Externalities, consumption constraints and regular economies ⋆

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Abstract

We consider a general model of pure exchange economies with consumption externalities. Households may have different consumption sets and each consumption set is described by a function called *possibility function*. Utility and possibility functions depend on the consumptions of all households. Our goal is to give sufficient conditions for the regularity of such economies. We prove that, generically, economies are regular in the space of endowments and possibility functions.

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1 Introduction

We consider a class of pure exchange economies with consumption externalities. Our goal is to provide sufficient conditions for the generic regularity when externalities appear both in utility functions and in consumption sets.

In presence of externalities, competitive equilibria are not necessarily Pareto optimal. It is therefore an open and important issue to study Pareto improving policies. Our work is a first step to study that issue. [3]

There is a large and growing literature on general equilibrium models with externalities. Regarding recent contributions, we can quote Cornet and Topuzu (2005), Scotchmer (2005), Noguchi and Zame (2006), del Mercato (2006), Carvajal (2007), Geanakoplos and Polemarchakis (2007), Heidhues and Riedel (2007), Kung (2007). We follow Laffont and Laroque (1972) and Laffont (1976, 1977, 1988) by incorporating consumption externalities not only in the preferences but also in the consumption sets. So, we consider individual consumption sets (which embody consumption constraints independently from wealth constraints) depending on the consumptions of the others. We provide some evidences of this dependency.

- *Noise.* "... The consumption by consumer i's neighbor of loud music at three in the morning may prevent her from sleeping" (Mas-Colell et al., 1995). So, without loud music the consumption set of consumer i allows any sleeping or working consumption up to 24 hours per day. But, the consumption of loud music m_{-i} by consumer i's neighbor decreases the physiologically possible daily consumption of sleeping or working by consumer i to a level $s_i(m_{-i})$ less than 24 hours.
- *Pollution.* (a) The global consumption of some goods pollutes rivers and seas. Without pollution the consumption set of household h allows any swimming daily consumption up to 24 hours. On the other hand, the presence of polluting consumptions by households other that h, z_{-h}, decreases the physiologically possible daily consumption of swimming by household h to a level $\alpha_h(z_{-h})$ less than 24 hours. However, the level can increase with the consumption of health cares (Laffont, 1988). (b) A high level of pollution due to the global consumption of energy may induce authorities to impose some upper limit on the possibility of physical exercise of households.
- *Congestion.* The available quantity and quality of a service in a network industry like telephone, internet or electricity depend on the global consumption. Externalities in consumption possibilities may naturally arise due to the negative effect of congestion. For instance, the consumption set of an Italian household usually allows any daily consumption of natural gas.

[3] See, in different settings, Geanakoplos and Polemarchakis (1986) and Citanna, Kajii and Villanacci (1998).

During Winter 2006, the Italian government has imposed upper bounds on the possible daily consumption of natural gas by households, public schools and public administrations, due to the high level of global consumption of natural gas by Russia and Ukraine.[4]

We consider *smooth* consumption sets, that is, each consumption set is described by an inequality on a differentiable function (called *possibility function*), in the spirit of Smale's work (1974b). In the same line, on the financial markets side, many different authors have considered portfolio sets described in terms of functions, Siconolfi (1986a, 1986b), Balasko, Cass and Siconolfi (1990), Cass (1992), Polemarchakis and Siconolfi (1997), Cass, Siconolfi and Villanacci (2001), Carosi and Villanacci (2006).

We follow Smale's *extended approach* where the study is directly focused on households characteristics (i.e., utility and possibility functions) and consequently, the equilibrium function is explicitly described using Kuhn-Tucker conditions without involving individual demand functions.[5] In presence of externalities, this approach can be useful since otherwise one should work with individual demand functions which depend on the individual demand functions of the others, which depend on the individual demand functions of the others, and so on. Assumptions 1 and 2 on utility and possibility functions are enough to get the non-emptiness and the compactness of the equilibrium set associated with a given economy (del Mercato, 2006).[6]

But, we provide an example of an exchange economy with externalities and no consumption constraints where all endowments are singular with an infinite equilibrium set.[7] So, for our purpose, that is regularity results, we introduce Assumption 10.

Point 1 of Assumption 10 means that the effect of change in the consumptions of the others on the marginal utility of an individual (with respect to his own consumption) is "dominated" by the effect of change in his own consumption. Point 2 of Assumption 10 is an analogous condition on possibility functions. Assumption 10 is satisfied when no externalities are taken into account or when utility and possibility functions are additively separable with respect to the own consumption and the consumption of the others. This latter condition is often assumed on utility functions (Crès, 1996, Geanakoplos

[4] The Russian Company Gazprom provides 27 percent of Italy's gas. In 2007, the governments of Russia and Italy signed a protocol to cooperate on the improvement of gas pipelines from Russia to Europe.

[5] See Smale (1974a, 1974b, 1981). The reader can also find a survey of this approach in Villanacci et al. (2002).

[6] In this paper, possibility functions also depend on endowments. For the sake of clarity, here we consider only externalities.

[7] We thank Andreu Mas-Colell and Paolo Siconolfi who have greatly help us for it.

and Polemarchakis, 2007).

We have also to overcome a difficulty coming from the fact that the equilibrium function is not everywhere differentiable, since equilibrium consumptions may be on the boundary of the consumption sets. We follow the strategy laid out in Cass, Siconolfi and Villanacci (2001), where general portfolio sets are encompassed while still permitting differential techniques. But, the presence of externalities in the possibility functions leads to a possible strong lack of continuity in the set of household's supporting prices. Then, we also consider *simple perturbations* of the possibility functions, that is, translations of the possibility functions (see major details in Section 5).

Our main result are Theorem 16 and and Corollary 18. Theorem 16 states that almost all *perturbed economies* are regular and Corollary 18 provides the generic regularity result in the space of endowments and possibility functions. [8] As usual, a regular economy has only a finite number of equilibria which locally depend on the parameters describing the economy in a continuous or a differentiable manner. [9]

There is a large and huge literature on regularity results without externalities. For major and exhaustive expositions see Smale (1981), Dierker (1982), Debreu (1983), Mas-Colell (1985), and Balasko (1988). Regarding analysis encompassing various sorts of consumption constraints, in Smale (1974a), Villanacci (1993) and Shannon (1994), consumption sets allow zero consumptions and in Bonnisseau and Rivera Cayupi (2006), consumptions may be restricted to be above minimal levels. Most related to our purpose is Smale (1974b), where households have general consumption sets described in terms of inequalities on functions. But, in Smale (1974b), no externalities are taken into account.

