Market imperfections and endogenous fluctuations: a general approach

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February 21, 2008

Abstract

We provide a methodology to study the role of distortions and market failures on endogenous fluctuations. We extend the well-known Woodford (1986) model to account for market distortions, introducing general specifications for three crucial functions: real rental cost of capital, real wage and workers offer curve. The elasticities of these three functions have a key role on local dynamics and, using them, we are able to identify the several parameters' configurations under which local indeterminacy and bifurcations occur. Most of the specific market imperfections considered in the related literature become particular cases of our general framework, and by comparing them we show that several types of market distortions are equivalent in terms of the local dynamics, sharing therefore the same indeterminacy mechanisms. We further provide examples of distortions leading to new configurations. We also found that indeterminacy is possible with arbitrarily small levels of distortions in real wage and/or in workers offer curve, but it requires extremely high values for the elasticityof substitution between inputs and for the elasticity of the labor supply curve.

JEL classification: C62, E32.

Keywords: Indeterminacy, endogenous fluctuations, market imperfections, externalities, imperfect competition, taxation.

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1 Introduction

Using dynamic general equilibrium models, several papers have stressed the role of specific market distortions, like externalities, imperfectly competitive markets or balanced-budget policy rules, on the occurrence of local indeterminacy, bifurcations and endogenous cycles driven by expectations.[1] Indeed, market distortions, changing the behavior of agents or introducing a gap between private and aggregate supplies/demands, can revert the typical result of uniqueness and determinacy, and engender mechanisms through which multiplicity and indeterminacy of equilibria arise.

In this paper, our purpose is to provide a general study of the role of distortions and market failures on local endogenous business cycles, without specifying a priori their source. To provide an answer to this program, we cannot consider a particular model based on some specific micro-foundations, but we rather have to consider a model as general as possible, suitable to encompass most of the distortions. Introducing general specifications for product, capital and labor market imperfections, we will be able to identify the main channels through which indeterminacy occurs, and compare the role of different market imperfections on the emergence of indeterminacy and endogenous fluctuations.

Although our methodology can be applied to any dynamic general equilibrium model, the framework used in this paper is based on the well-known Woodford (1986) model, which was later on developed by Grandmont et al. (1998) to take into account inputs' substitution. These two models considered a perfectly competitive economy with heterogeneous agents, workers, who face finance constraints, and capitalists. We develop this set up, introducing imperfections in the three markets (product, capital, labor), hence departing from the perfectly competitive case in several directions. First, we assume that the real prices of production factors (real wage, real interest rate) may no longer equal the corresponding marginal productivities at the firm level, being instead defined by more general expressions able to encompass imperfect competition, taxation or productive externalities. Second, the intertemporal choice of workers between future consumption and leisure is defined in a more complex way: the usual offer curve is replaced by a more general function, the generalized offer curve, and private consumption by a more general concept, effective consumption. The specification we use will admit as particular cases, among others, models with externalities in

[1] In section 5, we refer to several of these works. For an overview and better understanding of the relation between indeterminacy, bifurcations and (endogenous) fluctuations driven by volatile self fulfilling expectations, see, for instance, the survey of Benhabib and R. Farmer (1999) and Grandmont et al. (1998).

preferences due to aggregate consumption, aggregate leisure or government spending, and labor market imperfections due to the presence of unions or efficiency wages.

In order to analyze the occurrence of indeterminacy and bifurcations we study the local stability properties of the equilibrium dynamics. As it is well known, what matters for those properties are the elasticities of the equilibrium dynamic equations.[2] Therefore, the methodology used in this work basically proposes a general formulation for these elasticities, that allows us to represent most of the distortions affecting product, capital and labor markets as well as the standard perfectly competitive case. More specifically, we define general expressions for the elasticities of three crucial functions that characterize our two dimensional equilibrium dynamic system: the real interest rate, the real wage or equivalently effective consumption per unit of labor, and the generalized offer curve. Hence, our model admits as particular cases several forms of market distortions already addressed in the literature, but also covers other configurations, leading to new results on indeterminacy. Moreover, it allows us to identify which specific market imperfections share the same indeterminacy mechanisms. Our general framework can also be used to obtain an overview on the indeterminacy results in the more complex cases of several simultaneous specific distortions.

We analyze the local dynamic properties of equilibrium in the presence of market imperfections, focusing on not too weak values of the elasticity of capital-labor substitution. Indeed, weak values of this elasticity are not empirically relevant.[3] Also, in such a case, indeterminacy and endogenous cycles already occur in the perfectly competitive Woodford (1986)-Grandmont et al. (1998) models, whereas this is not possible when the wage bill is increasing in labor, which requires a not too low substitution between inputs. Moreover, in contrast to several existing works, we do not restrict our analysis to the case of an infinitely elastic individual labor supply. Indeed, as we shall see, even if under some types of distortions indeterminacy prevails for arbitrarily high values of the labor supply elasticity, it will become clear that, under other types of distortions, this result no longer holds. Therefore, by imposing an infinitely elastic labor supply, one may fail in accounting for some relevant phenomena and obtain a wrong idea of the implications of certain types of distortions on the occurrence of indeterminacy.

When capital and labor are not weak substitutes, we show that, in contrast to the perfectly competitive economy, indeterminacy and endogenous

[2] Indeed, papers where specific forms of market imperfections were considered, obtained new results because those imperfections introduced a modification of these elasticities with respect to the perfectly competitive case.

[3] See Hamermesh

fluctuations can emerge when there are market imperfections. Indeterminacy requires both, a sufficiently high elasticity of capital-labor substitution in production and a sufficiently high elasticity of the individual labor supply.[4] However, the lower bounds for these elasticities are not the same for all types and degrees of distortions. We also show that, depending on the market distortions considered, and as suggested above, indeterminacy may be ruled out if the elasticity of the individual labor supply becomes arbitrarily large.

Considering first distortions that do not affect the generalized offer curve, we show that, when capital and labor are not weak substitutes, indeterminacy requires a response of effective consumption to a variation of labor greater than the response of capital income to a variation of capital. Moreover, market failures modifying effective consumption seem to be more relevant for indeterminacy than those affecting the real interest rate, in particular if the distortions in effective consumption depend positively on capital and labor. It is also worth noticing that endogenous fluctuations cannot occur, when inputs are sufficiently substitutes, if distortions on the real interest rate and on the effective consumption depend negatively on capital and labor, whereas on the contrary, indeterminacy may emerge when these distortions depend positively on both capital and labor.

Let us now discuss our results when market imperfections also affect the generalized offer curve. We show that distortions modifying the generalized offer curve play a crucial role on local indeterminacy, since other configurations, in terms of the required bounds for the elasticity of the individual labor supply and capital-labor substitution, can then become relevant. Considering, for simplicity, that imperfections do not affect the real interest rate, we then have only two types of distortions (one on the effective consumption and another one on the generalized offer curve), both modifying only the equilibrium condition of workers' intertemporal arbitrage. We prove that, when inputs are not weak substitutes, indeterminacy requires that the global degree of distortions modifying effective consumption has to be larger than the one modifying the generalized offer curve. Hence, while distortions on real interest rate do not seem to play a major role on the occurrence of indeterminacy when inputs are sufficiently substitutes, distortions on the generalized offer curve that negatively depend on capital and labor and distortions on the effective consumption that depend positively on capital and labor seem to help the fulfillment of requirements for the possible occurrence of indeterminacy.

[4] These results were already obtained in other works that considered some specific forms of distortions (See the papers referred in examples of Section 5). Here we we generalize them to any form of specific distortion that fitts in our general formulation.

In a second step, we apply all these results to several examples. We start with examples where market distortions do not affect the generalized offer curve, as imperfect competition on the product market (Dos Santos Ferreira and Lloyd-Braga (2005), Jacobsen (1998), Kuhry (2001), Seegmuller (2007a,b), Weder (2000a)), externalities in production (Barinci and Chéron (2001), Benhabib and Farmer (1994), Cazzavillan (2001), Cazzavillan et al. (1998)) and consumption preferences (Alonso-Carrera et al. (2005), Gali (1994), Ljungqvist and Uhlig (2000), Weder (2000b)), balanced-budget rules and variable tax rates (Dromel and Pintus (2004), Giannitsarou (2005), Guo and Lansing (1998), Gokan (2005), Lloyd-Braga et al. (2006), Pintus (2003), Schmitt-Grohé and Uribe (1997)). We conclude that with negative productive externalities or capital taxation, the steady state is never indeterminate. We also prove that, in terms of local dynamics, labor income and consumption taxation are equivalent to consumption externalities. Moreover, we show that many models with product market imperfections characterized by business formation, mark-up variability and taste for variety can be seen as particular cases of the model with positive externalities in the production. Hence, even if the economic interpretation of all these examples are different, they share a common channel through which indeterminacy occurs. Indeterminacy requires a same lower bound for the elasticity of capital-labor substitution, and this lower bound is only below unity for relatively important distortions, i.e. sufficiently high positive externalities in production and in consumption preferences, rates of taxation large enough or sufficiently decreasing with their tax base.

Once examples with distortions affecting the offer curve are considered, more of our results may be illustrated. These examples cover labor market imperfections, like unemployment benefits and efficiency wages (Coimbra (1999), Nakajima (2006), Grandmont (2006)) or unions (Lloyd-Braga and Modesto (2006), Dufourt et al. (2006)), and also externalities in preferences, due to aggregate labor or public spending for instance (Benhabib and Farmer (2000), Weder (2004)). The conditions for indeterminacy are not the same in all these examples. However, although indeterminacy still requires a lower bound for the elasticity of capital-labor substitution, it can emerge under a Cobb-Douglas technology with plausible degrees of distortions in most of the examples. Finally, comparing these different examples, we note that, from a local dynamic point of view, the model with unemployment benefits and efficiency wages can be seen as a particular case of the model where the disutility of labor is negatively affected by labor externalities.

The rest of the paper is organized as follows. In the next section, we present the model and we define the respective perfect foresight equilibria. In Section 3, we begin the analysis of the local dynamic properties of the model, using the geometrical method developed by Grandmont et al. (1998). Our main results on the emergence of indeterminacy and bifurcations are presented and discussed in Section 4. In Section 5 we present examples with specific market distortions that are particular cases of our general framework, and analyze them using it. In Section 6 we provide concluding remarks. Many proofs and technical details are given in the Appendix.

2 The Model

The model developed in this paper extends the Woodford (1986) framework to take into account market imperfections. To underline the implications of imperfections on the equilibrium, we begin by a brief exposition of the one-sector Woodford model with perfect competition.[5] We present our framework, developed in order to deal with market imperfections in a second step.

According to the perfectly competitive economy studied by Woodford (1986) and Grandmont et al. (1998), in each period $t \in N^*$, a final good is produced under a constant returns to scale technology $AF(K_{t-1}, L_t)$, where $A > 0$ is a scale parameter, F is a strictly increasing function, concave and homogeneous of degree one in capital, $K > 0$, and labor, $L > 0$. From profit maximization, the real interest rate ρ_t and the real wage ω_t are respectively equal to the marginal productivities of capital and labor , i.e. $\rho_t = AF_K(K_{t-1}, L_t) \equiv A\rho(K_{t-1}/L_t)$ and $\omega_t = AF_L(K_{t-1}, L_t) \equiv A\omega(K_{t-1}/L_t)$.

There are two types of infinite-lived consumers, workers and capitalists. Both consume the final good, and can save through two assets, money and productive capital. However, only workers supply labor and they are more impatient than capitalists. Moreover, workers face a finance constraint which prevents them from borrowing against their wage earnings. Focusing on equilibria where the finance constraint is binding and capital is the asset with the greatest return, we obtain as a result that only workers hold money (they save all wage income in money), and capitalists hold the entire stock of capital. Therefore, the program that a representative worker solves each period t, can be summarized as:

$$Max\ U\left(C^w_{t+1}/B\right) - V(L_t) \tag{1}$$

$$s.t. P_{t+1}C_{t+1} = w_t L_t \tag{2}$$

where P_t is the price of the final good, w the nominal wage, $V(L)$ the disutility of labor in $L \in [0, L^*]$, $C^w_{t+1} \geq 0$ the worker's consumption of next

[5] For more details, one can refer to Grandmont, Pintus and de Vilder (1998) and Woodford (1986).

period, $B > 0$ a scaling parameter, and $U(x)$ the utility of consumption, with $-xU''(x)/U'(x) < 1$, (implying that consumption and leisure are gross substitutes), where $x \equiv C^w_{t+1}/B$.[6] From the first order condition, one obtains the intertemporal arbitrage condition for workers:

$$u\left(C^w_{t+1}/B\right) = v(L_t), \tag{3}$$

where C_{t+1} is given by (2) and $u(x) = xU'(x)$ and $v(L) = LV'(L)$. Since consumption and leisure are gross substitutes, from (3) we can obtain a function $\gamma \equiv u^{-1} \circ v$, the offer curve, such that $C^w_{t+1}/B = \gamma(L_t)$.

The representative capitalist maximizes the log-linear lifetime utility function $\sum_{t=1}^{\infty} \beta^t \ln C^c_t$, where C^c_t represents his consumption at period t and $\beta \in (0,1)$ his subjective discount factor. Since he does not save through money balances, he faces the budget constraint $C^c_t + K_t = R_t K_{t-1}$, where $R_t \equiv 1 - \delta + r_t/P_t$ is the real interest factor, r_t the nominal interest rate and $\delta \in (0,1)$ the depreciation rate of capital. Solving the capitalist's problem we obtain the capital accumulation equation $K_t = \beta R_t K_{t-1}$.

In each period, equilibrium on the labor market requires $w_t/P_t = \omega_t$ and equilibrium in the capital market requires that $r_t/P_t = \rho_t$. Let $M > 0$ be the constant money supply. Since workers save wage income in money, equilibrium in the money market requires $M = w_t L_t$ for all t, which using (2) leads to $C^w_{t+1} = \omega_{t+1} L_{t+1}$.[7] This, together with the worker's offer curve and the solution of capitalist's problem, motivates the following definition:

Definition 1 *A perfect foresight intertemporal equilibrium of the economy with perfect competition is a sequence* $(K_{t-1}, L_t) \in \mathbb{R}^2_{++}$, $t = 1, 2, ..., \infty$, *that satisfies*

$$K_t = \beta \left[1 - \delta + \rho_t\right] K_{t-1} \tag{4}$$

$$(1/B)\omega_{t+1} L_{t+1} = \gamma\left(L_t\right) \tag{5}$$

where $\rho_t = AF_K(K_{t-1}, L_t) = A\rho(K_{t-1}/L_t)$ *and* $\omega_t = AF_L(K_{t-1}, L_t) = A\omega(K_{t-1}/L_t)$.

The two dimensional dynamic system (4)-(5) is composed by the equilibrium capital accumulation equation determined by savings of capitalists and

[6] It is assumed that $U\left(C^w_{t+1}/B\right)$ is a continuous function of $C^w_{t+1} \geq 0$, and C^r, with r high enough, $U' > 0, U'' \leq 0$ for $C^w_{t+1} > 0$. Also, $V(l)$ is a continuous function for $[0, L^*]$, and C^r, with r high enough, $V' > 0, V'' \geq 0$ for $(0, L^*)$. We also assume that $lim_{L \to L^*} V'(L) = +\infty$, with L^* (the worker's endowment) possibly infinite.

[7] The good market equilibrium is ensured by Walras law.

the workers' intertemporal arbitrage condition between future consumption and leisure.[8]

We now present our general framework with market imperfections, underlying the main differences with respect to the perfectly competitive economy. Later on, we present examples of specific distortions that are particular cases of our general framework, and we will see that many models with market imperfections will provide microeconomic foundations for our general formulation.

We propose the following definition for the intertemporal equilibrium of an economy with market imperfections:

Definition 2 *A perfect foresight intertemporal equilibrium of the economy with market imperfections is a sequence* $(K_{t-1}, L_t) \in \mathbb{R}^2_{++}$, $t = 1, 2, ..., \infty$, *that satisfies*

$$K_t = \beta \left[1 - \delta + \varrho_t\right] K_{t-1} \tag{6}$$

$$(1/B)\Omega_{t+1}L_{t+1} = \Gamma_t \tag{7}$$

where $\varrho_t \equiv A\varrho(K_{t-1}, L_t)$, $\Omega_t \equiv A\Omega(K_{t-1}, L_t)$ *and* $\Gamma_t \equiv \Gamma(K_{t-1}, L_t)$.

We further assume that $\varrho(K, L)$, $\Omega(K, L)$ and $\Gamma(K, L)$ satisfy:

Assumption 1 *The functions* $\varrho(K, L)$, $\Omega(K, L)$ *and* $\Gamma(K, L)$ *are positively valued and differentiable as many times as needed for* $(K, L) \in \mathbb{R}^2_{++}$.

As in the case of the perfectly competitive economy, the dynamics of this economy are governed by a two dimensional system. Equation (6) describes capital accumulation and equation (7) the intertemporal choice of workers, where ϱ_t is the real interest rate relevant to capitalists' decisions, Γ_t a generalized offer curve, and $\Omega_t L_t$ effective consumption. Note that we recover the perfectly competitive case (Definition 1) for $\varrho(K, L) = F_K(K, L)$, $\Omega(K, L) = F_L(K, L)$ and $\Gamma(K, L) = \gamma(L)$. However, with market distortions ϱ_t may not coincide with the perfectly competitive marginal productivity of capital, $\Omega_t L_t$ may not coincide with the perfectly competitive wage bill and Γ_t may differ from the private offer curve $\gamma(L_t)$ used in Definition 1. Indeed, we assume that ϱ_t, Ω_t and Γ_t are given by general functions of K_{t-1} and L_t, without choosing a particular specification for them, so that they encompass a large class of market imperfections. In many models characterized by

[8]Note that capital is a predetermined variable, whose value is determined by past savings of capitalists (see (4), obtained from the solution of capitalist's problem), while labor is a non predetermined variable whose value depends on expectations for future consumption (see (5), obtained from the worker's offer curve).

market imperfections, the real interest rate and/or the real wage relevant to the consumers' decisions are no longer equal to the marginal productivities of capital and labor used at the firm level. This will happen for example in the cases of productive externalities, imperfect competition in the product market or with consumption, labor or capital taxation (see the examples in Section 5.1), which introduces a difference in the functions ϱ, and/or Ω, with respect to the one obtained in the perfectly competitive economy. The other differences introduced affect only the intertemporal choice of workers. First, with some market imperfections, like in the case of consumption or government spending externalities influencing utility from consumption, the relevant intertemporal choice of workers becomes a choice between future effective consumption[9] (affecting the function Ω, that no longer coincides with the wage) and leisure. Second, in the presence of some labor market imperfections, such as efficiency wages or unions, or with leisure externalities, the private offer curve derived for the perfectly competitive economy is no longer valid at the social level, affecting the function Γ (see the examples provided in Section 5.2).

Since we will focus on local dynamics, market imperfections will play a role on indeterminacy and on the occurrence of endogenous cycles because they modify the elasticities of these three functions with respect to the perfect competition case. Assuming that a normalized steady state $(K^*, L^*) = (1, 1)$ always exists, as it is established in the Appendix through a scaling procedure, we denote by $\varepsilon_{\varrho,K}$, $\varepsilon_{\varrho,L}$, $\varepsilon_{\Omega,K}$, $\varepsilon_{\Omega,L}$, $\varepsilon_{\Gamma,K}$ and $\varepsilon_{\Gamma,L}$ the elasticities of $\varrho(K, L)$, $\Omega(K, L)$ and $\Gamma(K, L)$ with respect to K and L evaluated at this steady state. In the perfectly competitive economy, since $\varrho(K, L) = F_K(K, L)$, $\Omega(K, L) = F_L(K, L)$ and $\Gamma(K, L) = \gamma(L)$, these elasticities are given by:

$$\begin{aligned} &\varepsilon_{\varrho,K} = -\frac{1-s}{\sigma} \ , \ \varepsilon_{\varrho,L} = \frac{1-s}{\sigma} \\ &\varepsilon_{\Omega,K} = \frac{s}{\sigma} \ , \ \varepsilon_{\Omega,L} = -\frac{s}{\sigma} \\ &\varepsilon_{\Gamma,K} = 0 \ , \ \varepsilon_{\Gamma,L} = \varepsilon_{\gamma}, \end{aligned} \tag{8}$$

where $\varepsilon_{\gamma} - 1 \geqslant 0$ represents the inverse of the elasticity of labor supply at the individual level, $s \in (0, 1)$ the elasticity of the production function with respect to capital at the individual firm level, and $\sigma > 0$ the elasticity of capital-labor substitution at the individual firm level, all evaluated at the steady state. For more details see Grandmont et al. (1998). When

[9]By effective consumption we mean the argument of the utility for consumption, which in the presence of consumption or public spending externalities on preferences will also include them.

there are market imperfections these six elasticities have more complicated expressions. In order to be able to take into account all the different types of market distortions referred before, we assume that these elasticities are modified with respect to the perfectly competitive case in the following way:

Assumption 2

$$\varepsilon_{\varrho,K} = \alpha_{K,K} + \frac{\beta_{K,K}}{\sigma} - \frac{1-s}{\sigma}\ ,\ \varepsilon_{\varrho,L} = \alpha_{K,L} + \frac{\beta_{K,L}}{\sigma} + \frac{1-s}{\sigma}$$
$$\varepsilon_{\Omega,K} = \alpha_{L,K} + \frac{\beta_{L,K}}{\sigma} + \frac{s}{\sigma}\ ,\ \varepsilon_{\Omega,L} = \alpha_{L,L} + \frac{\beta_{L,L}}{\sigma} - \frac{s}{\sigma}$$
$$\varepsilon_{\Gamma,K} = \alpha_{\Gamma,K} + \frac{\beta_{\Gamma,K}}{\sigma}\ ,\ \varepsilon_{\Gamma,L} = \alpha_{\Gamma,L} + \frac{\beta_{\Gamma,L}}{\sigma} + \varepsilon_{\gamma}$$

with $\alpha_{i,j} \in \mathbb{R}$ *and* $\beta_{i,j} \in \mathbb{R}$ *for* $i = K,\ L,\ \Gamma$ *and* $j = K, L$.

