_{\mathcal{E}_2}^M$ such that $x_1^1 \geq 0$, $x_1^2 \geq 9$, and $x_1^3 > 3$. But these inequalities apparently contradict to the relations

$$\sum_{i\in N} x_1^i = \sum_{i\in N} w_1^i = 12,$$

following from the obvious inclusion $X_{\mathcal{E}_2}^M \subseteq X(N)$.

So, our economy $\mathcal{E}_2$ possesses at least one weak totally contractual M-allocation. As to the competitive equilibrium, from the famous "irreducibility" criterion for the linear exchange economies, proposed by D. Gale [**?**], it follows immediately that $W(\mathcal{E}_2) = \emptyset$ (the latter fact can be verified directly, as well, just by applying the definition of equilibrium and taking account that coalition $S = \{2,3\}$ is too "self-sufficient"). Hence, we get required: $W(\mathcal{E}_2) = \emptyset$ and $D_*(\mathcal{E}_2) \neq \emptyset$.

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