It could be surprising, but to the best of our knowledge, there are only few results on regularity in presence of consumption externalities and no one considers consumption constraints. In Crès (1996) and in Geanakoplos and Polemarchakis (2007), each utility function is additively separable. In Heidhues and Riedel (2007), utility functions have a functional form which generalizes the additive separable case. In both cases, point 1 of Assumption 10 is clearly satisfied. Nevertheless, the main purpose of these papers concerns Pareto optimality issues. In Bonnisseau (2003), preferences satisfy Assumption DP which is more general than point 1 of Assumption 10. We extend Assumption DP to our setting. Its extension is weaker than Assumption 10 but the economic

[8] Following Smale (1974a), *almost all* means in an open and full measure subset. Following Mas-Colell (1985), *generic* means in an open and dense subset, since the space of possibility functions is not a finite dimensional space.

[9] Note that if the parameters are elements of an arbitrary topological space, then differentiability is a meaningless idea, but one still gets continuity.

interpretation would be hazardous (see the details in Subsection 4.1). In Kung (2007), there is no specific assumption on the utility functions, but the author has to perturb the utility functions.

The paper is organized as follows. Section 2 is devoted to the model and to the basic assumptions. In Section 3, we present the definitions of competitive equilibrium and of equilibrium function. Theorem 9 recalls non-emptiness and compactness results. In Section 4, we provide an example, then we state Assumption 10 and we compare it with assumptions previously made in literature. In Section 5, we state the definitions of regular economy and of perturbed economy. In Section 6, we present the main results of this paper, Theorem 16 and Corollary 18. In Section 7, we prove our main results. Finally, all proofs of the lemmas are gathered in Appendix A. In Appendix B, the reader can find results from differential topology used in our analysis.

2 Model and basic assumptions

There is a finite number C of physical commodities labeled by the superscript $c \in \{1, ..., C\}$. The commodity space is $\mathbb{R}^C_{++}$. There is a finite number H of households labeled by subscript $h \in \mathcal{H} := \{1, ..., H\}$. Each household $h \in \mathcal{H}$ is characterized by an endowment of commodities, a possibility function and a utility function. Possibility and utility function depend on the consumptions of all households.

The notations are summarized below.

- x^c_h is the consumption of commodity c by household h; $x_h := (x^1_h, .., x^c_h, .., x^C_h)$ denotes household h's consumption and $x_{-h} := (x_k)_{k \neq h}$ the consumptions of households other than h called *environment* of household h; $x := (x_h)_{h \in \mathcal{H}}$.
- e^c_h is the endowment of commodity c by household h; $e_h := (e^1_h, .., e^c_h, .., e^C_h)$ denotes household h's endowment; $e := (e_h)_{h \in \mathcal{H}}$.
- As in general equilibrium model à la Arrow-Debreu, each household h has to choose a consumption in his consumption set X_h, i.e., in the set of all consumption alternatives which are *a priori possible* for him. In the spirit of Smale's work (1974b), the consumption set of household h is described in terms of an inequality on a function χ_h. We call χ_h *possibility function*. Observe that this idea is usual for smooth economies with production where each production set is described by an inequality on a function called *transformation function* (see Villanacci et al. (2002), for instance).

  Also note that in Smale (1974b), each consumption set is described by more than one function. Our results can be extended to this case, but this is not our main objective. The main innovation of this paper comes from the dependency of the consumption set of each household with respect to the consumptions of households other than h, i.e., given $x_{-h} \in \mathbb{R}^{C(H-1)}_{++}$, the consumption set of household h is the following set,

$$X_h(x_{-h}) = \left\{ x_h \in \mathbb{R}^C_{++} \ : \ \chi_h(x_h, x_{-h}) \geq 0 \right\}$$

where the possibility function χ_h is a function from $\mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+}$ to $\mathbb{R}$.

- Each household $h \in \mathcal{H}$ has preferences described by a utility function u_h from $\mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+}$ to $\mathbb{R}$, and $u_h(x_h, x_{-h})$ is household h's utility level associated with the consumption x_h and the environment x_{-h}.
- $(u_h, \chi_h, e_h)_{h \in \mathcal{H}}$ is an economy.
- $p^c \in \mathbb{R}_{++}$ is the price of one unit of commodity c; $p := (p^1, .., p^c, .., p^C) \in \mathbb{R}^C_{++}$.
- Given a vector $w = (w^1, .., w^c, .., w^C) \in \mathbb{R}^C$, we denote $w^\backslash := (w^1, .., w^c, .., w^{C-1}) \in \mathbb{R}^{C-1}$.

From now on, we make the following assumptions on utility and possibility functions taken from del Mercato (2006).

Assumption 1 *For all $h \in \mathcal{H}$,*

(1) u_h is continuous on $\mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+}$ and C^2 in the interior of its domain.
(2) for each $x_{-h} \in \mathbb{R}^{C(H-1)}_{++}$, the function $u_h(\cdot, x_{-h})$ is differentiably strictly increasing, i.e., for every $x_h \in \mathbb{R}^C_{++}$, $D_{x_h} u_h(x_h, x_{-h}) \in \mathbb{R}^C_{++}$.
(3) For each $x_{-h} \in \mathbb{R}^{C(H-1)}_{++}$, the function $u_h(\cdot, x_{-h})$ is differentiably strictly quasi-concave, i.e., for every $x_h \in \mathbb{R}^C_{++}$, $D^2_{x_h} u_h(x_h, x_{-h})$ is negative definite on $\operatorname{Ker} D_{x_h} u_h(x_h, x_{-h})$.[10]
(4) For each $x_{-h} \in \mathbb{R}^{C(H-1)}_{+}$ and for every $u \in \operatorname{Im} u_h$, $cl_{\mathbb{R}^C}\{x_h \in \mathbb{R}^C_{++} : u_h(x_h, x_{-h}) \geq u\} \subseteq \mathbb{R}^C_{++}$.