Note first that, when $\alpha_{i,j} = \beta_{i,j} = 0$, with $i = K, L, \Gamma$ and $j = K, L$, we recover the elasticities under perfect competition (8), so that, in each equality, $\alpha_{i,j} + \beta_{i,j}/\sigma$ summarizes the role of market imperfections. Remark that market imperfections add two new components to the different elasticities: a first one through $\alpha_{i,j}$ which corresponds to the level of market imperfections when inputs are high substitutes in production (σ high) and a second one through $\beta_{i,j}$ which provides a measure of the importance of market imperfections when inputs are weak substitutes in production (σ weak).

Let us also remark that, as we shall see, this specification for the elasticities allow us to focus separately on only either labor market distortions (which influence only the Ω and Γ functions), or capital market distortions (which influence only the ϱ function) or output market distortions (which affect only the Ω and ϱ functions).

In the rest of the paper, we consider that $\left|\beta_{i,j}\right| < s < 1/2$, for all $i = K, L, \Gamma$ and $j = K, L$. This assumption covers the most interesting cases presented in the literature and is quite convenient from a technical point of view, simplifying our analysis. This assumption is also a plausible assumption, since empirical works usually show that market imperfections are not too big.

The other assumptions on the parameters that we will consider in this paper, and that we present below, are also supported by what we observe in the real world. Empirical studies tell us that the wage bill is increasing in labor. Without market imperfections this means that consumption is increasing in labor. By analogy (and continuity) we choose to extend this assumption to the case of imperfect competition. Therefore we assume that effective consumption is increasing in labor, i.e. that $1 + \epsilon_{\Omega,l} > 0$. Using

Assumption 2, we can see that this implies $\alpha_{LL} > -1$ and $\sigma > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$. Note that this last condition covers the most empirical relevant situations and collapses into $\sigma > s$ in the absence of distortions.[10] We also know from empirical works that capital income (RK) is increasing in capital. Hence, again by analogy and continuity, we keep this assumption in the case of market imperfections, requiring that $1 + \theta\epsilon_{\varrho,K} > 0$, where $\theta \equiv 1 - \beta(1-\delta) \in (0,1)$. Therefore, using Assumption 2, this implies $\alpha_{K,K} > -1/\theta$ and $\sigma > \frac{\theta(1-s-\beta_{KK})}{1+\theta\alpha_{KK}}$. We further assume that $s > \theta(1-s)$ and $s < 1/2$, which are usual assumptions in Woodford economies.[11] We also consider that $\frac{s-\beta_{LL}}{1+\alpha_{LL}} > \frac{\theta(1-s-\beta_{KK})}{1+\theta\alpha_{KK}}$. This is not a restrictive assumption, since empirical values of θ are rather small[12], and it becomes $s > \theta(1-s)$ in the absence of market imperfections.

We summarize all the conditions discussed above in the following Assumption:

Assumption 3 *1.* $0 < s < 1/2$ *and* $0 < \theta(1-s) < s$.

2. $|\beta_{i,j}| < s$*, for all* $i = K, L, \Gamma$ *and* $j = K, L$*;*

$\alpha_{LL} > -1$, $\alpha_{K,K} > -1/\theta$ *and* $\frac{s-\beta_{LL}}{1+\alpha_{LL}} > \frac{\theta(1-s-\beta_{KK})}{1+\theta\alpha_{KK}}$.

3. $\sigma > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$.

3 Geometrical Method

To study the role of market imperfections on local indeterminacy and the occurrence of endogenous cycles, we analyze the local stability properties of the dynamic system purposed in Definition 2. In order to do that, we first linearize the system (6) and (7) around the steady state $(K^*, L^*) = (1,1)$.[13] Then, we deduce the trace T and the determinant D of the associated Jacobian matrix, which correspond respectively to the product and the sum of the roots of the characteristic polynomial $P(\lambda) \equiv \lambda^2 - T\lambda + D = 0$. The values taken by T and D depend on the elasticities of the functions ϱ, Ω and Γ, evaluated at the steady state, which are defined in terms of the relevant parameters of the model within Assumption 2.[14]

[10] Empirical studies point to values of σ greater than 0.4. See Hamermesh () and Duffy and Papageorgiou (2000).

[11] See, for instance, Grandmont and al. (1998), Cazzavillan et. al. (1998), Barinci and Chéron (2001), Lloyd-Braga and Modesto (2006), Dufourt et al. (2006)

[12] Under usual parametrization, θ is around 0.0?.

[13] The existence of such a steady state is established in the Appendix.

[14] Some details are given in the Appendix.

To simplify our task, we assume in the rest of the paper that:

Assumption 4

(i) $(\beta_{L,K}+s)=\frac{(1-s-\beta_{K,K})(s-\beta_{L,L})}{(1-s+\beta_{K,L})}$;

(ii) $\beta_{\Gamma,K}=-\beta_{\Gamma,L}\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}}$.

This is equivalent to impose that the numerator and the denominator of T and D linearly depend on the elasticity of capital-labor substitution σ. This assumption is satisfied in models with no distortion and by all the works considered in the literature and presented here as applications. Under Assumption 4, T and D can then be written as:

$$\begin{aligned}
&T=\frac{\sigma}{\sigma(1+\alpha_{L,L})-(s-\beta_{L,L})}(\varepsilon_{\gamma}-1)+T_1 \text{ with}\\
&T_1=1+\{\sigma[1+\alpha_{\Gamma,L}+\theta(\alpha_{K,K}(1+\alpha_{L,L})-\alpha_{L,K}\alpha_{K,L})]+\beta_{\Gamma,L}\\
&-\theta[(1+\alpha_{L,L})(1-s-\beta_{K,K})+\alpha_{K,K}(s-\beta_{L,L})+\alpha_{L,K}(1-s+\beta_{K,L})\\
&+\alpha_{K,L}\frac{(1-s-\beta_{K,K})(s-\beta_{L,L})}{1-s+\beta_{K,L}}]\}/\{\sigma(1+\alpha_{L,L})-(s-\beta_{L,L})\}
\end{aligned} \tag{9}$$

$$\begin{aligned}
&D=\frac{\sigma(1+\theta\alpha_{K,K})-\theta(1-s-\beta_{K,K})}{\sigma(1+\alpha_{L,L})-(s-\beta_{L,L})}(\varepsilon_{\gamma}-1)+D_1 \text{ with}\\
&D_1=\{\sigma[(1+\theta\alpha_{K,K})(1+\alpha_{\Gamma,L})-\theta\alpha_{\Gamma,K}\alpha_{K,L}]+\beta_{\Gamma,L}(1+\theta\alpha_{K,K})\\
&-\theta[(1-s-\beta_{K,K})(1+\alpha_{\Gamma,L})+\alpha_{\Gamma,K}(1-s+\beta_{K,L})\\
&-\alpha_{K,L}\beta_{\Gamma,L}\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}}]\}/\{\sigma(1+\alpha_{L,L})-(s-\beta_{L,L})\}
\end{aligned} \tag{10}$$

The stability properties, defined by the location of the eigenvalues with respect to the unit circle, depend on the values taken by T and D. As in Grandmont, Pintus and de Vilder (1998), we proceed by analyzing the variations of T and D in the plane (T,D), as some parameters of the model are made to continuously vary in their admissible range (see Figures 1-6). On the line (AB), one eigenvalue is equal to -1, i.e. $P(-1)\equiv 1+T+D=0$. On the line (AC), one eigenvalue is equal to 1, i.e. $P(1)\equiv 1-T+D=0$. On the segment $[BC]$, the two eigenvalues are complex conjugates with a unit modulus, i.e. $D=1$ and $|T|<2$. It can be deduced that the steady state is a sink (asymptotically stable) when $D<1$ and $|T|<1+D$, i.e., (T,D)

is inside the triangle (ABC). It is a saddle-point when $|1+D| < |T|$. Otherwise, it is a source (locally unstable). Since only one variable (capital) is predetermined, the steady state is locally indeterminate if and only if (T, D) is inside the triangle (ABC), and is locally determinate otherwise. This geometrical method is also convenient to analyze the occurrence of bifurcations, i.e., the occurrence of a change in the stability properties when some parameter, whose value is made to continuously vary, crosses some critical value.[15] Considering, for instance, that ε_γ is running the interval $[1, +\infty)$, a transcritical bifurcation generically occurs when (T, D) crosses the line (AC), i.e. when ε_γ crosses the critical value ε_{γ_T}.[16] When (T, D) crosses the line (AB), ε_γ crossing the critical value ε_{γ_F}, a flip bifurcation generically occurs. When (T, D) crosses the segment $[BC]$ in its interior, ε_γ crossing the critical value ε_{γ_H}, a Hopf bifurcation generically occurs.[17]

We proceed now, precisely, by analyzing the variations of T and D in the plane (T, D), as ε_γ is running the interval $[1, +\infty)$.

3.1 The half-line Δ

From (9) and (10), we see that, in the plane (T, D), the locus of points $(T(\varepsilon_\gamma), D(\varepsilon_\gamma))$ for $\varepsilon_\gamma \in [1, +\infty)$ describes a half-line Δ, starting at (T_1, D_1) when $\varepsilon_\gamma = 1$, and with a slope S equal to:

$$S = 1 + \theta\alpha_{K,K} - \theta\frac{1 - s - \beta_{K,K}}{\sigma} \tag{11}$$

Using (10) and (11), we immediately obtain the following Lemma:

[15] When the steady state is locally indeterminate, or when it undergoes a local bifurcation, it is possible to construct stochastic and/or deterministic endogenous cycles, driven by self-fulfilling volatile expectations, that stay in a neighborhood of the steady state. For more details see for instance Grandmont, Pintus and de Vilder.

[16] Theoretically, when an eigenvalue crosses the value 1, either a transcritical, or a saddle node or a pitchfork bifurcation occurs, all of them being associated with the existence of multiple steady states. The case of a saddle node bifurcation (by which the existence of the steady state under analysis disappears) is ruled out, since we apply our analysis to $(K^*, L^*) = (1, 1)$ which existence is persistent under the usual scaling procedure. However, when ε_γ crosses the critical value ε_{γ_T}, a pitchfork bifurcation could instead occur, with the appearance of two other steady states. **Here, we assume that pitcfork bifurcations are ruled out, as a mere exposition device. Notice that several works in related literature have studied the existence of multiple steady states in parameterized economies with constant elasticities ϵ_γ and σ. See, for instance, Cazzavillan et al. (1998) and Khury (2001) . They found at most two steady states, which rules out the case of a pitchfork bifurcations.**

[17] The expressions of ε_{γ_T}, ε_{γ_F} and ε_{γ_H} are given in the Appendix.

Lemma 1 *Under Assumption 3, we have $D'(\varepsilon_\gamma) > 0$ and $S > 0$. Moreover, when σ goes from $(s - \beta_{LL})/(1 + \alpha_{LL})$ to $+\infty$, S increases to $1 + \theta\alpha_{K,K}$.*

Since D increases with ε_γ, the half-line Δ, which is positively sloped, points upwards to the right, as ε_γ increases from 1 to $+\infty$. To locate the half-line Δ in the plane (T, D), it is important to know, not only the level of its slope S, but also the position of the starting point (T_1, D_1), which depends on the level of different parameters and will be analyzed in detail in the next sections. We start by discussing the behavior of (T_1, D_1) as σ varies.

3.2 The half-line Δ_1

The locus of points $(T_1(\sigma), D_1(\sigma))$ obtained as σ decreases from $+\infty$ to $(s - \beta_{LL})/(1 + \alpha_{LL})$ describes a half-line Δ_1, starting at $(T_1(+\infty), D_1(+\infty))$ determined by:

$$T_1(+\infty) = 1 + \frac{1 + \alpha_{\Gamma,L} + \theta(\alpha_{K,K}(1 + \alpha_{L,L}) - \alpha_{L,K}\alpha_{K,L})}{1 + \alpha_{L,L}} \tag{12}$$

$$D_1(+\infty) = \frac{1 + \alpha_{\Gamma,L} + \theta[\alpha_{K,K}(1 + \alpha_{\Gamma,L}) - \alpha_{\Gamma,K}\alpha_{K,L}]}{1 + \alpha_{L,L}} \tag{13}$$

In the rest of the paper, we focus on configurations where this starting point $(T_1(+\infty), D_1(+\infty))$ is on the line (AC), i.e., satisfying $1 - T_1(+\infty) + D_1(+\infty) = 0$.[18] As the reader can check later on in Section 5, most of the distortions considered in the literature satisfy this condition. Using (12) and (13) this leads to the following assumption:

Assumption 5 $\alpha_{K,K}(\alpha_{\Gamma,L} - \alpha_{L,L}) = \alpha_{K,L}(\alpha_{\Gamma,K} - \alpha_{L,K})$, *i.e.,* $1 + D_1(+\infty) - T_1(+\infty) = 0$.

Under this assumption the slope S_1 of the half-line Δ_1, is given by:

$$S_1 = \frac{D_1'(\sigma)}{T_1'(\sigma)} = 1 + \theta\frac{I_2}{I_4 - I_3}, \tag{14}$$

[18] Most of the existing papers have not considered simultaneously distortions on several markets. In our model with only product market imperfections, we have $\alpha_{\Gamma,i} = 0$, $\alpha_{L,K} = \alpha_{K,K}$ and $\alpha_{L,L} = \alpha_{K,L}$; with only a capital market distortion, we have $\alpha_{\Gamma,i} = \alpha_{L,i} = 0$; and with only labor market imperfections or externalities in preferences, we have $\alpha_{K,i} = 0$. By direct inspection of (12) and (13), we see that $1 - T_1(+\infty) + D_1(+\infty) = 0$ in all these cases. In this work, although we analyze simultaneously the different distortions, we still suppose that this last equality is satisfied.

with

$$\begin{aligned}
I_2 &= -(1+\alpha_{L,L})[(1-s-\beta_{K,K})(\alpha_{L,L}-\alpha_{\Gamma,L}) \\
&\quad +\alpha_{K,K}(s-\beta_{L,L}+\beta_{\Gamma,L})+(\alpha_{L,K}-\alpha_{\Gamma,K})(1-s+\beta_{K,L}) \\
&\quad +\alpha_{K,L}\frac{(1-s-\beta_{K,K})(s-\beta_{L,L}+\beta_{\Gamma,L})}{1-s+\beta_{K,L}}]; \\
I_4-I_3 &= \theta(1+\alpha_{L,L})[(1+\alpha_{L,L})(1-s-\beta_{K,K}) \\
&\quad +\alpha_{L,K}(1-s+\beta_{K,L}) \\
&\quad +\alpha_{K,L}\frac{(1-s-\beta_{K,K})(s-\beta_{L,L})}{1-s+\beta_{K,L}}] \\
&\quad -(s-\beta_{L,L})[1+\alpha_{\Gamma,L}-\theta\alpha_{L,K}\alpha_{K,L}] \\
&\quad -(1+\alpha_{L,L})\beta_{\Gamma,L}. \qquad (15)
\end{aligned}$$

Note that the half-line Δ_1, that starts on the line (AC), may point upwards or downwards, as σ decreases from $+\infty$ to $(s-\beta_{LL})/(1+\alpha_{LL})$, according to whether $D_1'(\sigma)<0$ or $D_1'(\sigma)>0$. We proceed our analysis by considering these two cases separately. Recall that the half-line Δ, for each given value of σ, starts on the half-line Δ_1 and, under Lemma 1, points upwards as ε_γ increases, with a positive slope. As σ decreases from $+\infty$, the half-line Δ becomes less steeper, but its starting point shifts upwards or downwards along the half-line Δ_1 depending on whether $D_1'(\sigma)<0$ or $D_1'(\sigma)>0$.[19]

4 Indeterminacy and bifurcations

4.1 Case 1: $D_1'(\sigma)<0$

We notice that $D_1(\sigma)$ is decreasing in σ under the following assumption:

Assumption 6 *$I_4-I_3+\theta I_2<0$.*

Remark that this is the relevant case under perfect competition, for which Assumptions 1-5 and Equations (9)-(10) apply with the restriction $\alpha_{i,j}=\beta_{i,j}=0$, $i=L,K,\Gamma$ and $j=K,L$. Indeed, under this restriction, Assumption 3 is now read as $\sigma>s>\theta(1-s)$. Hence, from (15), we have

[19] Note that the point (T,D) fall on the critical line (AC) when either $\sigma=+\infty$ and $\varepsilon_\gamma=1$, or $\sigma=+\infty$ and $\alpha_{KK}=0$. Therefore, in these situations, an eigenvalue takes the value 1, and, in view of the Hatman-Grobman theorem, we exclude from our local dynamic analysis these two particular parameter's configurations.

$I_4 - I_3 + \theta I_2 = \theta(1-s) - s < 0$. We further have $I_2 = 0$, so that $S_1 = 1$, and $D_1(+\infty) = 1$. This implies that the half-line Δ_1 is on line (AC), starting at point C and pointing upwards. The half-line Δ, which starts on the half-line Δ_1, also points upwards with a slope $S \in (0,1)$, by Lemma 1. Hence, we can easily see that the half-line Δ is above (AB) and below (AC), which allows us to recover the Grandmont et al. (1998) result:

Proposition 1 *(Perfect Competition) Under Assumptions 1-3, for $\alpha_{i,j} = \beta_{i,j} = 0$ with $i = L, K, \Gamma$ and $j = K, L$, the steady state is always a saddle.*

This proposition underlines the fact that without market imperfections, local indeterminacy is not possible for sufficiently high and plausible values of σ. Indeed, indeterminacy requires $D < 1$. Since D increases with ε_γ (see Lemma 1) the lowest value for D is $D_1(\sigma)$. Also, since in the case studied in this section $D_1(\sigma)$ is decreasing, the lowest value for $D_1(\sigma)$ is $D_1(+\infty)$. Therefore, $D_1(+\infty) < 1$ is a necessary condition for indeterminacy. This inequality is not met under perfect competition but can be satisfied in the presence of market imperfections. Since we want to investigate the occurrence of indeterminacy, we will impose it throughout the rest of this section. As explained below, we will also assume that $D_1(+\infty) > -1$:

Assumption 7

1. *$D_1(+\infty) < 1$, i.e., $\alpha_{L,L} - \alpha_{\Gamma,L} > \theta[\alpha_{K,K}(1+\alpha_{\Gamma,L}) - \alpha_{\Gamma,K}\alpha_{K,L}]$;*
2. *$D_1(+\infty) > -1$, i.e., $\theta[\alpha_{K,K}(1+\alpha_{\Gamma,L}) - \alpha_{\Gamma,K}\alpha_{K,L}] > -(2 + \alpha_{L,L} + \alpha_{\Gamma,L})$.*

The second inequality is obviously satisfied without distortions ($\alpha_{i,j} = \beta_{i,j} = 0$ for $i = K, L, \Gamma$ and $j = K, L$) and, therefore, is also verified for values of $\alpha_{i,j}$ not arbitrarily large. Since we are not interested in unrealistically strong market imperfections, this seems to be a reasonable assumption.