Assumption 2 *For all $h \in \mathcal{H}$,*

(1) χ_h is continuous on $\mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+}$ and C^2 in the interior of its domain.
(2) (Convexity of the consumption set) For each $x_{-h} \in \mathbb{R}^{C(H-1)}_{+}$, the function $\chi_h(\cdot, x_{-h})$ is quasi-concave. [11]

[10] Let v and v' be two vectors in $\mathbb{R}^n$, $v \cdot v'$ denotes the *inner product* of v and v'. Let A be a real matrix with m rows and n columns, and B be a real matrix with n rows and l columns, AB denotes the *matrix product* of A and B. Without loss of generality, vectors are treated as row matrix and A denotes both the matrix and the following linear application $A : v \in \mathbb{R}^n \to A(v) := Av^T \in \mathbb{R}^{[m]}$ where v^T is the transpose of v and $\mathbb{R}^{[m]} := \{w^T : w \in \mathbb{R}^m\}$. When $m = 1$, $A(v)$ coincides with the inner product $A \cdot v$, treating A and v as vectors in $\mathbb{R}^n$.

[11] Since χ_h is C^2 in the interior of its domain, then for each $x_{-h} \in \mathbb{R}^{C(H-1)}_{++}$, the function $\chi_h(\cdot, x_{-h})$ is differentiably quasi-concave, i.e., and for every $x_h \in \mathbb{R}^C_{++}$, $D^2_{x_h} \chi_h(x_h, x_{-h})$ is negative semidefinite on $\operatorname{Ker} D_{x_h} \chi_h(x_h, x_{-h})$.

(3) (Survival condition) There exists $\bar{x}_h \in \mathbb{R}^C_{++}$ such that $\chi_h(\bar{x}_h, x_{-h}) \geq 0$ for every $x_{-h} \in \mathbb{R}^{C(H-1)}_{+}$.

(4) (Non-satiation) For each $x_{-h} \in \mathbb{R}^{C(H-1)}_{++}$ and for every $x_h \in \mathbb{R}^C_{++}$,
(a) $D_{x_h}\chi_h(x_h, x_{-h}) \neq 0$; (b) $D_{x_h}\chi_h(x_h, x_{-h}) \notin -\mathbb{R}^C_{++}$.

(5) (Global desirability) For each $x \in \mathbb{R}^{CH}_{++}$ and for each $c \in \{1, ..., C\}$ there exists $h(c) \in \mathcal{H}$ such that $D_{x^c_{h(c)}}\chi_{h(c)}(x_{h(c)}, x_{-h(c)}) \in \mathbb{R}_+$.

For the interpretation of points 1, 2, 4 and 5 of Assumption 2 we refer to del Mercato (2006), pp. 529-530.

Point 3 of Assumption 2 guarantees that there exists one point which belongs to the consumption set $X_h(x_{-h})$ of household h whatever are the consumptions $x_{-h} \in \mathbb{R}^{C(H-1)}_{++}$ of others. Hence we call point 3 of Assumption 2 "survival condition". Moreover, by Definition 4 and the condition (1) given below, it follows that point 3 of Assumption 2 implies the existence of endowments satisfying the *survival assumption* whatever is the environment.[12]

Note that in order to get compactness and properness results, in points 1 and 4 of Assumption 1 and in points 1, 2 and 3 of Assumption 2, we consider $x_{-h} \in \mathbb{R}^{C(H-1)}_{+}$, which gives some information on the boundary behavior of u_h and χ_h (see the proof of Lemmas 21 and 24 in Appendix A).

Definition 3 *$\mathcal{U}$ denotes the set of $u = (u_h)_{h\in\mathcal{H}}$ which satisfies Assumption 1, and $\mathcal{X}$ denotes the set of $\chi = (\chi_h)_{h\in\mathcal{H}}$ which satisfies Assumption 2.*

We now define the set of endowments, which satisfy the survival assumption for given possibility functions.

Definition 4 *Let $\chi \in \mathcal{X}$. Define the set $E_\chi := \prod_{h\in\mathcal{H}} E_{\chi_h} \subseteq \mathbb{R}^{CH}_{++}$ where*

$$E_{\chi_h} := \left(\bigcap_{x_{-h}\in\mathbb{R}^{C(H-1)}_{+}} \left\{ x_h \in \mathbb{R}^C_{++} \ : \ \chi_h(x_h, x_{-h}) \geq 0 \right\} \right) + \mathbb{R}^C_{++}$$

From point 3 of Assumption 2, $E_\chi \neq \emptyset$. Moreover, by definition, E_χ is an open subset of $\mathbb{R}^{CH}_{++}$. Observe that if $e = (e_h)_{h\in\mathcal{H}} \in E_\chi$, then the following condition is satisfied for every $h \in \mathcal{H}$.

$$\forall\, x_{-h} \in \mathbb{R}^{C(H-1)}_{+},\ \exists\, \widetilde{x}_h \in \mathbb{R}^C_{++} : \chi_h(\widetilde{x}_h, x_{-h}) > 0 \text{ and } \widetilde{x}_h \ll e_h \qquad (1)$$

Note that he above condition is nothing else than the survival condition given by point 3 of Assumption 2 in del Mercato (2006).

[12] The survival assumption states that for each household there is an interior point of his consumption set, which is strictly smaller than his endowment.

Remark 5 *From now on, $u \in \mathcal{U}$ is kept fixed and an economy is parameterized by (χ, e) taken in the following set.*

$$\Theta := \{(\chi, e) \in (C^{0-2}(T, \mathbb{R}))^H \times \mathbb{R}^{CH}_{++} : \chi \in \mathcal{X} \text{ and } e \in E_\chi\} \tag{2}$$

where $T := \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+}$ and $C^{0-2}(T, \mathbb{R})$ is defined by (19) in Appendix B.

3 Competitive equilibrium

The definitions and the results stated in this section are direct transpositions of the ones in del Mercato (2006).

Without loss of generality, commodity C is the *numéraire good.* Then, given $p^\backslash \in \mathbb{R}^{C-1}_{++}$ with innocuous abuse of notation we denote $p := (p^\backslash, 1) \in \mathbb{R}^C_{++}$.