To analyze local dynamics, we need more information about the slope of the Δ_1 line. Using (14), we can make a classification according to the level of the slope of Δ_1 and obtain then four different relevant configurations:

- Configuration (i): $S_1 \in (0,1)$ if $I_4 - I_3 < -\theta I_2 < 0$;

- Configuration (ii): $|S_1| > 1$ if:

(a) either $I_4 - I_3 < 0$ and $I_2 < 0$, where $S_1 > 1$;

(b) or $0 < I_4 - I_3 < -\theta I_2/2$, where $S_1 < -1$;

- Configuration (iii): $S_1 \in (-1, S_B)$ if
  $0 < (1 - S_B)(I_4 - I_3) < -\theta I_2 < 2(I_4 - I_3)$

- Configuration (iv): $S_1 \in (S_B, 0)$ if
  $0 < I_4 - I_3 < -\theta I_2 < (1 - S_B)(I_4 - I_3)$;

where we denote by $S_B < 0$ the critical value of S_1 such that the Δ_1 line goes through B given by:

$$S_B = 1 + \frac{4(1 + \alpha_{L,L})}{-3(1 + \alpha_{L,L}) - (1 + \theta\alpha_{K,K})(1 + \alpha_{\Gamma,L}) + \theta\alpha_{\Gamma,K}\alpha_{K,L}}$$

We now proceed with the full discussion of local stability properties and bifurcations, analyzing the location of the half-line Δ for each of these configurations. We begin by a more detailed presentation of configuration (i), and continue with configurations $(ii) - (iv)$. A relevant issue in the discussion below is whether the slope of the half-line Δ is higher or lower than 1. Using (11), straightforward computations lead to the following Lemma:[20]

Lemma 2 *Under Assumption 3, if $\alpha_{KK} \leq 0$ then $S < 1$, with S tending to 1 when σ tends to $+\infty$ and $\alpha_{KK} = 0$, whereas if $\alpha_{KK} > 0$, then $S > 1$ if and only if $\sigma > \sigma_T$, and $S = 1$ if and only if $\sigma = \sigma_T$, with $\sigma_T \equiv (1 - s - \beta_{KK})/\alpha_{KK}$.*

4.1.1 Configuration (i) ($S_1 \in (0, 1)$)

In this configuration, the half line Δ_1 starts (for $\sigma = +\infty$) on the line (AC) between A and C (see Assumption 7), with a slope lower than 1, i.e., lower than the slope of (AC), and points upwards, thereby lying on the right of (AC). Two main cases can arise (see also Figure 1).

If $\alpha_{KK} \leq 0$, then $S < 1$ (Lemma 2) and the half-line Δ is entirely below line (AC) and above line (AB). Hence, the steady state is a saddle.

If $\alpha_{KK} > 0$, there exists the critical value σ_T such that $S = 1$ (Lemma 2). Hence, for $\sigma \leq \sigma_T$, the same as before happens, since the half-line Δ, starting on a point at the right of (AC), has a slope $S \leq 1$. However, if $\sigma > \sigma_T$, then $S > 1$, and the half-line Δ will cross (AC). Consider the definition of the following critical value of σ:

[20] Note that, under Assumption 3, σ_T is only relevant for our analysis if $\sigma_T > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$, which requires that α_{KK} is bounded above by a positive value, i.e., $\alpha_{KK} < \frac{(1+\alpha_{LL})(1-s-\beta_{KK})}{s-\beta_{LL}}$.

Definition 3 *σ_{H_2} is a critical value of σ such that the half-line Δ goes through the point $(T, D) = (2, 1)$, i.e., goes through point C.*[21]

For $\sigma = +\infty$, the half-line Δ starting on (AC) points upwards with a slope higher than 1 for $\alpha_{KK} > 0$. By continuity, the critical value σ_{H_2}, in this configuration greater than σ_T, exists.[22] For $\sigma > \sigma_{H_2}$, Δ crosses $[BC]$ after crossing (AC), i.e., the steady state is a saddle for $1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T}$, undergoes a transcritical bifurcation at $\varepsilon_\gamma = \varepsilon_{\gamma_T}$, becomes a sink for $\varepsilon_{\gamma_T} < \varepsilon_\gamma < \varepsilon_{\gamma_H}$, undergoes a Hopf bifurcation at $\varepsilon_\gamma = \varepsilon_{\gamma_H}$, and becomes a source for $\varepsilon_\gamma > \varepsilon_{\gamma_H}$. For $\sigma_T < \sigma < \sigma_{H_2}$, the Hopf bifurcation disappears and the steady state is either a saddle ($1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T}$) or a source ($\varepsilon_\gamma > \varepsilon_{\gamma_T}$).

These results can be summarized as follows:[23]

Proposition 2 *($S_1 \in (0, 1)$) Under Assumptions 1-7, for $I_4 - I_3 < -\theta I_2 < 0$, the following results for the steady state generically hold:*

1. *If $\alpha_{K,K} \leq 0$: saddle.*
2. *If $\alpha_{K,K} > 0$, then:*
   - *(i) when $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma \leq \sigma_T$: saddle;*
   - *(ii) when $\sigma_T < \sigma < \sigma_{H_2}$: saddle ($1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T}$) - transcritical ($\varepsilon_\gamma = \varepsilon_{\gamma_T}$) - source ($\varepsilon_\gamma > \varepsilon_{\gamma_T}$);*
   - *(iii) when $\sigma > \sigma_{H_2}$: saddle ($1 \leq \varepsilon_\gamma < \varepsilon_{\gamma T}$) - transcritical ($\varepsilon_\gamma = \varepsilon_{\gamma_T}$) - sink ($\varepsilon_{\gamma T} < \varepsilon_\gamma < \varepsilon_{\gamma_H}$)- Hopf ($\varepsilon_\gamma = \varepsilon_{\gamma_H}$) - source ($\varepsilon_\gamma > \varepsilon_{\gamma_H}$).*

4.1.2 Configuration (ii) ($|S_1| > 1$)

Since the half-line Δ_1 starts on the line (AC), between A and C, and points upwards with a slope S_1 strictly greater than 1 or strictly smaller than -1, it crosses neither (AB), nor (AC). However, since Δ_1 crosses the segment $[BC]$, we now give the following definition:

Definition 4 *σ_{H_1} is the critical value of σ such that $D_1(\sigma_{H_1}) = 1$.*[24]

[21] In the Appendix, we show conditions for its existence and uniqueness. Recalling the definitions of ε_{γ_T} and ε_{γ_H}, note that $\varepsilon_{\gamma_T} = \varepsilon_{\gamma_H}$ for $\sigma = \sigma_{H_2}$.

[22] In the Appendix, we show the uniqueness of σ_{H_2} in the configuration under analysis.

[23] Of course, if $\alpha_{KK} > \frac{(1+\alpha_{LL})(1-s-\beta_{KK})}{s-\beta_{LL}}$ Proposition 2.2 (i) becomes irrelevant since $\sigma_T < \frac{s-\beta_{LL}}{1+\alpha_{LL}}$.

[24] The expression for σ_{H_1} is given in the Appendix.

As in the previous configuration, the analysis depends on the value of $\alpha_{K,K}$. Consider first that $\alpha_{K,K} \leq 0$, which means that $S < 1$. If $\sigma \leq \sigma_{H_1}$, the half-line Δ starts above $[BC]$ and crosses (AC). For $\sigma > \sigma_{H_1}$, $(T_1(\sigma), D_1(\sigma))$ is inside (ABC). When $\sigma_{H_1} < \sigma < \sigma_{H_2}$, Δ crosses first the segment $[BC]$ and then line (AC), above point C. For $\sigma > \sigma_{H_2}$, Δ only crosses (AC) below point C.

Assuming now that $\alpha_{K,K} > 0$, the critical value $\sigma_T > 0$ exists (see Lemma 1). We assume that σ_T is sufficiently big,[25] so that $\sigma_T > \sigma_{H_1}$, i.e., the slope of the half-line Δ at σ_{H_1} is lower than 1. This is ensured by[26]:

Assumption 8 *If $\alpha_{K,K} > 0$, then $-1 - T_1(\sigma_T) < D_1(\sigma_T) < 1$.*

As before, when $\sigma \leq \sigma_{H_1}$, $(T_1(\sigma), D_1(\sigma))$ is above or on the segment $[BC]$ and the half-line Δ only crosses (AC). Now, to simplify the analysis, we consider that:[27]

Assumption 9 *If $\sigma > \sigma_{H_1}$ and $\alpha_{K,K} > 0$, then $\varepsilon_{\gamma_H} < \varepsilon_{\gamma_T}$.*

As a consequence, when $\sigma_{H_1} < \sigma < \sigma_T$, Δ crosses first the segment $[BC]$ and then line (AC) above C. When $\sigma \geq \sigma_T$, the half-line Δ only crosses $[BC]$.

The results obtained under this configuration can be summarized in the following proposition:

Proposition 3 *($|S_1| > 1$) Under Assumptions 1-9, for either $I_4 - I_3 < 0$ and $I_2 < 0$, or $0 < I_4 - I_3 < -\theta I_2/2$, the following results for the steady state generically hold:*

1. *If $\alpha_{K,K} \leq 0$, then:*

   (i) when $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma \leq \sigma_{H_1}$: source ($1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T}$) - transcritical ($\varepsilon_\gamma = \varepsilon_{\gamma_T}$) - saddle ($\varepsilon_\gamma > \varepsilon_{\gamma_T}$);

   (ii) when $\sigma_{H_1} < \sigma < \sigma_{H_2}$: sink ($1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_H}$) - Hopf ($\varepsilon_\gamma = \varepsilon_{\gamma_H}$) - source ($\varepsilon_{\gamma H} < \varepsilon_\gamma < \varepsilon_{\gamma_T}$) - transcritical ($\varepsilon_\gamma = \varepsilon_{\gamma_T}$) - saddle ($\varepsilon_\gamma > \varepsilon_{\gamma_T}$);

[25] This assumption is also considered in Cazzavillan et al. (1998), see their Assumption 4.1. Remark also that for $\sigma > \sigma_T$ we have that ρ is increasing in K (see Assumption 2), which is at odds with empirical results. Hence, considering a high value for σ_T shrink the range of values for σ that are less relevant.

[26] More precisely, Assumption 8 ensures that, for configurations (ii), (iii) and (iv), the point $(T_1(\sigma_T), D_1(\sigma_T))$ lies within the triangle (ABC) when $\alpha_{KK} > 0$.

[27] See Appendix 7.5 on the existence of σ_{H_2}. This condition is ensured, by assuming that (32) is satisfied.

(*iii*) *when* $\sigma > \sigma_{H_2}$*: sink* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma T})$.[28]

2. *If* $\alpha_{K,K} > 0$*, then:*

(*i*) *when* $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma \leq \sigma_{H_1}$*: source* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$*;*

(*ii*) *when* $\sigma_{H_1} < \sigma < \sigma_T$*: sink* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_H})$ *- Hopf* $(\varepsilon_\gamma = \varepsilon_{\gamma_H})$ *- source* $(\varepsilon_{\gamma H} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$*;*

(*iii*) *when* $\sigma \geq \sigma_T$*: sink* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_H})$ *- Hopf* $(\varepsilon_\gamma = \varepsilon_{\gamma_H})$ *- source* $(\varepsilon_\gamma > \varepsilon_{\gamma_H})$.

4.1.3 Configuration (*iii*) ($S_1 \in (-1, S_B)$)

In this configuration the slope S_1 is negative and greater than -1, and the half-line Δ_1, that starts on (AC) and points upwards to the left, crosses (AB) above point B. Let us then define the following critical value:

Definition 5 *The critical value* σ_F *is defined by* $1 + D_1(\sigma_F) + T_1(\sigma_F) = 0$.[29]

Consider first that $\alpha_{K,K} \leq 0$, i.e., the slope of Δ is always smaller or equal to 1. When $\sigma < \sigma_F$, $(T_1(\sigma), D_1(\sigma))$ is below line (AB) and above segment $[BC]$. Since the half-line Δ points upwards it does not cross $[BC]$, but crosses (AB) before crossing (AC). When $\sigma_F \leq \sigma \leq \sigma_{H_1}$, $(T_1(\sigma), D_1(\sigma))$ is above (AB) and above $[BC]$. Then, Δ only crosses (AC). When $\sigma > \sigma_{H_1}$, the point $(T_1(\sigma), D_1(\sigma))$ is inside the triangle (ABC). As in the previous configuration, σ_{H_2}, as defined in Definition 3, is greater than σ_{H_1}. Therefore, for $\sigma_{H_1} \leq \sigma < \sigma_{H_2}$, the half-line Δ crosses first $[BC]$ and then (AC) above C, and for $\sigma > \sigma_{H_2}$, the half-line Δ only crosses (AC) below C.

Consider now that $\alpha_{K,K} > 0$. In this case, the critical value $\sigma_T > 0$ exists and, under Assumption 8, we have $\sigma_T > \sigma_{H_1} > \sigma_F$. Therefore, when $\sigma < \sigma_{H_1}$, we obtain the same results as before. When $\sigma_{H_1} < \sigma < \sigma_T$, $(T_1(\sigma), D_1(\sigma))$ is inside the triangle (ABC) and, under Assumption 9, Δ crosses first $[BC]$, and then (AC) above point C. When $\sigma \geq \sigma_T$, S becomes greater than 1, which means that Δ only crosses $[BC]$.

The following proposition gives the results under this configuration.

[28] Note that when σ_{H_2} does not exist (see the Appendix), case 1.(*ii*) applies for all $\sigma > \sigma_{H_1}$ and case 1.(*iii*) disappears.

[29] The expression for σ_F is given in the Appendix.

Proposition 4 *($S_1 \in (-1, S_B)$) Under Assumptions 1-9, for* $0 < (1 - S_B)(I_4 - I_3) < -\theta I_2 < 2(I_4 - I_3)$, *the following results for the steady state generically hold:*

1. *If* $\alpha_{K,K} \leq 0$, *then:*

   (*i*) *when* $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma_F$: *saddle* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_F})$ *- flip* $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$ *- source* $(\varepsilon_{\gamma F} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$;

   (*ii*) *when* $\sigma_F \leq \sigma \leq \sigma_{H_1}$: *source* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T})$*- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$;

   (*iii*) *when* $\sigma_{H_1} < \sigma < \sigma_{H_2}$: *sink* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_H})$ *- Hopf* $(\varepsilon_\gamma = \varepsilon_{\gamma_H})$ *- source* $(\varepsilon_{\gamma H} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$;

   (*iv*) *when* $\sigma > \sigma_{H_2}$: *sink* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T})$*- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma T})$.[30]

2. *If* $\alpha_{K,K} > 0$, *then:*

   (*i*) *when* $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma_F$: *saddle* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_F})$ *- flip* $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$ *- source* $(\varepsilon_{\gamma F} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$;

   (*ii*) *when* $\sigma_F \leq \sigma \leq \sigma_{H_1}$: *source* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T})$*- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$;

   (*iii*) *when* $\sigma_{H_1} < \sigma < \sigma_T$: *sink* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_H})$ *- Hopf* $(\varepsilon_\gamma = \varepsilon_{\gamma_H})$ *- source* $(\varepsilon_{\gamma H} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$;

   (*iv*) *when* $\sigma \geq \sigma_T$: *sink* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_H})$*- Hopf* $(\varepsilon_\gamma = \varepsilon_{\gamma_H})$ *- source* $(\varepsilon_\gamma > \varepsilon_{\gamma_H})$.

4.1.4 Configuration (*iv*) ($S_1 \in (S_B, 0)$)

In this configuration, the slope S_1 is negative and greater than -1, and the half line Δ_1, that points upwards to the left, crosses line (AB) below point B. In this configuration, a new critical value, σ_{H_3}, becomes relevant:

[30] Note that when σ_{H_2} does not exist (see the Appendix), case 1.(*iii*) applies for all $\sigma > \sigma_{H_1}$ and case 1.(*iv*) disappears.

Definition 6 *σ_{H_3} is the critical value of σ such that the half line Δ goes through the point $(T, D) = (-2, 1)$, i.e., goes through point B.*[31]

We begin by assuming $\alpha_{K,K} \leq 0$, which implies S smaller than 1. For $\sigma < \sigma_{H_3}$, Δ starts on the left-side of (AB), crosses (AB) above B and (AC). For $\sigma_{H_3} < \sigma < \sigma_F$, Δ also starts on the left-side of (AB), but crosses (AB) below B, the segment $[BC]$, and (AC) above C. Recall that when σ_{H_2} exists, the Δ line crosses point C (Definition 3). Then, for $\sigma_F \leq \sigma < \sigma_{H_2}$, $(T_1(\sigma), D_1(\sigma))$ is now inside (ABC), and Δ crosses $[BC]$ and (AC) above C. For $\sigma > \sigma_{H_2}$, $(T_1(\sigma), D_1(\sigma))$ is still inside (ABC) and Δ crosses (AC) below C.

We consider now the case where $\alpha_{K,K} > 0$. Since we assume that σ_T is sufficiently big, we have that $\sigma_T > \sigma_F$ $(> \sigma_{H_3})$ (Assumption 8). Then, for $\sigma < \sigma_{H_3}$, the half-line Δ crosses (AB) above B and (AC). For $\sigma_{H_3} < \sigma < \sigma_F$, Δ crosses (AB) below B, the segment $[BC]$ and (AC) above C (Assumption 9). For $\sigma_F \leq \sigma < \sigma_T$, Δ starts inside (ABC) with a slope smaller than 1. Then, it crosses $[BC]$ and (AC) above C. For $\sigma \geq \sigma_T$, the slope S being greater than 1, Δ only crosses $[BC]$.

The results obtained under this configuration are summarized in the following proposition:

Proposition 5 *($S_1 \in (S_B, 0)$) Under Assumptions 1-9, for $0 < I_4 - I_3 < -\theta I_2 < (1 - S_B)(I_4 - I_3)$, the following results for the steady state generically hold:*

1. *If $\alpha_{K,K} \leq 0$, then:*

   (i) when $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma_{H_3}$: saddle $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_F})$ - flip $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$ - source $(\varepsilon_{\gamma F} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ - transcritical $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ - saddle $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$;

   (ii) when $\sigma_{H_3} < \sigma < \sigma_F$: saddle $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_F})$ - flip $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$ - sink $(\varepsilon_{\gamma_F} < \varepsilon_\gamma < \varepsilon_{\gamma H})$ - Hopf $(\varepsilon_\gamma = \varepsilon_{\gamma H})$ - source $(\varepsilon_{\gamma_H} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ - transcritical $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ - saddle $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$;

   (iii) when $\sigma_F \leq \sigma < \sigma_{H_2}$: sink $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_H})$ - Hopf $(\varepsilon_\gamma = \varepsilon_{\gamma_H})$ - source $(\varepsilon_{\gamma H} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ - transcritical $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ - saddle $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$;

[31] Recalling the definitions of ε_{γ_F} and ε_{γ_H}, note that $\varepsilon_{\gamma_F} = \varepsilon_{\gamma_H}$ for $\sigma = \sigma_{H_3}$. In the Appendix, we prove that in this configuration, there exists a unique critical value $\sigma_{H_3} \in (\sigma_{H_1}, \sigma_F)$ such that the half-line Δ goes through point B and crosses $[BC]$ on the right of B for $\sigma > \sigma_{H_3}$.

(*iv*) *when* $\sigma > \sigma_{H_2}$*: sink* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma T})$.[32]

2. *If* $\alpha_{K,K} > 0$*, then:*

(*i*) *when* $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma_{H_3}$*: saddle* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_F})$*- flip* $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$ *- source* $(\varepsilon_{\gamma F} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$*;*

(*ii*) *when* $\sigma_{H_3} < \sigma < \sigma_F$*: saddle* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_F})$ *- flip* $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$ *- sink* $(\varepsilon_{\gamma_F} < \varepsilon_\gamma < \varepsilon_{\gamma H})$ *- Hopf* $(\varepsilon_\gamma = \varepsilon_{\gamma H})$ *- source* $(\varepsilon_{\gamma_H} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$*;*

(*iii*) *when* $\sigma_F \leq \sigma < \sigma_T$*: sink* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_H})$ *- Hopf* $(\varepsilon_\gamma = \varepsilon_{\gamma_H})$ *- source* $(\varepsilon_{\gamma H} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$*;*

(*iv*) *when* $\sigma \geq \sigma_T$*: sink* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_H})$ *- Hopf* $(\varepsilon_\gamma = \varepsilon_{\gamma_H})$*- source* $(\varepsilon_\gamma > \varepsilon_{\gamma_H})$.

4.2 Case 2: $D_1'(\sigma) > 0$

Since in this case $D_1'(\sigma) > 0$, we have:

Assumption 10 $I_4 - I_3 + \theta I_2 > 0$.

Note that this case is not possible when the Γ function is not affected by market imperfections, i.e. when $\alpha_{\Gamma,i} = \beta_{\Gamma,i} = 0$.[33] Therefore, throughout this section, distortions on the Γ function are required. Moreover, using (15), we can easily see that the sign of $I_4 - I_3 + \theta I_2$ is not strongly affected by $\alpha_{K,i}$ and $\beta_{K,i}$, under θ small. Hence, to simplify the analysis, in the remainder of this section we will not consider distortions on the ϱ function ($\alpha_{K,i} = \beta_{K,i} = 0$). Note that when $\alpha_{K,i} = \beta_{K,i} = 0$, the condition $D > T - 1$ required for indeterminacy becomes $I_2 < 0$, i.e. $\alpha_{L,L} - \alpha_{\Gamma,L} > \alpha_{\Gamma,K} - \alpha_{L,K}$[34] Therefore we

[32]Note that when σ_{H_2} does not exist (see the Appendix), case 1.(*iii*) applies for all $\sigma > \sigma_F$ and case 1.(*iv*) disappears.

[33]Indeed, we then have $I_4 - I_3 + \theta I_2 = \theta(1+\alpha_{L,L})(1-s-\beta_{K,K}) - (s-\beta_{L,L})(1+\theta\alpha_{K,K})$, which is always strictly negative under Assumption 3. Hence, Case 1 is the relevant one when the Γ function is not affected by market imperfections.

[34]With $\alpha_{K,i} = \beta_{K,i} = 0$, $D > T - 1$ becomes $\theta(1-s)\frac{[(\alpha_{L,L}+\alpha_{L,K})-(\alpha_{\Gamma,L}+\alpha_{\Gamma,K})]-(\epsilon_\gamma-1)}{\sigma(1+\alpha_{L,L})-(s-\beta_{L,L})} > 0$. Since $\epsilon_\gamma - 1 > 0$, this condition can only be satisfied when $(\alpha_{L,L}+\alpha_{L,K}) - (\alpha_{\Gamma,L}+\alpha_{\Gamma,K}) > 0$, which for $\alpha_{K,i} = \beta_{K,i} = 0$ $(i = K, L)$ implies $I_2 \equiv -(1+\alpha_{LL})(1-s)[(\alpha_{L,L}+\alpha_{L,K}) - (\alpha_{\Gamma,L}+\alpha_{\Gamma,K})] < 0$, since $1 + \alpha_{LL} > 0$ by Assumption 3.

impose the latter, which, together with Assumption 10, implies $I_4 - I_3 > 0$ and $0 < S_1 < 1$.

In Case 1 we introduced Assumption 7 which implied that $D_1(+\infty) \in (-1, 1)$. In this section, since $D_1'(\sigma) > 0$, $D_1(+\infty) < 1$ is no longer a necessary condition for indeterminacy. However, we keep it for the sake of comparability. Also, since we are not interested in unrealistically strong distortions, we continue to assume that $D_1(+\infty) > -1$. We summarize the conditions considered in this section in the following assumption:

Assumption 11 *1. $\alpha_{K,i} = \beta_{K,i} = 0$, for $i = K, L$, and $I_2 < 0$, i.e. $\alpha_{L,L} - \alpha_{\Gamma,L} > \alpha_{\Gamma,K} - \alpha_{L,K}$;*

2. $D_1(+\infty) < 1$, i.e., $\alpha_{L,L} - \alpha_{\Gamma,L} > 0$ and $D_1(+\infty) > -1$, i.e., $2 + \alpha_{L,L} + \alpha_{\Gamma,L} > 0$.

Since, from Assumption 5, the starting point $(T_1(+\infty), D_1(+\infty))$ of Δ_1 is on (AC) and, from Assumption 10 and 11.1, $D_1'(\sigma) > 0$ and $0 < S_1 < 1$, Δ_1 is a half-line lying on the left of (AC). Moreover, as $D_1(+\infty) > -1$ (see Assumption 11.2.), Δ_1 crosses (AB) between A and B at the critical value $\sigma_F \in (\frac{s-\beta_{L,L}}{1+\alpha_{L,L}}, +\infty)$.

The half-line Δ, beginning on line Δ_1 for $\varepsilon_\gamma = 1$, points upwards by Lemma 1. Since $D_1'(\sigma) > 0$, it must then cross (AB) at $\varepsilon_{\gamma_F} > 1$ if and only if $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma_F$. Moreover, Δ always crosses (AC) at a value $\varepsilon_{\gamma_T} > 1$, since by Lemma 1 and Assumption 11.1 it has a slope $S \in (0, 1)$, with S tending to 1 when σ tends to $+\infty$. Notice also that, since $D_1(+\infty) < 1$ and $D_1'(\sigma) > 0$, the half-line Δ, pointing upwards, also always crosses the line (BC), defined by $D = 1$, at $\varepsilon_{\gamma H} > 1$. However, whether Hopf bifurcations occur or not, depend on whether Δ crosses the segment $[BC]$ in its interior or not. The following Lemma, proved in the Appendix, will help us with this question:

Lemma 3 *Let $S_D \equiv 1 - \frac{\theta(1+\alpha_{LL})(1-s)}{s-\beta_{LL}}$.*

1. If $S_1 < S_D$, (i) when $\alpha_{\Gamma,K} \leq \alpha_{L,K}$ then $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$; (ii) when $\alpha_{\Gamma,K} > \alpha_{L,K}$, then $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ for $\frac{s-\beta_{L,L}}{1+\alpha_{L,L}} < \sigma < \sigma_{H_2}$ and $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$ for $\sigma > \sigma_{H_2}$.

2. If $S_1 > S_D$, (i) when $\alpha_{\Gamma,K} \geq \alpha_{L,K}$ then $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$; (ii) when $\alpha_{\Gamma,K} < \alpha_{L,K}$, then: $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$ for $\frac{s-\beta_{L,L}}{1+\alpha_{L,L}} < \sigma < \sigma_{H_2}$ and $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ for $\sigma > \sigma_{H_2}$.

According to this Lemma, it is convenient to analyze the local dynamics considering separately the two following sub-configurations, defined in terms of the slope of Δ_1:

- Configuration (v): $S_1 \in (0, S_D)$ if $-\frac{(I_4-I_3)}{\theta} < I_2 < -\frac{(1+\alpha_{LL})(1-s)}{s-\beta_{LL}}(I_4 - I_3) < 0$;

- Configuration (vi): $S_1 \in (S_D, 1)$ if $-\frac{(1+\alpha_{LL})(1-s)}{s-\beta_{LL}}(I_4 - I_3) < I_2 < 0$.

4.2.1 Configuration (v) ($S_1 \in (0, S_D)$)

In the Appendix (Lemma 4), we show that in this configuration $S_1 < S$. Hence, for $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma_F$, Δ crosses (AC) only after having crossed line (AB), i.e. $\varepsilon_{\gamma_T} > \varepsilon_{\gamma F} > 1$ (see Figure 5).

For $\sigma < \sigma_{H_3}$ Δ crosses line (BC) on the left of point B, i.e., $\varepsilon_{\gamma_H} < \varepsilon_{\gamma F}$, whereas for $\sigma > \sigma_{H_3}$ it lies on the right of point B, i.e., $\varepsilon_{\gamma H} > \varepsilon_{\gamma_F}$.[35] In the first case ($\sigma < \sigma_{H_3}$) there can be no Hopf bifurcations, since the crossing point lies out, on the left, of segment $[BC]$. However, for $\sigma > \sigma_{H_3}$, this crossing point can be either on the left or on the right of point C, according to whether $\varepsilon_{\gamma_H} < \varepsilon_{\gamma_T}$ or $\varepsilon_{\gamma_H} > \varepsilon_{\gamma_T}$. In the latter situation there are again no Hopf bifurcations, since the half-line Δ will not cross the interior of segment $[BC]$. Under Lemma 3, this will depend not only on the sign of $\alpha_{\Gamma,K} - \alpha_{L,K}$ but also on whether σ is higher or lower than σ_{H_2}. With the help of geometrical arguments we can see that when $\sigma_{H_2} > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$ exists, then $\sigma_{H_2} > \sigma_{H_3}$.[36] However σ_{H_2} may be higher or lower than σ_F. To simplify the exposition we summarize the results for this configuration considering only that $\sigma_{H_2} > \sigma_F$.[37]

Proposition 6 *($S_1 \in (0, S_D)$) Under Assumptions 1-5, and 10-11, for* $-\frac{(I_4-I_3)}{\theta} < I_2 < -\frac{(1+\alpha_{LL})(1-s)}{s-\beta_{LL}}(I_4 - I_3) < 0$, *the following results for the steady state generically hold when* $\sigma_{H_2} > \sigma_F$*:*

1. If $\alpha_{\Gamma,K} - \alpha_{L,K} > 0$, *then*

(i) *when* $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma_{H_3}$: *saddle* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_F})$ *- flip* $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$ *- source* $(\varepsilon_{\gamma_F} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$.

(ii) *when* $\sigma_{H_3} < \sigma < \sigma_F$: *saddle* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_F})$ *- flip* $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$ *- sink* $(\varepsilon_{\gamma_F} < \varepsilon_\gamma < \varepsilon_{\gamma_H})$ *- Hopf* $(\varepsilon_\gamma = \varepsilon_{\gamma H})$ *- source* $(\varepsilon_{\gamma H} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$.

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[35] See **Definition 6 for σ_{H_3}. We can** see geometrically that $\sigma_{H_3} < \sigma_F$. In the Appendix, more details are given on σ_{H_3}.

[36] Suppose on the contrary that $\sigma_{H_2} < \sigma_{H_3}$. For $\sigma_{H_2} < \sigma < \sigma_{H_3}$, Δ could not cross the line (BC) on the right of point C because for $\sigma < \sigma_{H_3}$, as shown in the Appendix, it must cross line (BC) on the left of point B.

[37] Using geometrical considerations, the reader can easily adapt Proposition 6 to the case where $\sigma_F > \sigma_{H_2}$.

(*iii*) *when* $\sigma_F < \sigma < \sigma_{H_2}$*: sink* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_H})$ *- Hopf* $(\varepsilon_\gamma = \varepsilon_{\gamma_H})$ *- source* $(\varepsilon_{\gamma H} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$.

(*iv*) *when* $\sigma > \sigma_{H_2}$*: sink* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$.

2. *If* $\alpha_{\Gamma,K} - \alpha_{L,K} \leq 0$, *then*

(*i*) *when* $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma_{H_3}$*: saddle* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma F})$ *- flip bifurcation* $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$ *- source* $(\varepsilon_{\gamma F} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle*$(\varepsilon_\gamma > \varepsilon_{\gamma_T})$.

(*ii*) *when* $\sigma_{H_3} < \sigma < \sigma_F$*: saddle* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_F})$ *- flip bifurcation* $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$ *- sink* $(\varepsilon_{\gamma F} < \varepsilon_\gamma < \varepsilon_{\gamma_H})$ *- Hopf* $(\varepsilon_\gamma = \varepsilon_{\gamma_H})$ *- source* $(\varepsilon_{\gamma_H} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$.

(*iii*) *when* $\sigma > \sigma_F$*: sink* $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_H})$ *- Hopf* $(\varepsilon_\gamma = \varepsilon_{\gamma_H})$ *- source* $(\varepsilon_{\gamma H} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ *- transcritical* $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ *- saddle* $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$.

4.2.2 Configuration (vi) $(S_1 \in (S_D, 1))$

In this configuration, as shown in the Appendix (Lemma 4), $S < S_1$ for all $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma_{S_1}$ and $S > S_1$ for all $\sigma > \sigma_{S_1}$, where σ_{S_1} is the value of σ for which $S = S_1$.[38] Consider also the following critical value of σ:

Definition 7 *σ_{S_2} is the critical value of σ such that the half line Δ goes through the point $(T, D) = (-1, 0)$, i.e., goes through point A.*[39]

When $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma_{S_2}$, the half-line Δ, pointing upwards with having a slope lower than Δ_1 (we can see geometrically that $\sigma_{S_2} < \sigma_{S_1}$), crosses first (AC) at $\varepsilon_{\gamma_T} > 1$, then it crosses (AB) at $\varepsilon_{\gamma_F} > 1$, both below point A. What happens for $\sigma > \sigma_{S_2}$ depends on whether Δ goes through (AB), which will only happen when $\sigma < \sigma_F$, and on whether Δ crosses the segment $[BC]$ in its interior or not.

We will assume that (T_1, D_1) for $\sigma = \sigma_{S_1}$ is inside the triangle (ABC):

[38] The expression of σ_{S_1} is given in the Appendix 7.7

[39] The expression for σ_{S_2} is given in the Appendix. As the slope of Δ increases and its initial point shifts upwards along Δ_1, $\varepsilon_{\gamma T} < \varepsilon_{\gamma F}$ for $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma^{S_2}$, and $\varepsilon_{\gamma T} > \varepsilon_{\gamma F}$ for $\sigma > \sigma^{S_2}$. Easy analytical computations show that $\sigma_{S_2} \in \left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \infty\right)$ exists if $\frac{s-\beta_{LL}}{1+\alpha_{LL}}(1 + \alpha_{LL} + \alpha_{LK} - \alpha_{\Gamma K}) > \theta(1-s)(1 + \alpha_{LL} + \alpha_{LK}) - \beta_{\Gamma L}$. Hence, if this condition is not met, then $\varepsilon_{\gamma T} > \varepsilon_{\gamma F}$ for all $\sigma > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$.

Assumption 12 $D_1(\sigma_{S_1}) > -1 - T_1(\sigma_{S_1})$

Hence, $\sigma_{S_1} > \sigma_F$.[40] Then, for $\sigma_{S_2} < \sigma < \sigma_F$, Δ still has a slope lower than Δ_1, but it first crosses (AB) and then (AC). For $\sigma > \sigma_F$, Δ no longer crosses (AB). Whether Δ goes through (BC) on the left or on the right of point C depends, according to Lemma 3, on the sign of $\alpha_{\Gamma K} - \alpha_{LK}$ and on whether σ is higher or lower than σ_{H_2}. We can see geometrically that $\sigma_{H_2} > \sigma_{S_1}$ and therefore $\sigma_{H_2} > \sigma_F$. The following proposition summarizes these results:[41]

Proposition 7 *$(S_1 \in (S_D, 1))$Under Assumptions 1- 5, 10-11 and 12, for $-\frac{(1+\alpha_{LL})(1-s)}{s-\beta_{LL}}(I_4 - I_3) < I_2 < 0$, the following results for the steady state generically hold:*

1. *If $\alpha_{\Gamma,K} - \alpha_{L,K} \geq 0$, then*

(i) when $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma^{S_2}$: saddle $(1 < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ - transcritical $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ - source $(\varepsilon_{\gamma_T} < \varepsilon_\gamma < \varepsilon_{\gamma F})$ - flip $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$ - saddle $(\varepsilon_\gamma > \varepsilon_{\gamma_F})$;

(ii) when $\sigma^{S_2} < \sigma < \sigma_F$: saddle $(1 < \varepsilon_\gamma < \varepsilon_{\gamma_F})$ - flip $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$, - sink $(\varepsilon_{\gamma F} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ - transcritical $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ - saddle $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$,

(iii) when $\sigma > \sigma_F$: sink $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T})$ - transcritical $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ - saddle $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$.

2. *If $\alpha_{\Gamma,K} - \alpha_{L,K} < 0$, then*

(i) when $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma^{S_2}$: saddle $(1 < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ - transcritical $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ - source $(\varepsilon_{\gamma_T} < \varepsilon_\gamma < \varepsilon_{\gamma F})$ - flip $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$ - saddle $(\varepsilon_\gamma > \varepsilon_{\gamma_F})$;

(ii) when $\sigma^{S_2} < \sigma < \sigma_F$: saddle $(1 < \varepsilon_\gamma < \varepsilon_{\gamma_F})$ - flip $(\varepsilon_\gamma = \varepsilon_{\gamma_F})$, - sink $(\varepsilon_{\gamma F} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ - transcritical $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ - saddle $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$

(iii) when $\sigma_F < \sigma < \sigma_{H_2}$: sink $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T})$ - transcritical $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ - saddle $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$.

(iv) when $\sigma > \sigma_{H_2}$: sink $(1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_H})$ - Hopf $(\varepsilon_\gamma = \varepsilon_{\gamma_H})$ - source $(\varepsilon_{\gamma H} < \varepsilon_\gamma < \varepsilon_{\gamma_T})$ - transcritical $(\varepsilon_\gamma = \varepsilon_{\gamma_T})$ - saddle $(\varepsilon_\gamma > \varepsilon_{\gamma_T})$.

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[40] Note that then we have $\varepsilon_{\gamma_H} > \varepsilon_{\gamma_F}$ for all $\sigma > \frac{s-\beta_{LL}}{(1+\alpha_{LL})}$. See Definition 6 and Appendix on the existence of σ_{H_3}. Indeed, there cannot exist $\sigma_{H_3} > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$, since its existence would require that $\sigma_{S_1} < \sigma_{H_3} < \sigma_F$, which is ruled out by Assumption 12.

[41] Of course, when $\frac{s-\beta_{LL}}{1+\alpha_{LL}}(1+\alpha_{LL}+\alpha_{LK}-\alpha_{\Gamma K}) > \theta(1-s)(1+\alpha_{LL}+\alpha_{LK}) - \beta_{\Gamma L}$, 1$(i)$ and 2(i) of Proposition 7 are not relevant, since in that case $\sigma_{S_2} < \frac{s-\beta_{LL}}{1+\alpha_{LL}}$ (see footnote on the definition of σ_{S_2}).

4.3 Discussion of the results

When capital and labor are sufficiently substitutable in production, indeterminacy emerges in the presence of market imperfections, in contrast to the perfectly competitive case (see Propositions 1-7).

Indeed, Propositions 2 to 7 show that indeterminacy may occur if the elasticity of substitution between capital and labor (σ) is higher than a lower bound. However, the value of this lower bound depends on the different configurations obtained. In Case 1, where $D_1'(\sigma) < 0$, this lower bound is higher or equal than σ_{H_1}, because $\sigma > \sigma_{H_1}$ is a necessary condition for indeterminacy.[42] In fact, in configurations (i) and (iv), indeterminacy respectively requires[43] $\sigma > \sigma_{H_2} > \sigma_{H_1}$ and $\sigma > \sigma_{H_3} > \sigma_{H_1}$, while in configurations (ii) and (iii), σ_{H_1} is the lower bound. In Case 2, where $D_1'(\sigma) > 0$, although $\sigma > \sigma_{H_1}$ is no longer a necessary condition for indeterminacy, indeterminacy also requires a lower bound for σ ($\sigma > \sigma_{H_3}$ in configuration (v), and $\sigma > \sigma_{S_2}$ in configuration (vi)).

Another result is that indeterminacy never emerges when ε_γ is sufficiently high, i.e. ε_γ has to be lower than an upper bound whose value (either $\epsilon_{\gamma H}$ or $\epsilon_{\gamma T}$) depends on the configuration considered. However, except in configurations (ii) and (iii) of Case 1, indeterminacy may be ruled out if ϵ_γ is sufficiently close to 1. Therefore, imposing an infinitely elastic labor supply at the individual level ($\varepsilon_\gamma = 1$) may not be appropriate to study the implications of market distortions on local indeterminacy. This is a new interesting result because, as it will be illustrated in the examples (see Section 5.2), we find that if distortions affecting effective consumption and/or the offer curve are strong enough, indeterminacy may only be possible for intermediate values of the elasticity of substitution between inputs, if the (private) labor supply curve is not infinitely elastic.

We will now discuss the role played by the different distortions on indeterminacy. We shall focus on the role of the distortions represented by $\alpha_{i,j}$, which are the most relevant ones when high levels of σ are considered (see Assumption 2), as assumed in our analysis (Assumption 3).

Consider first the case without distortions on the Γ function, so that $\alpha_{\Gamma,i} = \beta_{\Gamma,i} = 0$, as it happens, for example, when we only have product or capital market imperfections (see Section 5.1). Then, only configurations $(i) - (iv)$ of Case 1, and Propositions 2-5, can be obtained. Indeed, with $\alpha_{\Gamma,i} = \beta_{\Gamma,i} = 0$, Assumption 3 implies that $I_4 - I_3 + \theta I_2 < 0$, as already

[42] Indeterminacy requires $D < 1$. Since D is increasing with ε_γ this implies $D_1(\sigma) < 1$, which, under the Assumption $D_1'(\sigma) < 0$ of Case 1, is equivalent to $\sigma > \sigma_{H_1}$.

[43] Using geometrical arguments one can easily see that $\sigma_{H_2} > \sigma_{H_1}$ in configuration (i), and that and $\sigma_{H_3} > \sigma_{H_1}$ in configuration (iv).

discussed at the beginning of section 4.2.[44] It is interesting to note that the necessaary condition for indeterminacy $D_1(+\infty) < 1$, which becomes $\alpha_{L,L} > \theta\alpha_{K,K}$ when $\alpha_{\Gamma,i} = \beta_{\Gamma,i} = 0$ (see Assumption 7.1), has a suitable economic interpretation: the response of effective consumption to employment $(1 + \alpha_{L,L})$ must be stronger than the response of capital income to capital $(1 + \theta\alpha_{K,K})$, when $\sigma = +\infty$.

Although every configuration of Case 1 can a priori be obtained when there are no distortions on the Γ function, configurations (i) and $(ii)(a)$ are the most relevant ones, since empirical values of θ are rather small. Indeed, when θ is small,[45] we obtain $I_4 - I_3 \approx -(s - \beta_{L,L})$ which is negative by Assumption 3, so that only configurations (i) and $(ii)(a)$ of Case 1, and Propositions 2 and 3, apply. From Proposition 2, indeterminacy occurs only if $\alpha_{L,L} > \theta\alpha_{K,K} > 0$ and $I_2 > 0$, which implies $\alpha_{L,K} < 0$ and /or $\alpha_{K,L} < 0$. As it is illustrated in the next section, these conditions are never satisfied in the economic examples presented. On the contrary, the indeterminacy conditions required in Proposition 3 are often met by standard examples, for instance with product market imperfections.