Definition 6 *$(x^*, p^{*\backslash}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{C-1}_{++}$ is a competitive equilibrium for the economy (χ, e) if for all $h \in \mathcal{H}$, x^*_h solves the following problem*

$$\begin{aligned} \max_{x_h \in \mathbb{R}^C_{++}} \quad & u_h(x_h, x^*_{-h}) \\ \text{subject to } & \chi_h(x_h, x^*_{-h}) \geq 0 \\ & p^* \cdot x_h \leq p^* \cdot e_h \end{aligned} \tag{3}$$

and x^* satisfies market clearing conditions

$$\sum_{h \in \mathcal{H}} x^*_h = \sum_{h \in \mathcal{H}} e_h \tag{4}$$

Proposition 7 *Let $(\chi, e) \in \Theta$ be an economy, $x^*_{-h} \in \mathbb{R}^{C(H-1)}_{++}$ and $p^{*\backslash} \in \mathbb{R}^{C-1}_{++}$. Problem (3) has a unique solution. $x^*_h \in \mathbb{R}^C_{++}$ is the solution to problem (3) if and only if there exists $(\lambda^*_h, \mu^*_h) \in \mathbb{R}_{++} \times \mathbb{R}$ such that $(x^*_h, \lambda^*_h, \mu^*_h)$ is the unique solution of the following system.*

$$\begin{cases} (h.1) \; D_{x_h} u_h(x_h, x^*_{-h}) - \lambda_h p + \mu_h D_{x_h} \chi_h(x_h, x^*_{-h}) = 0 \\ (h.2) \; -p^* \cdot (x_h - e_h) = 0 \\ (h.3) \; \min\left\{\mu_h, \chi_h(x_h, x^*_{-h})\right\} = 0 \end{cases} \tag{5}$$

Define the set of endogenous variables as $\Xi := (\mathbb{R}^C_{++} \times \mathbb{R}_{++} \times \mathbb{R})^H \times \mathbb{R}^{C-1}_{++}$, with generic element $\xi := (x, \lambda, \mu, p^\backslash) := ((x_h, \lambda_h, \mu_h)_{h \in \mathcal{H}}, p^\backslash)$. We can now describe *extended equilibria* using system (5) and market clearing conditions (4). Observe that, from Definition 6 and Proposition 7, the market clearing condition for good C is "redundant" (see equations $(h.2)_{h\in\mathcal{H}}$ in (5)). The equilibrium function defined below takes into account this aspect. For each economy $(\chi, e) \in \Theta$, the equilibrium function $F_{\chi,e} : \Xi \to \mathbb{R}^{\dim \Xi}$

$$F_{\chi,e}(\xi) := \left(\left(F^{h.1}_{\chi,e}(\xi), F^{h.2}_{\chi,e}(\xi), F^{h.3}_{\chi,e}(\xi) \right)_{h\in\mathcal{H}}, F^{M}_{\chi,e}(\xi) \right) \tag{6}$$

is defined by

$F^{h.1}_{\chi,e}(\xi) := D_{x_h} u_h(x_h, x_{-h}) - \lambda_h p + \mu_h D_{x_h} \chi_h(x_h, x_{-h})$, $F^{h.2}_{\chi,e}(\xi) := -p \cdot (x_h - e_h)$, $F^{h.1}_{\chi,e}(\xi) := \min\{\mu_h, \chi_h(x_h, x_{-h})\}$, and $F^{M}_{\chi,e}(\xi) := \sum_{h\in\mathcal{H}} (x^{\setminus}_h - e^{\setminus}_h)$.

Remark 8 *$\xi^* \in \Xi$ is an extended competitive equilibrium for the economy $(\chi, e) \in \Theta$ if and only if $(x^*_h, \lambda^*_h, \mu^*_h)$ solves system (5) at $(x^*_{-h}, p^{*\setminus})$ for all $h \in \mathcal{H}$ and $\sum_{h\in\mathcal{H}} (x^{*\setminus}_h - e^{\setminus}_h) = 0$ or, in other words, $F^{-1}_{\chi,e}(\xi^*) = 0$. With innocuous abuse of terminology, we call ξ^* simply an equilibrium.*

Theorem 9 (Existence and compactness). *For each economy $(\chi, e) \in \Theta$, the equilibrium set $F^{-1}_{\chi,e}(0)$ is non-empty and compact.*

4 An additional assumption

To illustrate the fact that the two previous assumptions are not sufficient to get generic regularity, we consider the following example. Consider a two commodity-two household economy with consumption sets coinciding with the whole commodity space $\mathbb{R}^2_{++}$ and with the following utility functions.

$$u_1(x_1, x_2) := \ln((1+\varepsilon)x^1_1 + x^1_2) + x^2_1 + \frac{1}{1+\varepsilon} x^2_2 := u_2(x_2, x_1), \text{ with } \varepsilon > 0$$

One easily checks that for each $(e_1, e_2) \in (\mathbb{R}^2_{++})^2$, we have that

$$\left((e^1_1 - t_1, e^2_1 + t_2), (e^1_2 + t_1, e^2_2 - t_2), p^{*1} = \frac{1+\varepsilon}{(e^1_1 + e^1_2) + \varepsilon(e^1_1 - t_1)} \right)$$

is an equilibrium for every $(t_1, t_2) \in \mathbb{R}^2$ such that t_1 belongs to an appropriate neighborhood of 0 and $t_2 = t_1 p^{*1}$. So, no economy $(e_1, e_2) \in (\mathbb{R}^2_{++})^2$ has a finite number of equilibria, which implies that all economies are singular.

This phenomenon can be explained by the fact that the external effect of household 1 on household 2 is too strong with respect to the effect of household 2's own consumption. Indeed, let us consider the marginal rate of substitution

$\mathrm{MRS}_2(x_1, x_2)$ of household 2 at (x_1, x_2), which is equal to $\dfrac{(1+\varepsilon)}{(1+\varepsilon)x_1^1 + x_2^1}$. So

$$\left|\frac{\partial\,\mathrm{MRS}_2}{\partial x_1^1}(x_1, x_2)\right| = \frac{(1+\varepsilon)^2}{((1+\varepsilon)x_1^1 + x_2^1)^2} > \left|\frac{\partial\,\mathrm{MRS}_2}{\partial x_2^1}(x_1, x_2)\right| = \frac{(1+\varepsilon)}{((1+\varepsilon)x_1^1 + x_2^1)^2}$$

Thus, we introduce the following additional assumption.