Let us now discuss in more details the role of distortions affecting the ϱ function. For $\alpha_{K,K} < 0$, indeterminacy does not occur under configuration (i), which is the unique one where $I_2 > 0$ (see Proposition 2.1). Therefore, in this case, indeterminacy requires $I_2 < 0$. Note that, $I_2 < 0 \Leftrightarrow \frac{1-s-\beta_{K,K}}{s-\beta_{L,L}}\alpha_{L,L} + \alpha_{L,K}\frac{1-s+\beta_{K,L}}{s-\beta_{L,L}} + \alpha_{K,L}\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}} + \alpha_{K,K} > 0$. Hence, under Assumption 3, indeterminacy is ruled out if $\alpha_{L,L}$, $\alpha_{K,L}$, $\alpha_{K,L}$ and $\alpha_{K,K}$ all have a negative sign. Moreover, for $\alpha_{L,j} = \beta_{L,j} = 0$ (i.e. in the absence of distortions on the Ω function), indeterminacy, with $\alpha_{K,K} < 0$, requires a positive lower bound on $\alpha_{K,L}$, i.e., $\alpha_{K,L} > -\alpha_{K,K}\frac{1-s+\beta_{K,L}}{1-s-\beta_{K,K}} > 0$, and therefore the existence of cross effects $(\alpha_{K,L})$ with an opposite sign from the direct effects $(\alpha_{K,K})$ is also necessary. For $\alpha_{K,K} \geq 0$, given the necessary condition $\alpha_{L,L} > \theta\alpha_{K,K}$, indeterminacy requires a positive value for $\alpha_{L,L}$. We may then conclude that distortions affecting the ϱ function, i.e., affecting the capital accumulation equation, do not seem to play a crucial role for indeterminacy. Indeed, indeterminacy, in the presence of distortions on ρ, either requires opposite effects of capital and labor on the distortions affecting the capital accumulation equation and a distortion due to labor effects $(\alpha_{K,L})$ positive and bounded away from zero, or the presence of some other market failures, distorting the intertemporal arbitrage between future consumption and labor.

We now provide arguments explaining why distortions affecting the Ω

[44] See footnote 31.

[45] Note that this approximation is relevant with respect to empirical estimates. See footnote ??.

function are more relevant for indeterminacy. For $\alpha_{L,L} \leq 0$, the necessary condition for indeterminacy ($\alpha_{L,L} > \theta\alpha_{K,K}$) implies $\alpha_{K,K} < 0$, i.e., requires also distortions on the ϱ function. However, this is no longer true for $\alpha_{L,L} > 0$. Indeed, without distortions on the ϱ function, the necessary condition $\alpha_{L,L} > \theta\alpha_{K,K}$ is satisfied for all $\alpha_{L,L} > 0$. Moreover, $I_2 < 0$ is also required for indeterminacy,[46] which implies $\alpha_{L,K} > -\alpha_{L,L}\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}}$. Hence, opposite effects of capital and labor on the distortions influencing Ω are not nedeed. Moreover, indeterminacy does not require that $\alpha_{L,K} \neq 0$, and it may occur as soon as there is an arbitrarily small positive distortion on effective consumption due to labor effects. .

Taking now into account distortions on the Γ function, new results and configurations may appear. To simplify the exposition we assume that distortions on the ϱ function are absent ($\alpha_{K,i} = \beta_{K,i} = 0$), since, as discussed above, they do not seem to play a relevant role for indeterminacy. Hence, we only consider distortions on Ω and Γ, both solely influencing the equilibrium arbitrage condition of workers, as it happens, for instance, when we only have labor market imperfections (see Section 5.2). In the absence of distortions affecting capital accumulation, a necessary condition for indeterminacy is that $I_2 < 0$, i.e. $\alpha_{L,L} + \alpha_{L,K} > \alpha_{\Gamma,L} + \alpha_{\Gamma,K}$, as discussed at the beginning of Section 4.2. In economic terms, $\alpha_{L,L} + \alpha_{L,K}$ summarizes the global effect of distortions on the Ω function, and $\alpha_{\Gamma,L} + \alpha_{\Gamma,K}$ represents the global effect of distortions in the Γ function, when σ tends to $+\infty$, (see Assumption 2). Therefore, indeterminacy requires a positive difference between these two global distortion effects on Ω and Γ.

With $\alpha_{K,i} = \beta_{K,i} = 0$, we obtain $I_4 - I_3 + \theta I_2 = -(s - \beta_{L,L})(1 + \alpha_{\Gamma,L}) - \beta_{\Gamma,L}(1+\alpha_{L,L}) + \theta\,(1 - s)\,(1 + \alpha_{L,L})\,(1 + \alpha_{\Gamma,L} + \alpha_{\Gamma,K})$, which a priori can take a positive or negative sign. Hence, not only configurations of Case 1 are possible, but also configurations (v) and (vi) of Case 2 can emerge. Case 2 obtains when $\alpha_{\Gamma L} < -1 - \frac{\beta_{\Gamma,L}(1+\alpha_{L,L})-\theta(1-s)\left(1+\alpha_{L,L}\right)\alpha_{\Gamma,K}}{(s-\beta_{L,L})-\theta(1-s)\left(1+\alpha_{L,L}\right)} \equiv \alpha^*_{\Gamma L}$ and is not possible without effects through Γ, whereas it can appear without effects through Ω.[47] Indeterminacy is possible in the absence of distortions on effective consumption, even with arbitrarily low levels of distortions on on Γ. However, the new configurations of Case 2 can only emerge if some distortions are bounded away from zero, since $\alpha^*_{\Gamma L}$ is close to -1 when both $\beta_{\Gamma L}$ and $\alpha_{\Gamma K}$ are close to zero. If distortions on Γ are such that Case 1 applies, then[48]

[46] When $\alpha_{K,K} = 0$, indeterminacy does not occur under configuration (i), the unique configuration where $I_2 > 0$ (see Proposition 2).

[47] Note that, under Assumption 3, $(1 + \alpha_{LL})\,\theta < s - \beta_{Ll}$ when $\alpha_{K,i} = \beta_{K,i} = 0$.

[48] Recall that in Case 1 (i.e, in the case $I_4 - I_3 + \theta I_2 < 0$), indeterminacy requires that $D_1\,(+\infty) < 1$, which becomes $\alpha_{L,L} - \alpha_{\Gamma,L} > 0$ in the absence of distortions in the ϱ

$\alpha_{LL} > \alpha_{\Gamma L}$ and $\sigma > \sigma_{H_1}$ are also necessary conditions for indeterminacy, and a smaller value for $\alpha_{L,L} - \alpha_{\Gamma,L}$ leading to a higher σ_{H_1}, reduces the scope for indeterminacy.[49]

Finally, let us remark that indeterminacy with arbitrarily small distortions is possible if, as seen above, some of the distortions affect the effective consumption or the offer curve. However, it requires that the elasticity of the private labor supply and the elasticity of substitution between inputs take extreme values. Indeed, when θ take realistically small values (Assumption 3), $I_4 - I_3$ takes negative values in the absence of distortions, and will keep its sign if distortions are close to zero. Hence, only configurations (i) and (ii) of Case 1 become possible, and both, $\sigma > \sigma_{H_1}$ and either $\epsilon_\gamma < \epsilon_{\gamma H}$ or $\epsilon_\gamma < \epsilon_{\gamma T}$ are required for indeterminacy. As the limit value of σ_{H_1} tends to $+\infty$, and the limit value of $1/\left(\epsilon_{\gamma H} - 1\right)$ and $1/(\epsilon_{\gamma T} - 1)$ tends also to $+\infty$ when distortions are close to zero and σ close to $+\infty$, both elasticities should be close to infinity if indeterminacy occurs.

From the above discussion on the role of distortions, we conclude that, while general distortions on real interest rate do not seem to play a major role on the occurrence of indeterminacy when inputs are sufficiently substitutable, distortions on the generalized offer curve that negatively depend on capital and labor, and distortions on the effective consumption that depend positively on capital and labor, seem to help the fulfillment of requirements for the possible occurrence of indeterminacy. Also, a minimal degree of general distortions, in the real interest rate, in the generalized offer curve and/or the effective consumption, is a requirement for indeterminacy to occur, at least if plausible values for the elasticity of substitution between capital and labor are considered.

An important issue is to understand what type of specific distortions on output, capital and labor markets, are more relevant for the occurrence of indeterminacy. Therefore, we now proceed by applying our general methodology and results to several examples of specific market distortions.

5 Applications and discussion

In this section we present several examples of specific market distortions and that fit into our general formulation, so that they can be analyzed using our framework. These examples also provide microeconomic foundations for the

function.

[49]Note that σ_{H_1}, in the absence of distortions in the ϱ function, can be written as $\sigma_{H_1} \equiv \frac{s-\beta_{L,L}}{1+\alpha_{L,L}} - \frac{I_4-I_3+\theta I_2}{(1+\alpha_{L,L})(\alpha_{L,L}-\alpha_{\Gamma,L})}$. Note also that, since in Case 1 we have $I_4 - I_3 + \theta I_2 < 0$, we obtain that $\lim \sigma_{H_1} \to +\infty$ when $\alpha_{L,L} > \alpha_{\Gamma,L}$ tends to zero.

model developed in the previous sections. We first consider examples without distortions in the Γ function ($\alpha_{\Gamma,i} = \beta_{\Gamma,i} = 0$). In the second subsection, we present examples where distortions in the Γ function also appear ($\alpha_{\Gamma,i} \neq 0$ and/or $\beta_{\Gamma,i} \neq 0$). Many examples considered here have already been studied in the literature, but not always in a Woodford economy, while some of them are new and allow us to exhibit phenomena that are not illustrated otherwise.

5.1 Examples where $\alpha_{\Gamma,i} = \beta_{\Gamma,i} = 0$

In all these examples, we have $I_4 - I_3 + \theta I_2 < 0$ ($D_1'(\sigma) < 0$) and Case 1 applies. We will see that, even if the models presented below have different microeconomic foundations and economic meanings, most of them enter the same configuration (configuration (ii) of Case 1), meaning that the mechanisms for indeterminacy are not strongly different.

5.1.1 Externalities in Production

Externalities in production have often been introduced in macro-dynamic models. See among others Barinci and Chéron (2001), Benhabib and Farmer (1994), Cazzavillan (2001), Cazzavillan et al. (1998). For a survey see Benhabib and Farmer (1999). In these papers all markets are perfectly competitive, and the representative firm produces the final good using labor and capital with an internal constant returns to scale technology. However, production benefiting from positive externalities,[50] returns to scale are increasing at the social level. Here, we will extend this formulation, allowing also for negative productive externalities[51] so that, at the social level, returns to scale can be decreasing. We consider therefore that production is given by $y = ALf(a)\xi(K, L)$, where $\xi(K, L)$ stands for the externality function and $a = K/L$ the capital-labor ratio. Since firms take as given externalities when maximising profits we have:

$$
\begin{aligned}
\Omega_t &= A\omega(K_{t-1}/L_t)\xi(K_{t-1}, L_t) \\
\varrho_t &= A\rho(K_{t-1}/L_t)\xi(K_{t-1}, L_t)
\end{aligned}
$$

where $\rho(K/L)$ and $\omega(K/L)$ are given in Definition 1. Since $\Gamma(K_{t-1,}L_t) = \gamma(L_t)$, we obtain, using (2), $\alpha_{L,L} = \alpha_{K,L} = \varepsilon_{\xi,L}$, $\alpha_{L,K} = \alpha_{K,K} = \varepsilon_{\xi,K}$ and $\alpha_{\Gamma,i} = \beta_{j,i} = 0$, for $i = K, L$ and $j = K, L, \Gamma$, where $\varepsilon_{\xi,i}$ denotes the elasticity of the function ξ with respect to $i = K, L$ evaluated at the steady state. Therefore, denoting $\nu = \varepsilon_{\xi,L} + \varepsilon_{\xi,K}$, Assumption 5 is satisfied, and we obtain

[50] They are usually justified by learning by doing or matching problems on labor market.
[51] These can be justified, for instance, by congestion or pollution arguments.

$I_2 = -(1+\nu-\varepsilon_{\xi,K})\upsilon$ and $I_4 - I_3 = \theta(1+\nu-\varepsilon_{\xi,K})(1-s+\nu) - s(1+\theta\varepsilon_{\xi,K})$. Assumption 3 implies that $s(1+\theta\varepsilon_{\xi,K}) > \theta(1+\nu-\varepsilon_{\xi,K})(1-s)$, i.e. $I_4 - I_3 < 0$, so that $D_1'(\sigma) < 0$. Moreover, from Assumption 7, we have that $-2-\nu < (1+\theta)\varepsilon_{\xi,K} < \nu$.

Therefore, as assumed in Cazzavillan, Lloyd-Braga and Pintus (1998), positive externalities ensure $I_2 < 0$. Configuration (iia) of Case 1 where $S_1 > 1$ applies and indeterminacy can emerge for $\sigma > \sigma_{H1} = (s-\theta(1-s))/(\nu-(1+\theta)\varepsilon_{\xi,K})$.[52] Remark that σ_{H_1} can be below one. However this requires sufficiently strong labor externalities. Indeed, in the absence of capital externalities ($\varepsilon_{\xi,K} = 0$, $\nu = \varepsilon_{\xi,L}$), for indeterminacy to emerge in the Cobb-Douglas case, labor externalities must exceed $s-\theta(1-s)$, a value which is too high in empirical terms.

In the case of negative externalities, we have $I_2 > 0$. Configuration (i) of Case 1 where $0 < S_1 < 1$ applies. Since we have $\alpha_{K,K} < 0$, the steady state is always a saddle.[53]

5.1.2 Imperfect Competition, Mark-up Variabilty, Taste for Variety and Free Entry

Several authors have introduced imperfectly competitive product market in macro-dynamic models to analyze how this market failure can promote indeterminacy and endogenous cycles. As underlined by Benhabib an Farmer (1994) and Cazzavillan, Lloyd-Braga and Pintus (1998), imperfectly competitive economies with constant mark-up and decreasing marginal cost (increasing returns) have the same dynamic stability properties than perfectly competitive models with positive externalities in production, as presented above in Section 5.1.1. However, two other properties of imperfect competition can be exploited in a dynamic framework: mark-up variability and taste for variety. On the one hand, while several economic features can explain mark-up variability, we focus here on mark-up variability due to business formation (Dos Santos Ferreira and Lloyd-Braga (2005), Kuhry (2001), Seegmuller (2007a), Weder (2000a)). On the other hand, following Benassy (1996), we define taste for variety as the consumer utility gain of consuming one unit of all the N_t varieties of goods instead of consuming N_t units of a single variety (Jacobsen (1998), Seegmuller (2007b)). In these two types of

[52] See Proposition 3.2.
[53] See Proposition 2.1.

models,[54] increasing returns come from the existence of a fixed cost[55] and the number N_t of producers is determined by the usual zero profit condition. At equilibrium, one typically obtains the number of firms as an increasing function of aggregate production, i.e.

$$N_t = N(f(a_t)L_t) \tag{16}$$

with $\epsilon_N(Y) \equiv N'(Y)Y/N(Y) \geq 0$, and a same distortion $\mu(N_t)$, increasing with N_t, affects both the real wage and the real interest rate:

$$\begin{aligned}\Omega_t &= \mu(N_t)A\omega(K_{t-1}/L_t) &(17)\\ \varrho_t &= \mu(N_t)A\rho(K_{t-1}/L_t) &(18)\end{aligned}$$

with $\epsilon_\mu(N) \equiv \mu'(N)N/\mu(N) \geq 0$. In models with a counter-cyclical mark-up, $\mu(N_t)$ can be interpreted as the inverse of the mark-up factor, while when there is taste for variety, it represents the ratio between the price set by a single firm and the aggregate price. We notice that substituting (16) into (17) and (18), we obtain expressions for Ω_t and ϱ_t similar to those of the model with productive externalities (see Section 5.1.1). Moreover, since imperfect competition do not affect the Γ function, we also have $\Gamma(K_{t-1}, L_t) = \gamma(L_t)$. Defining $\nu \equiv \epsilon_\mu(N)\epsilon_N(Y)$, we obtain $\alpha_{L,L} = \alpha_{K,L} = (1-s)\nu$, $\alpha_{L,K} = \alpha_{K,K} = s\nu$ and $\alpha_{\Gamma,i} = \beta_{j,i} = 0$, for $i = K, L$ and $j = K, L, \Gamma$. Therefore, the models studied here correspond to a particular case of positive productive externalities, where $\varepsilon_{\xi,K}/\varepsilon_{\xi,L} = s/(1-s)$. We deduce that $I_4 - I_3 < 0$, $I_2 < 0$ and $D_1'(\sigma) < 0$, but Assumption 7 is satisfied for $1-s > \theta s$. These imperfectly competitve models enter configuration (iia) of Case 1 where $S_1 > 1$. Hence, indeterminacy can emerge for $\sigma > \sigma_{H1} = (s - \theta(1-s))/[\nu(1-(1+\theta)s)]$. Note that the range of elasticities of capital-labor substitution for indeterminacy increases (σ_{H1} decreases) with more counter-cyclical mark-ups, larger fixed cost or degree of taste for variety.

5.1.3 Fiscal Policy, Balanced Budget Rules and Variable Tax Rates

In this section we present another application that does not affect the Γ function. We consider a perfectly competitive economy characterized by a balanced budget rule and variable tax rates. We also introduce the possibility of government spending externalities in preferences. The example presented

[54] For sake of conciseness, we do not present models with mark-up variability or taste for variety in details, but rather give their main economic features. For more details, the reader can refer to the references cited just above.

[55] Using our notations, the production of a firm $i = 1, ..., N_t$ is given by $y_{it} = A(f(a_{it})l_{it} - \phi)$, where l_{it} represents labor hired by firm i and $\phi > 0$ a fixed cost.

follows closely, although extending it by considering capital income taxation, the work of Lloyd-Braga et al. (2006), and covers as particular cases those considered in Dromel and Pintus (2004), Giannitsarou (2005), Guo and Lansing (1998), Gokan (2005), Pintus (2003), Schmitt-Grohé and Uribe (1997), and also the case of constant tax rates.

We assume that the government chooses the tax policy and balances its budget at each period in time. The government can levy tax on capital income ($\rho_t K_{t-1}$), labor income ($\omega_t L_t$) and on private aggregate consumption ($C_t = C_t^w + C_t^c$). Therefore, real public spending in goods and services in period t, $G_t \geq 0$ is given by $G_t = \tau(\omega_t L_t)\omega_t L_t + \tau(C_t) C_t + \tau(\rho_t K_{t-1}) \rho_t K_{t-1}$, where the tax rates on labor and capital income, and on consumption are respectively given by $\tau(\omega_t L_t) = z_L (\omega_t L_t/\omega L)^{\phi_L}$, $\tau(\rho_t K_{t-1}) = z_K (\rho_t K_{t-1}/\rho K)^{\phi_K}$ and $\tau(C_t) = z_c (C_t/C)^{\phi_c}$, where ωL is the steady state value of the wage bill, ρK the steady state value of capital income and C the steady state level of consumption, and $z_i \in (0,1)$ for $i = L, K$ and $z_c > 0$ the level of the tax rates at the steady state. The parameters $\phi_j \in \mathbb{R}$, with $j = L, K, C$ denote the elasticities of the tax rates with respect to the tax bases. When $\phi_j = 0$ the tax rate is constant. Finally, we denote by $\eta > 0$ the elasticity of public spending externalities in preferences that affect workers' utility of consumption.[56] Assuming that agents take as given the tax rules and externalities, the functions Ω and ϱ are given by:

$$\begin{aligned}\Omega(K_{t-1}, L_t) &= A G_t^{\eta} \frac{1 - z_L (\omega_t L_t/\omega L)^{\phi_L}}{1 + z_c (C_t/C)^{\phi_c}} \omega(K_{t-1}/L_t) \\ \varrho(K_{t-1}, L_t) &= A[1 - z_K (\rho_t K_{t-1}/\rho K)^{\phi_K}] \rho(K_{t-1}/L_t)\end{aligned}$$

where $\rho(K/L)$ and $\omega(K/L)$ are given by Definition 1 and $\Gamma(K_{t-1,} L_t) = \gamma(L_t)$.

To ease the exposition we will present separately the different types of taxation. We start with the case of capital taxation, without considering public spending externalities in preferences. In this case, market imperfections only appear in the ϱ function so that, using (2) we have $\alpha_{j,i} = \beta_{j,i} = 0$ for $i = K, L$ and $j = L, \Gamma$, $\alpha_{K,K} = -\phi_K \frac{z_K}{1-z_K}$, $\alpha_{K,L} = 0$ and $\beta_{K,K} = -\alpha_{K,K}(1-s) = -\beta_{K,L}$. Therefore, Assumption 5 is satisfied. Under θ small, Assumptions 3 and 7 imply that $0 < \phi_K < \frac{(1-z_K)s}{z_K(1-s)}$ so that $I_2 > 0$, $I_4 - I_3 < 0$, and $I_4 - I_3 + \theta I_2 < 0$. This means that we are in configuration (i) of Case 1, where $0 < S_1 < 1$, and since we have $\alpha_{K,K} < 0$ the steady state is always a saddle (see Proposition 2).