Assumption 10 *Let* $(x, v) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CH}$ *such that* $v \in \prod_{h\in\mathcal{H}} \mathrm{Ker}\, D_{x_h} u_h(x_h, x_{-h})$ *and* $\sum_{h\in\mathcal{H}} v_h = 0$. *Then,*

(1) $v_h \sum_{k\in\mathcal{H}} D^2_{x_k x_h} u_h(x_h, x_{-h})(v_k) < 0$ *whenever* $v_h \neq 0$, *and*
(2) $v_h \sum_{k\in\mathcal{H}} D^2_{x_k x_h} \chi_h(x_h, x_{-h})(v_k) \leq 0$ *whenever* $v_h \in \mathrm{Ker}\, D_{x_h}\chi_h(x_h, x_{-h})$.

Point 1 of Assumption 10 means that the effect of changes in the consumptions $(x_k)_{k\neq h}$ of households other than h on the marginal utility $D_{x_h} u_h(x_h, x_{-h})$ of household h is "dominated" by the effect of changes in the consumption x_h of household h. Indeed, under point 3 of Assumption 1, point 1 of Assumption 10 states that the absolute value of $v_h D^2_{x_h} u_h(x_h, x_{-h})(v_h)$ is larger than the remaining term $v_h \sum_{k\neq h} D^2_{x_k x_h} u_h(x_h, x_{-h})(v_k)$. Under points 1*b* and 2 of Assumption 2, point 2 of Assumption 10 has an analogous meaning for the possibility functions.

4.1 Comparison with previous assumptions

First, let us consider the case where utility and possibility functions do not depend on the environment in the spirit of Smale (1974b), that is, when the following assumption holds true.

Assumption 11 *For all* $h \in \mathcal{H}$, u_h *and* χ_h *depend only on* x_h.

Then Assumption 10 is trivially satisfied, since utility functions are differentiably strictly quasi-concave and possibility functions are differentiably quasi-concave.

Now, when externalities are taken into account, there are only few results on regularity. Importantly, note that consumption constraints are never taken into account. In all of these results, every consumption set coincides with $\mathbb{R}^C_{++}$. Crès (1996), Bonnisseau (2003), and Geanakoplos and Polemarchakis (2007) concern exchange economy where preferences depend on the consumptions of all households. Whereas Kung (2007) concerns production economies where all households' consumptions (private and public) and all firms's productions enter into utility functions.

In Kung (2007), utility functions satisfy assumptions similar to the ones given in Assumption 1. There are no additional assumption on the utility functions but the author needs to perturb the utility functions.

In Crès (1996) and in Geanakoplos and Polemarchakis (2007), each utility function is additively separable, that is, for all $h \in \mathcal{H}$ the utility function u_h takes the following form

$$u_h(x_h, x_{-h}) = \widetilde{u}_h(x_h) + v_h(x_{-h})$$

But, their objective mainly concerns Pareto optimality issues.

However, first, observe that the equilibrium set associated with a given economy coincides with the one in which no externalities are taken into account, i.e., where the utility functions are $(\widetilde{u}_h)_{h \in \mathcal{H}}$.

Second, observe that point 1 of Assumption 10 is satisfied since in the above case for all $h \in \mathcal{H}$ and for every $(x_h, x_{-h}) \in \mathbb{R}^{CH}_{++}$,

$$D^2_{x_k x_h} u_h(x_h, x_{-h}) = 0, \;\; \forall\, k \neq h$$

Assumption 10 also holds true when the utility function has the following functional form, which generalizes the additively separable case:

$$u_h(x_h, x_{-h}) = V_h(\widetilde{u}_h(x_h), x_{-h})$$

where V_h is strictly increasing with respect to the first variable. We refer to Heidhues and Riedel (2007) where it is shown that the above functional form is a necessary and sufficient condition for the equality between the equilibrium set associated with the utility functions $(V_h)_{h \in \mathcal{H}}$ and the one in which no externalities are taken into account, i.e., where preferences are represented by the utility functions $(\widetilde{u}_h)_{h \in \mathcal{H}}$. So, once again, to study regular economies, one can apply the results of the literature concerning regularity results without externalities.

In an analogous way, under points 1 and 2 of Assumption 2, point 2 of Assumption 10 is satisfied if possibility functions are additively separable or, more generally, if they have the following functional form.

$$\chi_h(x_h, x_{-h}) = P_h(\widetilde{\chi}_h(x_h), x_{-h})$$

In Bonnisseau (2003), preferences are more general than the ones considered in this paper, since they are non transitive and non complete. In this general setting, the author obtains the result of regularity for almost all endowments, under a *geometric* assumption. When preferences are represented by smooth utility functions, this assumption can be restated as follows.

Assumption 12 (Assumption DP; Bonnisseau, 2003). For all $x \in \mathbb{R}^{CH}_{++}$,

$$\underset{K_u(x)}{\text{proj}} \left(\text{Im}\, \Gamma_u(x) + S_u(x) \right) = K_u(x)$$

where $K_u(x) := \prod_{h \in \mathcal{H}} \text{Ker}\, D_{x_h} u_h(x)$, $\Gamma_u(x) := D_x \left(D_{x_h} u_h(x) \right)_{h \in \mathcal{H}}$,

$S_u(x) := \left\{ \left(\frac{\partial u_h}{\partial x_h^C}(x) s \right)_{h \in \mathcal{H}} \Big|\ s \in \mathbb{R}^C \right\}$, $\underset{K_u(x)}{\text{proj}}(v) := \left(\underset{\text{Ker}\, D_{x_h} u_h(x)}{\text{proj}} (v_h) \right)_{h \in \mathcal{H}}$ for all $v \in \mathbb{R}^{CH}$.[13]

But, observe that Assumption 12 is mainly an assumption on the quantitative behavior of households' supporting prices with respect to an infinitesimal perturbation of consumption allocations, i.e., on $D_x \left(D_{x_h} u_h(x) \right)_{h \in \mathcal{H}}$. When externalities are present in both utility and possibility functions, it follows from Kuhn-Tucker conditions that supporting prices also depend on the Lagrange multipliers associated with the possibility functions, i.e., they are determined by $\left(D_{x_h} u_h(x) + \mu_h D_{x_h} \chi_h(x) \right)_{h \in \mathcal{H}}$. Therefore, Assumption 12 can not be extended to our setting without involving the Lagrange multipliers $(\mu_h)_{h \in \mathcal{H}}$. That is, Assumption 12 can be generalized as follows.