In the case of labor income taxation only, and considering public spending externalities in preferences, we have only imperfections through the Ω function, i.e. $\alpha_{j,i} = \beta_{j,i} = 0$ for $i = K, L$ and $j = K, \Gamma$, $\alpha_{L,L} = \eta(1+\phi_L) - \phi_L \frac{z_L}{1-z_L}$,

[56] See Lloyd-Braga, Modesto and Seegmuller (2006) for more details.

$\alpha_{L,K} = 0$ and $\beta_{L,L} = -\alpha_{L,L}s = -\beta_{L,K}$. As before, Assumption 5 is satisfied. Moreover, Assumptions 3 and 7 imply that $0 < \alpha_{L,L} < 1$, i.e. $-\frac{\eta(1-z_L)}{\eta(1-z_L)-z_L} < \phi_L < \frac{(1-\eta)(1-z_L)}{\eta(1-z_L)-z_L}$ if $\eta > \frac{z_L}{1-z_L}$, or $\frac{(1-\eta)(1-z_L)}{\eta(1-z_L)-z_L} < \phi_L < -\frac{\eta(1-z_L)}{\eta(1-z_L)-z_L}$ if $\eta < \frac{z_L}{1-z_L}$. In this case, $I_4 - I_3 + \theta I_2 < 0$, $I_2 < 0$, and $I_4 - I_3$ can be negative or positive but satisfying $I_4 - I_3 < -\theta I_2/2$.[57] This means that we are in configuration (ii) of Case 1 where $|S_1| > 1$, with $\alpha_{K,K} = 0$ so that Proposition 3.1 applies and indeterminacy may emerge if $\sigma > \sigma_{H1} = \frac{s(1+\alpha_{L,L})-\theta(1-s)}{\alpha_{L,L}} = \frac{(1-z_L)[s(1+\eta)-\theta(1-s)]+\phi_L s[\eta(1-z_L)-z_L]}{\eta(1-z_L)+\phi_L[\eta(1-z_L)-z_L]}$. Remark that $\sigma_{H1} < 1$ for $[s-\theta(1-s)]/(1-s) < \alpha_{L,L} < 1$, i.e., under a Cobb-Douglas production function, the distortion introduced and its variability cannot be too small.

From the previous discussion, it is easy to see that without public spending externalities in preferences ($\eta = 0$), indeterminacy is only possible if $-\frac{1-z_L}{z_L} < \phi_L < 0$, i.e., constant tax rates or tax rates that vary positively with the tax base promote determinacy. Moreover, in this case, indeterminacy for values of σ above one requires $-\frac{1-z_L}{z_L} < \phi_L < -\frac{1-z_L}{z_L}\frac{s-\theta(1-s)}{(1-s)}$, i.e., ϕ_L cannot be too close to zero.[58] However, by direct inspection of the expressions of $\alpha_{L,L}$ and σ_{H1}, we notice that these conclusions are no longer valid in the presence of public spending externalities in preferences ($\eta > 0$) where indeterminacy remains possible under a constant or a positively elastic tax rate.

In the case of consumption taxation only, imperfections again only appear in the Ω function, i.e. $\alpha_{j,i} = \beta_{j,i} = 0$ for $i = K, L$ and $j = K, \Gamma$, and Assumption 5 is satisfied. Considering for simplicity the case without public spending externalities in preferences ($\eta = 0$), we have $\alpha_{L,L} = -\frac{z_c\phi_c}{1+z_c(1+\phi_c)}\psi$, $\alpha_{L,K} = -\frac{z_c\phi_c}{1+z_c(1+\phi_c)}(1-\psi)$, $\beta_{L,L} = -\alpha_{L,L}\beta s$ and $\beta_{L,K} = -\beta_{L,L}$, where $\psi = \theta(1-s)/[\theta(1-s)+(1-\beta)s]$. Moreover, for θ sufficiently weak, Assumptions 3 and 7 imply that $1/\beta > \alpha_{L,L} > 0$, i.e., that $-\frac{(1+z_c)}{z_c}\frac{1}{1+\beta\psi} < \phi_c < 0$, so that $I_4 - I_3 + \theta I_2 < 0$ and $I_2 < 0$. Therefore, if ϕ_c is not too negative, $I_4 - I_3 < 0$ and we are in configuration (iia) of Case 1. On the contrary, if ϕ_c is sufficiently negative, we have $0 < I_4 - I_3 < -\theta I_2/2$ and configuration (iib) of Case 1 applies.[59] Using Proposition 3 with $\alpha_{K,K} = 0$, indeterminacy may emerge if $\sigma > \sigma_{H1} = \frac{(1+z_c)[s-\theta(1-s)]+z_c\phi_c[s(1-\beta\psi)-\theta(1-s)]}{-z_c\phi_c\psi}$. In the particular case where government spending is constant ($\phi_c = -1$), indeterminacy occurs for

[57] We assume that $2[s-\theta(1-s)] > \theta(1-s)$ which implicitly requires a sufficiently weak θ.

[58] Similarly, if we fix the value of $\phi_L < 0$, indeterminacy requires a sufficiently high value for z_L. For example, for $\eta = 0$ and $\phi_L = -1$ (the case considered in Schmitt-Grohé and Uribe (1997), Pintus (2003) and Gokan (2005) of a constant government spending), indeterminacy only emerges for $\sigma \geq 1$ if $z_L > [s-\theta(1-s)]/[1-\theta(1-s)]$.

[59] As before, this requires a sufficiently weak θ.

σ higher than one provided z_c is larger than a lower bound ($z_c > [s - \theta(1 - s)]/(1 - s\beta)\psi$).

5.1.4 Externalities in Consumption Preferences

Several works have modified consumers' preferences introducing consumption and/or leisure externalities, that affect respectively utility for consumption and leisure, to analyze the role of these externalities in preferences on equilibrium dynamics. In this section we discuss only the case of consumption externalities that will modify the Ω function. Leisure externalities, that rather affect the Γ function, will be discussed in the next section.

The existence of consumption externalities corresponds to the idea that individual utility of consumption is affected by the consumption of others (envy or altruism), so that aggregate or average consumption becomes an argument of the utility function (Alonso-Carrera et al. (2005), Gali (1994), Ljungqvist and Uhlig (2000), Weder (2000b)). In our framework, this amounts to consider that workers' utility is given by $U(C^w_{t+1}\varphi(\overline{C}_{t+1})/B) - V(L_t)$, where $\overline{C}_{t+1}$ denotes average consumption and $\varphi(\overline{C}_{t+1})$ the externalities function,[60] so that, at equilibrium, the Ω function is given by:

$$\Omega(K_{t-1}, L_t) = A\varphi(C_t)\omega(K_{t-1}/L_t)$$

where $\omega(K/L)$ is given in Definition 1. We also have $\varrho(K_{t-1}, L_t) = A\rho(K_{t-1}/L_t)$ and $\Gamma(K_{t-1,}L_t) = \gamma(L_t)$, as defined in Definition 1.

Since we have only imperfections through the Ω function, $\alpha_{j,i} = \beta_{j,i} = 0$ for $i = K, L$ and $j = K, \Gamma$. However, to compute the values of $\alpha_{L,i}$ and $\beta_{L,i}$ ($i = K, L$), we have to precise whether individual workers compare themselves to the average worker or to the average consumer, i.e., whether $C = C^w$ or $C = C^w + C^c$. Denote by υ the elasticity of φ with respect to C, evaluated at the steady state. In the first case ($C = C^w$), we have $\alpha_{L,L} = \upsilon$, $\alpha_{L,K} = 0$ and $\beta_{L,L} = -\alpha_{L,L}s = -\beta_{L,K}$. As in the previous examples, Assumption 5 is satisfied. Moreover, Assumptions 3 implies $-1 < \upsilon < 1$and Assumption 7, required for indeterminacy, imply that $\upsilon > 0$, so that $I_2 < 0$. Notice that indeterminacy is only possible when consumption externalities are of the "keeping-up with the Joneses" type, since $\upsilon > 0$ is required.[61] Considering θ small, [62] we have $I_4 - I_3 < -\theta I_2/2$, and configuration (ii) of Case 1 applies. From Proposition 3, taking into account that $\alpha_{K,K} = 0$,

[60] We do not introduce externalities into capitalists preferences because, since they have a log-linear utility function, such externalities would not affect the dynamics.

[61] $\partial^2 U/\partial C \partial \bar{C} > 0$, so that an increase in average consumption in the economy as a whole renders more valuable any addition to the individual own consumption.

[62] We assume that $2[s - \theta(1 - s)] > \theta(1 - s)$, which is ensured by θ sufficiently weak.

indeterminacy may emerge as soon as $\sigma > \sigma_{H1} = [s(1+\upsilon) - \theta(1-s)]/\upsilon$ (see Proposition 3.1). For $[s - \theta(1-s)]/(1-s) < \upsilon < 1$ we have $\sigma_{H1} < 1$. Therefore, in the Cobb-Douglas case, indeterminacy requires an elasticity of externalities higher than a positive lower bound, i.e., not too close to zero. This is important to note that the parameters $\alpha_{j,i}$ and $\beta_{j,i}$ are equivalent to those of labor taxation only, presented above.[63] Hence, from the point of view of local indeterminacy and dynamics, the two models are perfectly similar, i.e, the mechanisms operating for indeterminacy are the same, even if their economic interpretations are different.

When $C = C^w + C^c$, the results are slightly different. Now $\alpha_{L,L} = \upsilon\psi$, $\alpha_{L,K} = \upsilon(1-\psi)$ and $\beta_{L,L} = -\alpha_{L,L}s\beta = -\beta_{L,K}$, with $\psi = \theta(1-s)/[(1-\beta)s + \theta(1-s)]$, while $\alpha_{j,i} = \beta_{j,i} = 0$ for $i = K, L$ and $j = K, \Gamma$. Assumption 5 still holds. Taking into account that θ is small, Assumptions 3 and 7 imply that $0 < \upsilon < 1/\psi\beta$, so that $I_4 - I_3 + \theta I_2 < 0$ and $I_2 < 0$. Again we have consumption externalities of the "keeping-up with the Joneses" type. Since θ is assumed to be sufficiently weak, $I_4 - I_3 < 0$ or $0 < I_4 - I_3 < -\theta I_2/2$. We are in configuration (ii) of Case 1 and, as $\alpha_{K,K} = 0$, Proposition 3.1 applies. Indeterminacy may emerge if $\sigma > \sigma_{H1} = [s(1 + \upsilon\psi\beta) - \theta(1-s)]/(\upsilon\psi)$. Again, to get indeterminacy when the production function is Cobb-Douglas, υ can not be below a positive threshold, given by $[s - \theta(1-s)]/\psi(1-\beta s)$. As before, it is important to notice that this model is equivalent to the model with consumption taxation only, presented earlier.[64] The distortions introduced operate exactly in the same way, although they have very different microeconomic foundations.

We can therefore conclude that labor and consumption taxation on the one hand, and consumption externalities on the other hand, introduce the same type of distortions, sharing therefore the same mechanisms for indeterminacy.

5.2 Examples where $\alpha_{\Gamma,i} \neq 0$ and $\beta_{\Gamma,i} \neq 0$.

We start with examples where these imperfections only affect the Γ function (leisure externalities, efficiency wages). Then, we present examples were the imperfections introduced have also an influence on the Ω and ϱ functions (unions). With respect to the previous examples, we will see that, since the Γ function is modified, different configurations can apply in the models that we present below.

[63]Taking $\upsilon = \eta(1+\phi_L) - \phi_L \frac{z_L}{1-z_L}$, $\alpha_{j,i}$ and $\beta_{j,i}$ are exactly the same in both models.

[64]The parameters $\alpha_{j,i}$ and $\beta_{j,i}$ are identical in both models for $\upsilon = -\frac{z_c\phi_c}{1+z_c(1+\phi_c)}$.

5.2.1 Leisure externalities

The idea behind leisure externalities is that an individual's utility from leisure is affected by the amount of labor supplied by other people. Let the utility function of workers be written as $C^w_{t+1}/B - \overline{L}^{\mu}_t L^{\epsilon_\gamma}_t$, where $\mu \in \Re$ is a parameter and $\overline{L}_t$ denotes aggregate labor, which is taken as given by individual workers, but modifies their welfare. Solving the model with this utility function, imperfections only appear through the Γ function i.e. $\alpha_{j,i} = \beta_{j,i} = 0$ for $\{i, j\} = \{K, L\}$. Moreover, it is also easy to see that distortions only appear through the parameter $\alpha_{\Gamma,L} = \mu$, i.e., $\beta_{\Gamma,i} = \alpha_{\Gamma,K} = 0$.

Most of the papers that have introduced leisure externalities (Benhabib and Farmer (2000), Weder (2004)) assumed that the desutility of work is greater when an individual is working more than the other members of society, i.e., $\alpha_{\Gamma,L} < 0$. Note that this inequality is in fact required for indeterminacy when inputs are sufficiently substitutes, since it is imposed by our assumptions required for indeterminacy (Assumption 7 for Case 1 and Assumption 11.1 for Case 2). Indeed, from Assumption 7 for Case 1 and from Assumption 11 for Case 2, we will focus on the cases where $-2 < \alpha_{\Gamma,L} < 0$.[65]

Simple computations give us $I_2 = (1 - s)\alpha_{\Gamma,L} < 0$, $I_4 - I_3 = \theta(1 - s) - s(1 + \alpha_{\Gamma,L})$ and $I_4 - I_3 + \theta I_2 = -(1 + \alpha_{\Gamma,L})[s - \theta(1 - s)]$. Then, for $-1 < \alpha_{\Gamma,L} < 0$ we have $I_4 - I_3 + \theta I_2 < 0$, so that Case 1 applies. Let us define $\alpha^*_{\Gamma,L} \equiv [2[s - \theta(1 - s)] - 2\sqrt{s[s - \theta(1 - s)]}]/[\theta(1 - s)]$. If $\alpha^*_{\Gamma,L} < \alpha_{\Gamma,L} < 0$, configurations (ii) and (iii) apply. From Propositions 3.1 and 4.1, indeterminacy requires $\sigma > \sigma_{H1}$.[66] Since $\sigma_{H1} < 1$ when $\alpha_{\Gamma,L} < -[s - \theta(1-s)]/[1-\theta(1-s)]$, we can conclude that indeterminacy is possible in the Cobb-Douglas case for $\alpha^*_{\Gamma,L} < \alpha_{\Gamma,L} < -[s - \theta(1 - s)]/[1 - \theta(1 - s)]$.

For $\alpha_{\Gamma,L} < \alpha^*_{\Gamma,L}$, indeterminacy requires a different condition on the elasticity of capital-labor substitution. For $-1 < \alpha_{\Gamma,L} < \alpha^*_{\Gamma,L}$, configuration (iv) of Case 1 applies, so that indeterminacy may emerge as soon as $\sigma > \sigma_{H3}$. From Proposition 5.1. the steady state is locally indeterminate if $\varepsilon_{\gamma_F} < \varepsilon_\gamma < \varepsilon_{\gamma_H}$ when $\sigma_{H3} < \sigma < \sigma_F$, if $1 \le \varepsilon_\gamma < \varepsilon_{\gamma_H}$ when $\sigma_F < \sigma < \sigma_{H_2}$, and if $1 \le \varepsilon_\gamma < \varepsilon_{\gamma_T}$ when $\sigma > \sigma_{H_2}$. If $\alpha_{\Gamma,L} < -1$, as in Benhabib and Farmer (2000) that set $\alpha_{\Gamma,L} = -1.23$, we have $I_4 - I_3 + \theta I_2 > 0$, so that Case 2 applies. Simple computations show that we have $S_1 < S_D$, so that we are in configuration (v) of Case 2. Then, since $\alpha_{\Gamma,K} = \alpha_{L,K} = 0$, Proposition 6.2 applies and the conditions for indeterminacy are qualitatively similar to those obtained in Configuration (v). However, quantitatively the results are different. Note that $\sigma_F = \frac{2s+(1-s)\theta(2+\alpha_{\Gamma L})}{2(2+\alpha_{\Gamma L})}$. Hence $\sigma_F < 1$ for $\alpha_{\Gamma,L} > \alpha^{**}_{\Gamma,L} \equiv \frac{-2[2-s-\theta(1-s)]}{2-\theta(1-s)}$,

[65] We can easily see that Assumptions 3 and 5 are ensured.

[66] Note that configuration (iii) requires more negative values of $\alpha_{\Gamma,L}$ than configuration (ii).

with $-2 < \alpha^{**}_{\Gamma,L} < -1$. Hence, while in configuration (iv) indeterminacy emerges with a Cobb-Douglas tecnhology when the private labor supply is infinitely elastic, the same does not always apply in configuration (v), more precisely, when $\alpha_{\Gamma,L}$ takes values such that $-2 < \alpha_{\Gamma,L} < \alpha^{**}_{\Gamma,L}$. However, it is possible that, for values of externality degrees falling in this interval, indeterminacy occurs with a Cobb-Dougals technology, if $\epsilon_\gamma > \epsilon_F > 1$.[67] This shows that we shoud not limit our analysis to the case of an infinitely elastic labor supply, when we want to fully study the role of some labor market distortions on the occurrence of indeterminacy.

5.2.2 Efficiency Wages

Several papers have analyzed the role of efficiency wage arguments on the emergence of indeterminacy. One can refer to Coimbra (1999), Nakajima (2006) and Grandmont (2006). The example we present here follows closely Grandmont (2006), where efficiency wage, involuntary unemployment and unemployment insurance (unemployment compensation is a constant rate of wages, finaced by taxation on labor income at a uniform rate) are introduced in a otherwise standard finance constrained economy, as developed by Woodford (1986). The distortions introduced in Grandmont (2006) only affect the Γ function: $\Gamma(K_{t-1}, L_t) = g(L_t)$, where $g(L)$ stands for aggregate consumption of employed and unemployed workers. Note that $0 < \epsilon_g = Lg'(L)/g(L) < 1$, with ϵ_g close to 1 when there is weak unemployment insurance and is decreasing to zero when unemployment insurance becomes larger. Moreover, due to the existence of a constant reservation wage (not depending on the number of labor units supplied)[68] $\epsilon_\gamma = 1$. Hence, $-1 < \alpha_{\Gamma,L} = (\epsilon_g - 1) < 0$. Since market imperfections only affect the Γ function, we have $\alpha_{i,j} = \beta_{i,j} = 0$ for $\{i, j\} = \{K, L\}$. Moreover, $\beta_{\Gamma,L} = \beta_{\Gamma,K} = \alpha_{\Gamma,K} = 0$.

It is easy to see that Assumptions 3, 5 and 7 are satisfied. We also have $I_2 = (1 - s)\alpha_{\Gamma,L} < 0$, $I_4 - I_3 = \theta(1 - s) - s(1 + \alpha_{\Gamma,L})$ and $I_4 - I_3 + \theta I_2 = -(1 + \alpha_{\Gamma,L})[s - \theta(1 - s)] < 0$. This last inequality means that this model provides an example of Case 1. In fact, by direct inspection of the parameters $\alpha_{i,j}$ and $\beta_{i,j}$, this economy with efficiency wages and unemployment insurance is formally identical to the example, presented above, on leisure externalities with $-1 < \alpha_{\Gamma,L} < 0$, taking into account that here $\epsilon_\gamma = 1$. Therefore, the analysis is similar to the previous one and we obtain the same qualitative

[67] This will happen if $\sigma_{H3} < 1$.

[68] Each individual worker supplies one unit of labor with a labor disutility that depends on the level of effort. Since at equilibrium the level of effort is constant, the indiviudual labor supply is infinitely elastic.

results, provided we properly restrain the possible domain of values for $\alpha_{\Gamma,L}$.

For $-1 < \alpha_{\Gamma,L} < \alpha^*_{\Gamma,L}$, the model is in configuration (iv): given Proposition 5.1, and since $\epsilon_\gamma = 1$, indeterminacy emerges for $\sigma > \sigma_F = \frac{2s+(1-s)\theta(1+\epsilon_g)}{2(1+\epsilon_g)}$, with $\sigma_F < 1$, since $\epsilon_g > 0$. For $\alpha^*_{\Gamma,L} < \alpha_{\Gamma,L} < 0$, we are in configurations (ii) and (iii). Propositions 3.1 and 4.1 apply, i.e., indeterminacy emerges for $\sigma > \sigma_{H1}$. Since $\sigma_{H1} < 1$ requires $0 < \epsilon_g < (1-s)/[1-\theta(1-s)]$, we see that unemployment insurance cannot be arbitrarily weak when the technology is represented by the standard Cobb-Douglas production function.[69] However, the range of values for ϵ_g such that indeterminacy emerges for all $\sigma \geq 1$ is compatible with a wide and quite plausible range of values of unemployment and unemployment insurance rates.[70]

5.2.3 Unions

Efficiency wage is not the unique labor market imperfection considered in the literature. Using a financed constrained economy of the Woodford type, Lloyd-Braga and Modesto (2006) and Dufourt et al. (2006) have introduced union power and unemployment to analyze their role on local indeterminacy. In both papers, wages and employment are determined through an efficient bargaining mechanism between unions and firms. Unions are able to set wages above a reservation wage, with a markup factor $\mu(K,L) = \frac{1-\alpha s(a)}{1-s(a)} \geq 1$, $a = K/L$, increasing in the (constant) bargaining power of unions $1-\alpha \in [0,1)$. Moreover, employment is determined by the equality between the reservation wage and the marginal productivity of labor.[71]

Lloyd-Braga and Modesto (2006) do not only consider labor market imperfections, but also introduce productive labor externalities that positively affect the total productivity of factors. More specifically, they consider the production function $F(K,L) = ALf(a)\xi(L)$, where the strictly increasing function $\xi(L)$ stands for the externalities. In this model, market imperfections appear in the three functions Ω, ϱ and Γ, respectively given by:

$$\begin{aligned}
\Omega(K_{t-1}, L_t) &= A\mu(K_{t-1}/L_t)\xi(L_t)\omega_t \\
\varrho(K_{t-1}, L_t) &= A\alpha\xi(L_t)\rho_t \\
\Gamma(K_{t-1}, L_t) &= \mu(K_{t-1}/L_t)\gamma_t
\end{aligned}$$

[69] From the previous example on leisure externalities, remember that $\sigma_{H1} < 1$ requires $\alpha_{\Gamma,L} < -[s-\theta(1-s)]/[1-\theta(1-s)]$.

[70] See Grandmont (2006) for a more detailed discussion.

[71] Here the reservation wage is due to the existence of home production, workers supplying inelastically one unit of labor. Note also that the case of a perfectly competitive labor market would be obtained with $\mu(K,L) = 1$, i.e., $\alpha = 1$.

where ρ_t, ω_t and γ_t are defined in Definition 1. In this example, due to the existence of a reservation wage, we have that $\gamma_t = L_t$, i.e., $\epsilon_\gamma = 1$. Using the elasticity of the markup with respect to a, given by $\frac{s(1-\alpha)}{1-\alpha s}\frac{\sigma-1}{\sigma}$, and the elasticity of $\xi(L)$, given by $\varepsilon_{\xi,L} > 0$, both evaluated at the steady state, we obtain $\alpha_{\Gamma,K} = \alpha_{L,K} = -\alpha_{\Gamma,L} = -\beta_{\Gamma,K} = \beta_{L,L} = -\beta_{L,K} = \beta_{\Gamma,L} = \frac{s(1-\alpha)}{1-\alpha s} \in (0,s)$, $\alpha_{L,L} = \alpha_{\Gamma,L} + \varepsilon_{\xi,L}$, $\alpha_{K,L} = \varepsilon_{\xi,L}$ and $\alpha_{K,K} = \beta_{K,L} = \beta_{K,K} = 0$. One can see that Assumptions 5 and 7 are satisfied and $I_2 = -(\frac{1-s}{1-\alpha s} + \varepsilon_{\xi,L})\varepsilon_{\xi,L} < 0$. Assumption 3 implies that $\alpha s > \theta[1-s+\varepsilon_{\xi,L}(1-\alpha s)]$, so that $I_4 - I_3 + \theta I_2 < 0$ and Case 1 applies. Furthermore, under $I_4 - I_3 < 0$, as assumed in Lloyd-Braga and Modesto (2006), the model enters configuration (iia), i.e., indeterminacy emerges for $\sigma > \sigma_{H1} = \frac{[s-\theta(1-s)](1-\alpha s)+\theta s(1-\alpha)\varepsilon_{\xi,L}}{\varepsilon_{\xi,L}[1-\alpha s+\theta s(1-\alpha)]}$ (see Proposition 3.1). Remark finally that when the technology is Cobb-Douglas, indeterminacy requires $\sigma_{H1} < 1$, and thereby $\varepsilon_{\xi,L} > s - \theta(1-s)$, i.e., labor productive externalities not too small.

Dufourt et al. (2006) do not introduce productive externalities (the technology exhibits constant returns to scale), but, besides unions, they also consider the existence of imperfect unemployment insurance: a constant real unemployment benefit financed by taxes on those employed. In their case, the functions Ω, ϱ and Γ can be written as:

$$\begin{aligned}\Omega(K_{t-1}, L_t) &= A\mu(K_{t-1}/L_t)\omega_t \\ \varrho(K_{t-1}, L_t) &= A\alpha\rho_t \\ \Gamma(K_{t-1}, L_t) &= b\frac{\mu(K_{t-1}/L_t)}{L_t}\gamma_t\end{aligned}$$

where ρ_t, ω_t and γ_t are given in Definition 1, and $b > 0$ is the real unemployment benefit. In this example, again due to the existence of a reservation wage[72] we have that $\gamma_t = L_t$, i.e., $\epsilon_\gamma = 1$. Comparing this model with Lloyd-Braga and Modesto (2006), we can see that, except the existence or not of productive externalities, the main difference between them concerns $\Gamma(K, L)$. Indeed, the unemployment insurance scheme considered introduces another distortion that operates only through the parameter $\alpha_{\Gamma,L}$. Moreover, in Dufourt et al. (2006), there is a significant difference between the parameters $\alpha_{\Gamma,L}$ and $\alpha_{L,L}$.

Using (2), we get $\alpha_{K,j} = \beta_{K,j} = 0$ for $j = K, L$, $\alpha_{\Gamma,K} = \alpha_{L,K} = -\alpha_{L,L} = -\beta_{\Gamma,K} = \beta_{L,L} = -\beta_{L,K} = \beta_{\Gamma,L} = \frac{s(1-\alpha)}{1-\alpha s} \in (0,s)$, and $\alpha_{\Gamma,L} = \alpha_{L,L} - 1$. One can check that Assumption 5 is satisfied, Assumption 7 is ensured by

[72] In Dufourt et. al (2006) workers also supply inelastically 1 unit of labor. Due to the unemployment benefit there is a reservation wage below which individuals prefer not to work.

$s < 1/2$ and Assumption 3 by $\alpha s > \theta(1-s)$. Moreover, $I_1 = 0$, $I_2 = -\frac{(1-s)^2}{1-\alpha s} < 0$ and $I_4 - I_3 + \theta I_2 = -(1-s)s(1-\alpha)/(1-\alpha s) < 0$, implying that Case 1 applies. Then, for $\theta(1-s)/s < \alpha < 1 - [\theta(1-s)/s(4-\theta)]$, configurations (ii) and (iii) are the relevant ones and indeterminacy emerges for $\sigma > \sigma_{H1} = s$ (see Propositions 3.1 and 4.1). For $1 - [\theta(1-s)/s(4-\theta)] < \alpha \leq 1$, configuration (iv) applies. Since $\epsilon_\gamma = 1$, indeterminacy emerges when $\sigma > \sigma_F = \frac{2s[\alpha(2-s)-1]+\theta(1-s)(1-\alpha s)}{2[1-s(2-\alpha)]}$ (see Proposition 5.1). Remark that since, for $1 - [\theta(1-s)/s(4-\theta)] < \alpha \leq 1$, we have $s < \sigma_F < 1$. We deduce that the steady state of this economy is always indeterminate, if the technology is of a Cobb-Douglas type.

5.2.4 Public spending externalities in preferences

In this example, the standard Woodford model is modified introducing public spending, financed by taxation on capital and labor incomes through a balanced budget rule, which will provide illustrations of Proposition 7. Moreover, we assume that government expenditures (G) provide services that affect not only workers' utility for consumption, but also their desutility of labor. To focus only on the role of public spending externalities on local indeterminacy, tax rates on capital and labor incomes, that we respectively note $\tau_K \in (0,1)$ and $\tau_L \in (0,1)$, are supposed to be constant and the level of government spending is defined by:

$$G_t = \tau_K \rho_t K_{t-1} + \tau_L \omega_t L_t$$

Let the utility of the representative worker be defined by $G_{t+1}^{\eta} C_{t+1}^{w}/B - G_t^{\mu} L_t^{\epsilon_\gamma}$, where η and μ are parameters representing, respectively, the elasticity of government spending affecting utility for consumption and desutility of labor. Then, we obtain:

$$\begin{aligned}
\Omega(K_{t-1}, L_t) &= [\tau_K A \rho_t K_{t-1} + \tau_L A \omega_t L_t]^{\eta}(1-\tau_L) A \omega_t \\
\varrho(K_{t-1}, L_t) &= (1-\tau_K) A \rho_t \\
\Gamma(K_{t-1}, L_t) &= [\tau_K A \rho_t K_{t-1} + \tau_L A \omega_t L_t]^{\mu} \gamma_t,
\end{aligned}$$

where ρ_t and ω_t are given in Definition 1.

By direct inspection of $\varrho(K_{t-1}, L_t)$, we immediately deduce that its elasticities are not affected by the distortions introduced, i.e., $\alpha_{K,i} = \beta_{K,i} = 0$.[73] Let us define $\psi \equiv \tau_L(1-s)/[\tau_L(1-s) + s\tau_K] \in (0,1)$. Computing the elasticities of $\Omega(K_{t-1}, L_t)$ and $\Gamma(K_{t-1}, L_t)$ with respect to K_{t-1} and

[73]This means that in a local dynamics point of view, the model that we study in this example is as if the function $\varrho(K_{t-1}, L_t)$ is not affected by government intervention.

L_t, evaluated at the steady state, we get $\alpha_{L,L} = \eta\psi$, $\alpha_{L,K} = \eta(1-\psi)$, $\beta_{L,L} = (1-s-\psi)\eta = -\beta_{L,K}$, and $\alpha_{\Gamma,L} = \mu\psi$, $\alpha_{\Gamma,K} = \mu(1-\psi)$, $\beta_{\Gamma,L} = (1-s-\psi)\mu = -\beta_{\Gamma,K}$. Before analyzing the conditions for indeterminacy, and to simplify the presentation, we further suppose $\eta > \max\{\mu, 0, -2/\psi - \mu\}$ and $\psi < \min\{1 - s[(1+\eta)/(1-s+\eta)], (1-\theta)/(1+\theta\eta)\} < (1-s)$. Moreover Assumption 3 implies that $\eta < s/(1-s-\psi)$, $\mu > -s/(1-s-\psi)$, and $\theta < [s - \eta(1-s-\eta)]/(1+\eta\psi)(1-s)$.

Since $\alpha_{K,i} = \beta_{K,i} = 0$, $I_2 < 0$ is a necessary condition for indeterminacy. Using the expressions above, this is equivalent to $\eta > \mu$. Straightforward computations show that for $\mu > \mu^a = \frac{\theta(1+\eta\psi)(1+\eta)(1-s)-[s-(1-s-\psi)\eta]}{(1-s)(1-\psi)}$ we have $I_4 - I_3 < 0$, and for $\mu > \mu^b = \frac{\eta(1-s)(1-\psi)-(1+\eta\psi)[s-\theta(1-s)]}{(1-s)[1-\psi-\theta(1+\eta\psi)]}$, where $\mu^b < \mu^a$, we have $I_4 - I_3 + \theta I_2 < 0$. Therefore for $\mu > \mu^a$ we obtain configuration (iia) of Case 1, for $\mu^b < \mu < \mu^a$, configurations (iib), (iii) and (iv) of Case 1 are the relevant ones. Indeterminacy emerges for $\sigma > \sigma_{H1} = \frac{s-(1-s-\psi)(\eta-\mu)-\theta(1-s)(1+\mu)}{\psi(\eta-\mu)}$ in configurations (ii) and (iii) (see Propositions 3.1 and 4.1). In configuration (iv) of Case 1 (see Proposition 5.1), indeterminacy requires $\varepsilon_{\gamma F} < \varepsilon_\gamma < \varepsilon_{\gamma_H}$ for $\sigma_{H_3} < \sigma < \sigma_F$ and $1 < \varepsilon_\gamma < \min\{\varepsilon_{\gamma_H}, \varepsilon_{\gamma_T}\}$ for $\sigma > \sigma_F$, with $\sigma_F \equiv \frac{2[s-(1-s-\psi)(\eta+\mu)]+\theta(1-s)(2+\eta+\mu)}{2[2+\psi(\eta+\mu)]}$. When $\mu < \mu^b$, we obtain Case 2. Configuration (v) appears for $\mu^c < \mu < \mu^b$, where $\mu^c = \frac{\eta[s-(1-s-\psi)\eta-(1-s)(1-\psi)]+(1+\eta\psi)[s-\theta(1-s)(1+\eta)]}{s-(1-s-\psi)\eta-(1-s)(1-\psi)}$, while for $\mu < \mu^c$ we obtain configuration (vi).[74] In configurations (v), indeterminacy occurs under similar conditions than in configuration (iv) (see Proposition 6.2). In configuration (vi), indeterminacy requires $\varepsilon_{\gamma_F} < \varepsilon_\gamma < \varepsilon_{\gamma_T}$ for $\sigma^{S_2} < \sigma < \sigma_F$ and $1 \leq \varepsilon_\gamma < \min\{\varepsilon_{\gamma_H}, \varepsilon_{\gamma_T}\}$ for $\sigma > \sigma^{S_2}$ (see Proposition 7.2).[75]

5.3 Discussion of the Examples

The above examples highlight some important results already stressed before. In all the examples presented that did not introduce market distortions affecting the generalized offer curve, only configurations (i) and (ii) of Case 1 emerged (see section 5.1). Configuration (i) appeared with negative productive externalities and with capital taxation. In both cases $\alpha_{K,K} < 0$ so that the steady state was never indeterminate (a saddle). Therefore, without distortions affecting the generalized offer curve, indeterminacy only appeared in configuration (ii), requiring a lower bound for the elasticity of substitution: $\sigma > \sigma_{H1}$. Moreover, we have also seen that this lower bound was only below unity for relatively high levels of distortions. Indeed, in the standard

[74] Note that μ^c falls into the appropriate range of values for μ if η is sufficiently large.
[75] Note that $\alpha_{\Gamma,K} - \alpha_{L,K} = -(\eta-\mu)(1-\psi) < 0$.

case of a Cobb-Douglas technology, indeterminacy requires either sufficiently high positive production, consumption or public spending externalities, high rates of labor income/consumption taxation, or sufficiently (negative) elastic tax rates. This shows that the indeterminacy mechanism operating in all these last examples of market distortions is quite similar, even if the economic interpretation of distortions across these examples is different. More specifically, we have shown that many models with product market imperfections (mark-up variability, taste for variety) can be seen as a particular case of positive productive externalities and that labor and consumption taxation on the one hand, and consumption externalities on the other hand are perfectly equivalent from a local dynamics point of view.

Once distortions affecting the offer curve are considered, the situation changes drastically, since new configurations are obtained. Indeed, besides configurations (*ii*) of Case 1 as above, configurations (*iii*)-(*iv*) of Case 1, and (*v*) -(*vi*) of Case 2 may also appear. This is what happens for instance with leisure externalities and/or public spending externalities in preferences, where configurations of Case 2 emerge when the difference between the elasticities of the generalized and private offer curve with respect to labor supply is sufficiently negative. Let us remark, however, that in configurations (*v*) and (*vi*), which are obtained when the degree of externalities is sufficiently negative and bounded away from zero, indeterminacy may be excluded for reasonable values of the elasticity of substitution between inputs, if the elasticity of labor supply at the individual level is sufficiently high. This is a new result, and shows that we should be careful in not constrain our analysis, on the role of some labor market distortions for the occurrence of indeterminacy, to the situation where an infinitely elastic labor supply is considered, although this assumption is widely used in the literature.

The model with efficiency wages can be seen as a particular case of, negative and not too large, aggregate labor externalities in leisure utility, so that configuration of Case 2 do not emerge. With unions, also only Case 1 emerges. However, with labor market rigidities, not only configuration (*ii*) appear, but configurations (*iii*) and (*iv*) are also obtained in the presence of unemployment insurance, either when union power is close to zero (the case of unions with constant real unemployment benefit/taxes) or when unemployment insurance is sufficiently large (the case of efficiency wages with a constant rate of unemployment compensation/ tax rate over the wage rate). In both examples, indeterminacy occurs with a Cobb-Douglas technology without requiring high degrees of distortions: in the example with unions, the steady state is always indeterminate when the union power is sufficiently weak; in the example with efficiency wages, indeterminacy occurs for plausible levels of the unemployment compensation rate, although it is not possible when this rate is arbitrarily weak.

6 Concluding Remarks

Our work developed a useful general methodology to analyze the role of market distortions on the emergence of local indeterminacy, bifurcations and business cycles driven by expectations, which we applied to a Woodford economy. We fully characterized the local dynamics, according to several different configurations for the parameters of the model. We then applied our results to several examples of specific market distortions and compared them.

Some results, already latent in previous works, are here confirmed and put in evidence. In particular, we found that capital market distortions do not seem to play a role for the occurrence of indeterminacy. On the contrary, we have seen that indeterminacy emerges under labor market rigidities, in the financially constrained Woodford economy, without imposing strange or implausible values for the parameters, whereas if distortions mainly affect the output market, indeterminacy requires parameters values that might be considered as less relevant from an empirical point of view. These findings suggest that, in economies where workers are financially constrained, the functioning of labor markets, which in the real world show significant deviations from the competitive paradigm, may create additional volatility along business cycles caused by self fulfilling volatile expectations. Empirical analysis on this issue is therefore an important direction for further research.

Finally, other results show that several standard different types of macrodynamic models have some similarities in terms of local dynamic stability. We have shown that indeterminacy, in the Woodford economy, can only occur with arbitrarily small **d**istortions if the elasticity of capital-labor substitution is high enough and the elasticity of private labor supply is strong enough. This result is indeed similar to those obtained in standard Ramsey and overlapping generations economies with productive externalities.[76] Hence, this seems to be a consistent result in these type of models, which share in common the fact that only future expectations in consumption decisions of consumers/workers are relevant, opening the room for the existence of cycles driven by self fulfilling volatile expectations. This fact may indeed explain why standard distortions in the capital accumulation equilibrium dynamic equation do not play a role for the emergence of local indeterminacy,

[76] See Lloyd-Braga et al. (2007) and Pintus (2006). There it is also shown that positive productive externalities from labor affecting the intertemporal arbitrage curve of consumers/workers, α_{LL}, must exist, while externalities from capital affecting the capital accumulation equation, α_{KK}, are not nedeed.

while distortions affecting the intertemporal arbitrage equilibrium condition for consumers are important. However, in all these standard models, the modeling of the behavior for capital dynamics is quite simple, productive capital being just rented by consumers/capitalists to firms. Strategic considerations by firms owning productive capital, which are usually disregarded, may create a new channel for the relevance of future expectations of capitalists/producers and change the results. Hence, although some works have already considered some of these aspects,[77] further research on this issue is welcome.

An important final remark is wortwhile. Due to our methodology, we were able to find several equivalences across different types of specific distortions, in terms of the general equilibrium dynamic equations and the respective local dynamics. This has some strong implications. In particular, this means that, if what we observe is aggregate and equilibrium variables, we might not be able to distinguish productive externalities from the existence of taste for variety or mark up variability in the output market, since both models are observationally equivalent; also, we might not be able to distinguish aggregate consumption externalities in private utility of consumption from taxation on labor income/consumption spending; we might not be able to distinguish the existence of efficiency wages with unemployment insurance from aggregate labor externalities in private utility of leisure. Moreover, the level of a specific distortion required for indeterminacy may be considered as implausibly high, while the same conditions on indeterminacy are obtained with another equivalent specific distortion for which the required degree of distortions is considered reasonable.

7 Appendix

7.1 Existence of a Steady State

A stationary equilibrium of the dynamic system (6)-(7) is a solution $(K, L) = (K_{t-1}, L_t)$ for all t, that satisfies:

$$\begin{aligned} A\varrho(K, L) &= \theta/\beta & (19) \\ (A/B)\Omega(K, L)L &= \Gamma(K, L) & (20) \end{aligned}$$

with $\theta \equiv 1 - \beta(1 - \delta)$.