Assumption 13 (Generalization of Assumption DP). For all $x \in \mathbb{R}^{CH}_{++}$ and for all $\mu = (\mu_h)_{h \in \mathcal{H}} \in \mathbb{R}^H_+$ such that $\left(D_{x_h} u_h(x) + \mu_h D_{x_h} \chi_h(x) \right)_{h \in \mathcal{H}} \in \mathbb{R}^{CH}_{++}$,

$$\underset{K_{u,\mu,\chi}(x)}{\text{proj}} \left(\text{Im}\, \Gamma_{u,\mu,\chi}(x) + S_{u,\mu,\chi}(x) \right) = K_{u,\mu,\chi}(x)$$

where $K_{u,\mu,\chi}(x) := \prod_{h \in \mathcal{H}} \left(\text{Ker}\, D_{x_h} u_h(x) \cap \text{Ker}\, \mu_h D_{x_h} \chi_h(x) \right)$,

$\Gamma_{u,\mu,\chi}(x) := D_x \left(D_{x_h} u_h(x) + \mu_h D_{x_h} \chi_h(x) \right)_{h \in \mathcal{H}}$,

$S_{u,\mu,\chi}(x) := \left\{ \left(\frac{\partial u_h}{\partial x_h^C}(x) s + \mu_h \frac{\partial \chi_h}{\partial x_h^C}(x) s \right)_{h \in \mathcal{H}} \Big|\ s \in \mathbb{R}^C \right\}$, and

$\underset{K_{u,\mu,\chi}(x)}{\text{proj}}(v) := \left(\underset{\text{Ker}\, D_{x_h} u_h(x) \cap \text{Ker}\, \mu_h D_{x_h} \chi_h(x)}{\text{proj}} (v_h) \right)_{h \in \mathcal{H}}$ for all $v \in \mathbb{R}^{CH}$.

It can be shown that the above assumption is weaker than Assumption 10. The major drawback of the above assumption is that it involves the Lagrange

[13] $\underset{\text{Ker}\, D_{x_h} u_h(x)}{\text{proj}} (v_h)$ denotes the orthogonal projection of v_h on $\text{Ker}\, D_{x_h} u_h(x)$. Also note that Assumption 12 is weaker than the following assumption: for all $x \in \mathbb{R}^{CH}_{++}$ and for all $v \in K_u(x)$, $v \Gamma_u(x)(v) = 0$ implies $v = 0$.

multipliers associated with the possibility functions. Consequently, utility and possibility functions are not independently considered. So, the economic interpretation would be hazardous.

5 Regular economies and possibility perturbations

Let us start with the definition of *regular* economy.

Definition 14 $(\chi, e) \in \Theta$ *is a regular economy if for each* $\xi^* \in F_{\chi,e}^{-1}(0)$,

(1) $F_{\chi,e}$ *is a* C^1 *function around* ξ^**.* [14]
(2) The differential mapping $D_\xi F_{\chi,e}(\xi^*)$ *is onto.*

Our analysis is based on results from differential topology, in the spirit of Balasko, Debreu, Mas-Colell and Smale. Since nothing prevents the equilibrium consumptions to be on the boundary of the consumption sets, then for every $h \in \mathcal{H}$ the function $F_{\chi,e}^{h.3}(\xi) = \min\{\mu_h, \chi_h(x_h, x_{-h})\}$ is not C^1 if $\mu_h = 0$ and $\chi_h(x_h, x_{-h}) = 0$. Therefore, first of all, it shall be shown that this case is *exceptional* at each equilibrium. For that purpose we follow the strategy laid out by Cass, Siconolfi and Villanacci (2001). [15]

But, the presence of externalities in the possibility functions leads to a possible strong lack of continuity in the set of household's supporting prices. Indeed, consider the situation in which, at equilibrium, household h is on the boundary of his consumption set, i.e., $\chi_h(x_h^*, x_{-h}^*) = 0$. If the associated multiplier μ_h^* is equal to 0, household h's supporting prices belong to the positive half-line generated by $D_{x_h} u_h(x_h^*, x_{-h}^*)$. If, in every neighborhood of x_{-h}^*, households other than h can move to x_{-h} in a such way that x_h^* remains on the boundary of the consumption set, i.e., $\chi_h(x_h^*, x_{-h}) = 0$ then, the positive half-line of household h's supporting prices might spreads over the cone positively generated by $D_{x_h} u_h(x_h^*, x_{-h})$ and $D_{x_h} \chi_h(x_h^*, x_{-h})$.

If the associated multiplier μ_h^* is positive, household h's supporting prices belong to the cone positively generated by $D_{x_h} u_h(x_h^*, x_{-h}^*)$ and $D_{x_h} \chi_h(x_h^*, x_{-h}^*)$. If, in every neighborhood of x_{-h}^*, households other than h can move to x_{-h} in a such way that x_h^* is now in the interior of the consumption set, that is $\chi_h(x_h^*, x_{-h}) > 0$ then, the positive cone of household h's supporting prices collapses into the positive half-line generated by $D_{x_h} u_h(x_h^*, x_{-h})$.

[14] $F_{\chi,e}$ is a C^1 function around ξ^* means that there exists an open neighborhood $I(\xi^*)$ of ξ^* in Ξ such that the restriction of $F_{\chi,e}$ to $I(\xi^*)$ is a C^1 function.
[15] The reader can find a survey of this approach in Villanacci et al. (2002).

Therefore, we consider *simple perturbations* of the possibility functions, that is, translations of the possibility functions. Following is the definition of perturbed economies for a given $\chi \in \mathcal{X}$.

Definition 15 *A perturbed economy* $(\chi + a, e)$ *is parameterized by possibility levels* $a = (a_h)_{h\in\mathcal{H}} \in \mathbb{R}^H_{++}$ *and endowments* $e \in E_\chi$*, and it is defined by*

$$\chi + a := (\chi_h + a_h)_{h\in\mathcal{H}}$$

$\Lambda_\chi := \mathbb{R}^H_{++} \times E_\chi$ *denotes the set of perturbed economies.*

It is an easy matter to check that for every $(a, e) \in \Lambda_\chi$, the perturbed economy $(\chi + a, e) \in \Theta$.

Finally, note that for the reasons mentioned above, if Assumption 11 holds true, then perturbations of the possibility functions are not needed. Indeed, the set of household h's supporting prices does not depend on the consumptions of others.

6 Main results

We now state the main results of the paper: the regularity for almost all *perturbed economies* and the generic regularity in the space of endowments and possibility functions.