The existence of a steady state can be established by choosing appropriately the two scaling parameters $A > 0$ and $B > 0$ so as to ensure that one

[77] See for instance d'Aspremont et. al (2000).

steady state coincides with $(K^*, L^*) = (1,1)$. From equation (19), we obtain a unique solution $A > 0$, determined by:

$$A = \theta/(\beta\varrho(1,1)) \tag{21}$$

Substituting (21) into (20), we then obtain the unique solution for B:

$$B = \frac{\theta\Omega(1,1)}{\beta\varrho(1,1)\Gamma(1,1)} \tag{22}$$

Proposition 8 (Existence of the normalized steady state) *Under Assumption 1, $(K^*, L^*) = (1,1)$ is a stationary solution of the dynamic system (6)-(7) if and only if A and B are the unique solutions of (21) and (22).*

7.2 Trace T and determinant D of the Jacobian matrix

To determine the trace T and the determinant D of the Jacobian matrix, we first differentiate the dynamic system (4) and (5) in the neighborhood of $(K, L) = (1,1)$ (see Proposition 8):

$$\frac{dK_t}{K} = (\theta\varepsilon_{\varrho,K} + 1)\frac{dK_{t-1}}{K} + \theta\varepsilon_{\varrho,L}\frac{dL_t}{L} \tag{23}$$

$$\frac{dL_{t+1}}{L} = \frac{\epsilon_{\Gamma,K} - \varepsilon_{\Omega,K}(1+\theta\varepsilon_{\varrho,K})}{1+\varepsilon_{\Omega,L}}\frac{dK_{t-1}}{K} + \frac{\varepsilon_{\Gamma,L} - \theta\varepsilon_{\Omega,K}\varepsilon_{\varrho,L}}{1+\varepsilon_{\Omega,L}}\frac{dL_t}{L} \tag{24}$$

We deduce the trace T and the determinant D of the associated Jacobian matrix, which correspond respectively to the sum and the product of the two roots of the characteristic polynomial $P(\lambda) \equiv \lambda^2 - T\lambda + D = 0$:

$$T = 1 + \frac{\varepsilon_{\Gamma,L} + \theta(\varepsilon_{\varrho,K}(1+\varepsilon_{\Omega,L}) - \varepsilon_{\Omega,K}\varepsilon_{\varrho,L})}{1+\varepsilon_{\Omega,L}} \tag{25}$$

$$D = \frac{\varepsilon_{\Gamma,L}(1+\theta\varepsilon_{\varrho,K}) - \theta\varepsilon_{\Gamma,K}\varepsilon_{\varrho,L}}{1+\varepsilon_{\Omega,L}} \tag{26}$$

Substituting the expressions given in Assumption 2 in these two equations, we obtain (9) and (10).

7.3 Expressions for critical values of ε_γ

7.3.1 The value of ε_{γ_H}

ε_{γ_H} is such that $D = 1$, which is equivalent to:

$$\begin{aligned}\varepsilon_{\gamma_H} = 1 + \{&\sigma[\alpha_{L,L} - \alpha_{\Gamma,L} + \theta(\alpha_{\Gamma,K}\alpha_{K,L} - \alpha_{K,K}(1+\alpha_{\Gamma,L}))] \\ &+ \theta[(1-s-\beta_{K,K})(1+\alpha_{\Gamma,L}) + \alpha_{\Gamma,K}(1-s+\beta_{K,L}) \\ &- \alpha_{K,L}\beta_{\Gamma,L}\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}}] - \beta_{\Gamma,L}(1+\theta\alpha_{K,K}) - (s-\beta_{L,L})\} \\ &/[\sigma(1+\theta\alpha_{K,K}) - \theta(1-s-\beta_{K,K})]\end{aligned} \tag{27}$$

7.3.2 The value of ε_{γ_F}

ε_{γ_F} is such that $1 + T + D = 0$. After some computations, we obtain:

$$\begin{aligned}\varepsilon_{\gamma_F} = 1 + \{&\sigma[2(2+\alpha_{L,L}+\alpha_{\Gamma,L}) + \theta(\alpha_{K,K}(2+\alpha_{L,L}+\alpha_{\Gamma,L}) - \alpha_{K,L}(\alpha_{L,K}+\alpha_{\Gamma,K}))] \\ &- 2(s-\beta_{L,L}-\beta_{\Gamma,L}) - \theta[(1-s-\beta_{K,K})(2+\alpha_{L,L}+\alpha_{\Gamma,L}) \\ &+ \alpha_{K,K}(s-\beta_{L,L}-\beta_{\Gamma,L}) + (\alpha_{L,K}+\alpha_{\Gamma,K})(1-s+\beta_{K,L}) \\ &+ \alpha_{K,L}(s-\beta_{L,L}-\beta_{\Gamma,L})\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}}]\}/[\theta(1-s-\beta_{K,K}) - \sigma(2+\theta\alpha_{K,K})]\end{aligned} \tag{28}$$

7.3.3 The value of ε_{γ_T}

ε_{γ_T} is such that $1 - T + D = 0$. After some computations, we obtain:

$$\begin{aligned}\varepsilon_{\gamma_T} = 1 + \{&(1-s-\beta_{K,K})(\alpha_{L,L}-\alpha_{\Gamma,L}) + \alpha_{K,K}(s-\beta_{L,L}+\beta_{\Gamma,L}) + \\ &+ (\alpha_{L,K}-\alpha_{\Gamma,K})(1-s+\beta_{K,L}) + \alpha_{K,L}(s-\beta_{L,L}+\beta_{\Gamma,L})\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}}\} \\ &/(1-s-\beta_{K,K}-\sigma\alpha_{K,K})\end{aligned} \tag{29}$$

7.4 Expressions for critical values of σ

7.4.1 The value of σ_{H_1}

σ_{H_1} is the critical value of σ such that $D_1(\sigma_{H_1}) = 1$.

$$\sigma_{H_1} \equiv \frac{s - \beta_{L,L} + \beta_{\Gamma,L}(1 + \theta\alpha_{K,K})}{\alpha_{L,L} - \alpha_{\Gamma,L} - \theta[\alpha_{K,K}(1 + \alpha_{\Gamma,L}) - \alpha_{\Gamma,K}\alpha_{K,L}]} - \frac{\theta[(1 - s - \beta_{K,K})(1 + \alpha_{\Gamma,L}) + \alpha_{\Gamma,K}(1 - s + \beta_{K,L}) - \alpha_{K,L}\beta_{\Gamma,L}\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}}]}{\alpha_{L,L} - \alpha_{\Gamma,L} - \theta[\alpha_{K,K}(1 + \alpha_{\Gamma,L}) - \alpha_{\Gamma,K}\alpha_{K,L}]} \tag{30}$$

7.4.2 The value of σ_F

The critical value σ_F is defined by $1 + D_1(\sigma_F) + T_1(\sigma_F) = 0$.[78]

$$\sigma_F \equiv \frac{(s - \beta_{LL} - \beta_{\Gamma,L})\left[2 + \theta(\alpha_{KK} + \alpha_{KL}\frac{1-s-\beta_{KK}}{1-s+\beta_{KL}}\right]}{(2 + \theta\alpha_{KK})(2 + \alpha_{LL} + \alpha_{\Gamma,L}) - \theta\alpha_{KL}(\alpha_{LK} + \alpha_{\Gamma,K})} + \frac{\theta[(1 - s - \beta_{KK})(2 + \alpha_{LL} + \alpha_{\Gamma,L}) + (1 - s + \beta_{KL})(\alpha_{LK} + \alpha_{\Gamma,K})]}{(2 + \theta\alpha_{KK})(2 + \alpha_{LL} + \alpha_{\Gamma,L}) - \theta\alpha_{KL}(\alpha_{LK} + \alpha_{\Gamma,K})} \tag{31}$$

7.4.3 The value of σ_{S_2}

σ_{S_2} is the value of σ such that $\varepsilon_{\gamma T} = \varepsilon_{\gamma_F}$, with $\alpha_{Ki} = \beta_{Ki} = 0$.

$$\sigma_{S_2} \equiv \frac{\theta\,(1 - s)\,(1 + \alpha_{LL} + \alpha_{LK}) + (s - \beta_{LL} - \beta_{\Gamma L})}{2\,(1 + \alpha_{LL}) + \alpha_{LK} - \alpha_{\Gamma K}}$$

7.5 Existence of σ_{H_2} and Proof of Lemma 3

Recall that when $\sigma = \sigma_{H_2}$ we have $\epsilon_{\gamma_H} = \epsilon_{\gamma_T}$, i.e. the Δ line goes through the point C (see Definition 3). To discuss the existence and uniqueness of σ_{H_2}, we consider first the configurations where $S_1 \in (0, 1)$, and then the remaining ones.

1. Configurations where $S_1 \in (0, 1)$.

When $D_1'(\sigma) < 0$ (as in configuration (i) of Case 1), the existence of σ_{H_2} requires $\alpha_{K,K} > 0$. Since $D_1'(\sigma) < 0$ and $S(\sigma)$ increases with σ (with $S(+\infty) > 1$), we deduce by direct geometrical considerations the existence and uniqueness of $\sigma_{H_2} (> \sigma_T)$, such that $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ for $\sigma < \sigma_{H_2}$, and $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$ for $\sigma > \sigma_{H_2}$.

[78] Note that using Assumption 5 the denominator of σ_F can also be written as $2(2 + \alpha_{L,L} + \alpha_{\Gamma,L}) - 2\theta[\alpha_{\Gamma,K}\alpha_{K,L} - \alpha_{K,K}(1 + \alpha_{\Gamma,L}) > 0$ from Assumption 7.

When $D_1'(\sigma) > 0$, Lemma 3 is helpful. Let us prove it. With $\alpha_{K,i} = \beta_{K,i} = 0$, $i = K, L$ (as in Case 2), note that $\epsilon_{\gamma_H} > \epsilon_{\gamma_T} \Leftrightarrow (\sigma - \sigma_{H_2})(\alpha_{\Gamma,K} - \alpha_{L,K}) > 0$, where $\sigma_{H_2} \equiv \frac{s-\beta_{L,L}}{1+\alpha_{L,L}} - \frac{\frac{s-\beta_{L,L}}{(1+\alpha_{L,L})(1-s)}\frac{I_4-I_3}{\theta}(S_1-S_D)}{(\alpha_{\Gamma,K}-\alpha_{L,K})(1+\alpha_{L,L})}$ is the value of σ such that $\epsilon_{\gamma_H}(\sigma_{H_2}) = \epsilon_{\gamma_T}(\sigma_{H_2})$. Hence, when $S_1 < S_D$ and $\alpha_{\Gamma,K} < \alpha_{L,K}$ or when $S_1 > S_D$ and $\alpha_{\Gamma,K} > \alpha_{L,K}$, we have $\sigma_{H_2} \leq \frac{s-\beta_{L,L}}{1+\alpha_{L,L}}$ so that $\sigma > \sigma_{H_2}$ for all σ under consideration. Otherwise, we get $\sigma_{H_2} > \frac{s-\beta_{L,L}}{1+\alpha_{L,L}}$. In the case where $\alpha_{\Gamma,K} = \alpha_{L,K}$, $(\sigma - \sigma_{H_2})(\alpha_{\Gamma,K} - \alpha_{L,K}) = \frac{s-\beta_{L,L}}{(1+\alpha_{L,L})^2(1-s)}\frac{I_4-I_3}{\theta}(S_1 - S_D)$. Therefore, $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$ for $S_1 > S_D$ and $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ for $S_1 < S_D$. Lemma 3 immediately follows.

2. Configurations where $S_1 > 1$ or $S_1 < 0$.

Consider first the case where $\alpha_{K,K} < 0$. Note that the equation $\epsilon_{\gamma_H} = \epsilon_{\gamma_T}$ is a polynomial of degree 2, i.e. has at most two solutions. Since $S(+\infty) \in (0,1)$, we can see geometrically that a solution $\sigma_{H_2} \in (\sigma_{H_1}, +\infty)$ must exist and the number of these solutions is odd. We deduce the uniqueness of $\sigma_{H_2}(> \sigma_{H_1})$, such that $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ for $\sigma < \sigma_{H_2}$, and $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$ for $\sigma > \sigma_{H_2}$.

Consider now that $\alpha_{K,K} = 0$. Note that in this particular case, ϵ_{γ_T} does not depend on σ. The equation $\epsilon_{\gamma_H} = \epsilon_{\gamma_T}$ has at most one solution $\sigma_{H_2} \in (\sigma_{H_1}, +\infty)$ and this solution σ_{H_2} is by continuity such that again $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ for $\sigma < \sigma_{H_2}$, and $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$ for $\sigma > \sigma_{H_2}$.

Finally, consider that $\alpha_{K,K} > 0$. We can see geometrically that if there is a solution σ_{H_2} to $\epsilon_{\gamma_H} = \epsilon_{\gamma_T}$ then $\sigma_{H_2} \in (\sigma_{H_1}, \sigma_T)$. The inequality $\epsilon_{\gamma_H} \leq \epsilon_{\gamma_T}$ is equivalent to $g(\sigma) \geq 0$, where

$$\begin{aligned} g(\sigma) \equiv\, & \alpha_{K,K}[\alpha_{L,L} - \alpha_{\Gamma,L} + \theta(\alpha_{\Gamma,L}\alpha_{K,L} - \alpha_{K,K}(1+\alpha_{\Gamma,L}))](\sigma - \sigma_T)(\sigma - \sigma_{H_1}) \\ & - \frac{I_2}{1+\alpha_{L,L}}[\sigma(1+\theta\alpha_{K,K}) - \theta(1 - s - \beta_{K,K})] \end{aligned}$$

This function describes a convex parabola with $g(0) > 0$, $g(\sigma_{H_1}) > 0$, $g(\sigma_T) > 0$ and $g(+\infty) = +\infty$. Hence, either it can exist two solutions (if $g'(\sigma_{H_1}) < 0$) or none (if $g'(\sigma_{H_1}) \geq 0$) to the equation $g(\sigma) = 0$. As:

$$\begin{aligned} g'(\sigma) = \alpha_{K,K}&[\alpha_{L,L} - \alpha_{\Gamma,L} + \theta(\alpha_{\Gamma,L}\alpha_{K,L} - \alpha_{K,K}(1+\alpha_{\Gamma,L}))] \\ &[2\sigma - (\sigma_T + \sigma_{H_1})] - \frac{I_2}{1+\alpha_{L,L}}(1+\theta\alpha_{K,K}) \end{aligned}$$

We deduce that $g'(\sigma_{H_1}) \geq 0$ is equivalent to:

$$\begin{aligned} I_2 \leq\, & \alpha_{K,K}(1+\alpha_{L,L})[\alpha_{L,L} - \alpha_{\Gamma,L} + \theta(\alpha_{\Gamma,L}\alpha_{K,L} - \alpha_{K,K}(1+\alpha_{\Gamma,L}))] \\ & (\sigma_{H_1} - \sigma_T)/(1+\theta\alpha_{K,K}) \end{aligned} \tag{32}$$

Hence, when this inequality is satisfied, there is no solution to $g(\sigma) = 0$, because $g(\sigma_T) \geq g(\sigma_{H_1}) > 0$. This implies that $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ for all $\sigma > \sigma_{H_1}$, i.e. the half-line Δ always goes above point C.

7.6 Existence of σ_{H_3}

Using (27) and (28) we have that $\epsilon_{\gamma H} = \epsilon_{\gamma F} \Leftrightarrow h(\sigma) = 0$, and $\epsilon_{\gamma H} > \epsilon_{\gamma F} \Leftrightarrow h(\sigma) > 0$, where:

$$\begin{aligned} h(\sigma) \equiv & [\sigma(2 + \theta\alpha_{K,K}) - \theta(1 - s - \beta_{K,K})][\alpha_{L,L} - \alpha_{\Gamma,L} + \theta(\alpha_{\Gamma,K}\alpha_{K,L} \\ & - \alpha_{K,K}(1 + \alpha_{\Gamma,L}))](\sigma - \sigma_{H_1}) + 2[\sigma(1 + \theta\alpha_{K,K}) - \theta(1 - s - \beta_{K,K})] \\ & [2 + \alpha_{L,L} + \alpha_{\Gamma,L} - \theta(\alpha_{\Gamma,K}\alpha_{K,L} - \alpha_{K,K}(1 + \alpha_{L,L}))](\sigma - \sigma_F), \end{aligned}$$

By Definition 6, σ_{H_3} is a value of σ such that $\epsilon_{\gamma H} = \epsilon_{\gamma F}$ and therefore it must be a solution of $h(\sigma) = 0$.

Since $h(\sigma)$ is a polynomial of degree 2, the equation $h(\sigma) = 0$ has at most two solutions. Here we limit our analysis to configurations (iv) of Case 1, and (v) and (vi) of Case 2 since σ_{H_3} may only be relevant under these configurations. In all of them, since Δ is positively sloped (see Lemma 1) pointing upwards, it can only go through point B if its initial point in Δ_1 is on the left of line (AB), i.e., $\sigma_{H_3} < \sigma_F$. Also, in all these three configurations the polynomial $h(\sigma)$ is a convex function of σ since the coefficient of the quadratic term σ^2 is positive.[79]

Consider first configuration (iv) of Case 1. We can see geometrically that if there is a $\sigma_{H_3} > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$ then it must satisfy $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma_{H_1} < \sigma_{H_3} < \sigma_F$. Straight computations show that in this configuration $h(\sigma_F) > 0$ and $h(\sigma_{H_1}) < 0$. Therefore there is a unique $\sigma_{H_3} \in (\sigma_{H_1}, \sigma_F)$ such that $h(\sigma_{H_3}) = 0$. By continuity, we have that $\epsilon_{\gamma H} > \epsilon_{\gamma F}$ for $\sigma_F > \sigma > \sigma_{H_3}$, and $\epsilon_{\gamma H} < \epsilon_{\gamma F}$ for $\sigma_{H_1} < \sigma < \sigma_{H_3}$.

Consider now configurations (v) and (vi) of Case 2. As seen above if $\sigma_{H_3} > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$ exists it must satisfy $\sigma_{H_3} < \sigma_F$. Straight computations show that in these configurations $h(\sigma_F) > 0$. In configuration (v) of Case 2 we also have that $h\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}\right) < 0$, which proves existence and uniqueness of σ_{H_3}. We

[79] Indeed, this coefficient is given by

$$\begin{aligned} c \equiv & (2 + \theta\alpha_{KK})\{\alpha_{LL} - \alpha_{\Gamma L} + \theta[\alpha_{\Gamma K}\alpha_{KL} - \alpha_{KK}(1 + \alpha_{\Gamma L})]\} \\ & +2(1 + \theta\alpha_{KK})\{2 + \alpha_{LL} + \alpha_{\Gamma L} - \theta[\alpha_{\Gamma K}\alpha_{KL} - \alpha_{KK}(1 + \alpha_{LL})]\} \end{aligned}$$

In configuration (iv) $c > 0$, by Assumption 3 and 7. In configurations (v) and (vi) it is also positive since, by Assumption 11, $\alpha_{Ki} = \beta_{Ki} = 0$ and c becomes $c \equiv 4(1 + \alpha_{LL})$ which is positive by Assumption 3.

then have $\epsilon_{\gamma H} > \epsilon_{\gamma F}$ for $\sigma > \sigma_{H_3}$, and $\epsilon_{\gamma H} < \epsilon_{\gamma F}$ for $\sigma < \sigma_{H_3}$. In configuration (vi), on the contrary, $h\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}\right) > 0$. Therefore two cases are possible. Either there are two roots, $\sigma^a_{H_3}$ and $\sigma^b_{H_3}$, for the polynomial $h(\sigma_{H_3}) = 0$, such that $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma^a_{H_3} < \sigma^b_{H_3} < \sigma_F$, and in this case $\varepsilon_{\gamma_H} < \varepsilon_{\gamma_F}$ for $\sigma \in \left(\sigma_{H^a_3}, \sigma_{H^b_3}\right)$, and $\varepsilon_{\gamma_H} > \varepsilon_{\gamma_F}$ otherwise. Notice however that the existence of $\sigma_{H_3} \in \left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \sigma_F\right)$ in this configuration requires that $S > S_1$, which is ruled out by Assumption 12. Alternatively there is no $\sigma_{H_3} \in \left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \sigma_F\right)$ and $\varepsilon_{\gamma H} > \varepsilon_{\gamma_F}$ for all $\sigma > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$.

7.7 Configurations of Case 2 and Lemma 4

Lemma 4 *Under Assumptions 3, 10 and 11.1, if* $0 < S_1 < S_D$*, then* $S > S_1$ *for all* $\sigma > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$*; while if* $S_D < S_1 < 1$*, then* $S < S_1 \Leftrightarrow \frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma^{S_1}$*, where* $\sigma^{S_1} \equiv -\frac{(I_4-I_3)(1-s)}{I_2}$.

Proof: Notice that using (11), (14), Lemma 3, Assumptions 10 and 11.1, we can write $S_1 - S_D = -\frac{\theta I_2}{(I_4-I_3)}\frac{1+\alpha_{LL}}{s-\beta_{LL}}\left[\sigma^{S_1} - \frac{s-\beta_{LL}}{1+\alpha_{LL}}\right]$ and $S - S_1 = -\frac{\theta I_2}{\sigma(I_4-I_3)}\left[\sigma - \sigma^{S_1}\right]$. Hence, since $I_2 < 0$ and $I_4 - I_3 > 0$ under Assumptions 10 and 11.1, and since $\frac{1+\alpha_{LL}}{s-\beta_{LL}} > 0$ under Assumption 3, we see that $S > S_1 \Leftrightarrow \sigma > \sigma^{S_1}$, while $\sigma^{S_1} < \frac{s-\beta_{LL}}{1+\alpha_{LL}} \Leftrightarrow S_1 < S_D$. Therefore, when $S_1 < S_D$, $S > S_1$ for all $\sigma > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$. When $S_1 > S_D$ then $S < S_1 \Leftrightarrow \frac{s-\beta_{LL}}{(1+\alpha_{LL})} < \sigma < \sigma^{S_1}$.

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