Theorem 16 (Regularity for almost all perturbed economies). *Let* $\chi \in \mathcal{X}$*. The set* Λ^r_χ *of* $(a, e) \in \Lambda_\chi$ *such that* $(\chi + a, e)$ *is a regular economy is an open and full measure subset of* Λ_χ*.*

Under Assumption 11, that is when utility and possibility functions do not depend on the environment, then regularity holds for almost all endowments without perturbing possibility functions. It suffices to follow the same strategy and computations as in the proof of Theorem 16 (see Subsections 7.1 and 7.2).

Theorem 17 (Regularity for almost all endowments). *If utility and possibility functions satisfy Assumption 11, then set* E^r_χ *of endowments* $e \in E_\chi$ *such that* (χ, e) *is a regular economy is an open and full measure subset of* E_χ*.*

Now, endow the set $C^{0-2}(T, \mathbb{R})$ with the topology of the C^{0-2} uniform convergence on compacta (see Definition 26 in Appendix B), the set $\mathbb{R}^{CH}_{++}$ with the topology induced by the usual topology on $\mathbb{R}^{CH}$, and the set Θ with the topology induced by the product topology on $(C^{0-2}(T, \mathbb{R}))^H \times \mathbb{R}^{CH}_{++}$. As a consequence of Theorem 16 we obtain the following corollary.

Corollary 18 (Generic regularity). *The set $\mathcal{R}$ of $(\chi, e) \in \Theta$ such that (χ, e) is a regular economy is an open and dense subset of Θ.*

From Corollary 18, Theorem 9, a consequence of the Regular Value Theorem, and the Implicit Function Theorem (see Corollary 30 and Theorem 33 in Appendix B), we obtain the following proposition which provides the main properties of regular economies.

Proposition 19 (Properties of regular economies). *For each $(\chi, e) \in \mathcal{R}$,*

(1) the equilibrium set associated with the economy (χ, e) is a non-empty finite set, i.e.,

$$\exists\, r \in \mathbb{N} \setminus \{0\} : \; F_{\chi,e}^{-1}(0) = \{\xi^1, ..., \xi^r\}$$

(2) there exist an open neighborhood I of (χ, e) in Θ, and for each $i = 1, \dots, r$ an open neighborhood N_i of ξ^i in Ξ and a continuous function $g_i : I \to N_i$ such that

(a) $N_j \cap N_k = \emptyset$ if $j \neq k$,
(b) $g_i(\chi, e) = \xi^i$,
(c) for all $(\chi', e') \in I$, $F_{\chi',e'}^{-1}(0) = \{g_i(\chi', e') : \; i = 1, \dots, r\}$,
(d) the economies $(\chi', e') \in I$ are regular.

7 Proofs

The proof of Theorem 16 is divided into two steps: first, we prove that the equilibrium function is C^1 around each equilibrium for almost all perturbed economies. Second, we show that almost all perturbed economies are regular.

Corollary 18 is then deduced from Theorem 16 by using the particular form of perturbations. The proofs of the lemmas are gathered in Appendix A.

7.1 The equilibrium function is almost everywhere C^1

Take for fixed $\chi \in \mathcal{X}$ and consider the set of perturbed economies Λ_χ given in Definition 15. We prove the following statement.

Proposition 20 *The set Λ_χ^1 of $(a, e) \in \Lambda_\chi$ such that for each $\xi^* \in F_{\chi+a,e}^{-1}(0)$, $F_{\chi+a,e}$ is a C^1 function around ξ^* is an open and full measure subset of Λ_χ.*

From Assumptions 1 and 2, the equilibrium function $F_{\chi+a,e}$ is differentiable everywhere but not at any point ξ such that $\mu_h = \chi_h(x_h, x_{-h}) + a_h = 0$. To prove that this so-called border line case is exceptional, we consider a finite family of *auxiliary functions*. We then show that a border line case is a pre-image of 0 by one of these functions and that the set of perturbed economies for which the pre-image of 0 is non-empty is exceptional.

We consider the equilibrium function $\widetilde{F} : \Xi \times \Lambda_\chi \to \mathbb{R}^{\dim \Xi}$ defined by

$$\widetilde{F}(\xi, a, e) := F_{\chi+a,e}(\xi) \tag{7}$$

We also consider the mapping Φ which is the restriction to $\widetilde{F}^{-1}(0)$ of the projection of $\Xi \times \Lambda_\chi$ onto Λ_χ, that is:

$$\Phi : (\xi, a, e) \in \widetilde{F}^{-1}(0) \to \Phi(\xi, a, e) := (a, e) \in \Lambda_\chi$$

We state a fundamental property of Φ. A similar result is proved in del Mercato (2006).

Lemma 21 *The projection $\Phi : \widetilde{F}^{-1}(0) \to \Lambda_\chi$ is a proper function.*

For all $h \in \mathcal{H}$, we consider the set

$$\widetilde{B}_h := \left\{(\xi, a, e) \in \widetilde{F}^{-1}(0) : \ \mu_h = \chi_h(x_h, x_{-h}) + a_h = 0\right\} \text{ and } \widetilde{B} := \bigcup_{h\in\mathcal{H}} \widetilde{B}_h$$

We remark that

$$\Lambda^1_\chi = \Lambda_\chi \setminus \Phi(\widetilde{B}) \tag{8}$$

Then Λ^1_χ is open since $\widetilde{B}$ is clearly closed in $\widetilde{F}^{-1}(0)$ and $\Phi(\widetilde{B})$ is then closed by the properness of Φ.

We now show that $\Phi(\widetilde{B})$ is of measure zero in Λ_χ. For this, we consider the following finite set

$$\mathcal{P} := \left\{ \mathcal{J} = \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\} \;\middle|\; \begin{array}{l} \mathcal{J}_i \subseteq \mathcal{H}, \ \forall\, i = 1,2,3; \ \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3 = \mathcal{H}; \\ \mathcal{J}_i \cap \mathcal{J}_j = \emptyset, \ \forall\, i,j = 1,2,3, \ i \neq j. \end{array} \right\}$$

For all $\mathcal{J} \in \mathcal{P}$, $|\mathcal{J}_i|$ denotes the number of elements of $\mathcal{J}_i$ and

$$\Xi_{\mathcal{J}} := \mathbb{R}^{(C+1)H}_{++} \times (\mathbb{R}^{|\mathcal{J}_1|+|\mathcal{J}_3|} \times \mathbb{R}^{|\mathcal{J}_2|}_{++}) \times \mathbb{R}^{(C-1)}_{++} \tag{9}$$

The function $\widetilde{F}_{\mathcal{J}} : \Xi_{\mathcal{J}} \times \Lambda_\chi \to \mathbb{R}^{\dim \Xi_{\mathcal{J}}}$ is defined by

$$\widetilde{F}_{\mathcal{J}}(\xi, a, e) := ((\widetilde{F}^{h.1}(\xi, a, e), \widetilde{F}^{h.2}(\xi, a, e), \widetilde{F}^{h.3}_{\mathcal{J}}(\xi, a, e))_{h\in\mathcal{H}}, \widetilde{F}^M(\xi, a, e))$$

where $\widetilde{F}_{\mathcal{J}}$ differs from $\widetilde{F}$ defined in (7), for the domain and for the component $\widetilde{F}^{h.3}_{\mathcal{J}}$ defined by

$$\widetilde{F}^{h.3}_{\mathcal{J}}(\xi, a, e) := \begin{cases} \mu_h \text{ if } h \in \mathcal{J}_1 \cup \mathcal{J}_3, \\ \chi_h(x_h, x_{-h}) + a_h \text{ if } h \in \mathcal{J}_2 \end{cases} \tag{10}$$

When $\mathcal{J}_3$ is non-empty, for each $\bar{h} \in \mathcal{J}_3$, we define the function $\widetilde{F}_{\mathcal{J},\bar{h}} : \Xi_{\mathcal{J}} \times \Lambda_\chi \to \mathbb{R}^{\dim \Xi_{\mathcal{J}}+1}$ where

$$\begin{aligned} &\widetilde{F}_{\mathcal{J},\bar{h}}(\xi, a, e) := (\widetilde{F}_{\mathcal{J}}(\xi, a, e), \widetilde{F}^{\bar{h}.4}_{\mathcal{J}}(\xi, a, e)) \in \mathbb{R}^{\dim \Xi_{\mathcal{J}}+1} \\ &\text{where } \widetilde{F}^{\bar{h}.4}_{\mathcal{J}}(\xi, a, e) := \chi_{\bar{h}}(x_{\bar{h}}, x_{-\bar{h}}) + a_{\bar{h}} \end{aligned} \tag{11}$$

Note that $\widetilde{F}_{\mathcal{J}}$ and $\widetilde{F}_{\mathcal{J},\bar{h}}$ are C^1 functions on their domains. Furthermore, for all $\mathcal{J} \in \mathcal{P}$, $\dim \Xi_{\mathcal{J}} = \dim \Xi$. The key lemma of this step is the following one.

Lemma 22 *For every $\mathcal{J} \in \mathcal{P}$ such that $\mathcal{J}_3 \neq \emptyset$ and for each $\bar{h} \in \mathcal{J}_3$, 0 is a regular value for $\widetilde{F}_{\mathcal{J},\bar{h}}$.*

Since the dimension $\dim \Xi_{\mathcal{J}} + 1$ of the target space is strictly larger than the dimension of $\Xi_{\mathcal{J}}$, the Regular Value Theorem and a consequence of Sard's Theorem (see Theorems 29 and 31 in Appendix B) imply that for every $\mathcal{J} \in \mathcal{P}$ such that $\mathcal{J}_3 \neq \emptyset$ there exists a full measure subset $\Omega_{\mathcal{J},\bar{h}}$ of Λ_χ such that for each $(a, e) \in \Omega_{\mathcal{J},\bar{h}}$,

$$\{\xi \in \Xi_{\mathcal{J}} : \ \widetilde{F}_{\mathcal{J},\bar{h}}(\xi, a, e) = 0\} = \emptyset$$

Now, let us consider an element $(a, e) \in \Phi(\widetilde{B})$. Then, there exist $\xi \in \Xi$ such that $\widetilde{F}(\xi, a, e) = 0$ and $\bar{h} \in \mathcal{H}$ such that $\mu_{\bar{h}} = \chi_{\bar{h}}(x_{\bar{h}}, x_{-\bar{h}}) + a_{\bar{h}} = 0$. Let $\mathcal{J} \in \mathcal{P}$ defined by

$\mathcal{J}_1 := \{h \in \mathcal{H} : \ \mu_h = 0 \text{ and } \chi_h(x_h, x_{-h}) + a_h > 0\}$,
$\mathcal{J}_2 := \{h \in \mathcal{H} : \ \mu_h > 0 \text{ and } \chi_h(x_h, x_{-h}) + a_h = 0\}$,
$\mathcal{J}_3 := \{h \in \mathcal{H} : \ \mu_h = \chi_h(x_h, x_{-h}) + a_h = 0\}$.

One easily checks that $\widetilde{F}_{\mathcal{J},\bar{h}}(\xi, a, e) = 0$. So, $(a, e) \notin \Omega_{\mathcal{J},\bar{h}}$. Hence, we have prove that $\Phi(\widetilde{B})$ is included in the finite union of the complements of $\Omega_{\mathcal{J},\bar{h}}$ over all $\mathcal{J} \in \mathcal{P}$ such that $\mathcal{J}_3 \neq \emptyset$ and $\bar{h} \in \mathcal{J}_3$. Since each of these sets is of measure zero, so is $\Phi(\widetilde{B})$.

Take for fixed $\chi \in \mathcal{X}$. Observe that for given $(a,e) \in \Lambda_\chi$, by Definition 14, the economy $(\chi + a, e)$ is regular if (a,e) belongs to the open and full measure set Λ^1_χ given by (8) in the previous subsection, and

$$\forall\, \xi^* \in F^{-1}_{\chi+a,e}(0),\ \operatorname{rank} D_\xi F_{\chi+a,e}(\xi^*) = \dim \Xi$$

From now on, with innocuous abuse of notation:

- the domain of $\widetilde{F}$ defined in (7) will be $\Xi \times \Lambda^1_\chi$ instead of $\Xi \times \Lambda_\chi$,
- Φ denotes the restriction to $\widetilde{F}^{-1}(0)$ of the projection of $\Xi \times \Lambda^1_\chi$ onto Λ^1_χ.

Importantly, one easily checks that from (8), now $D_\xi \widetilde{F}(\xi, a, e) = D_\xi F_{\chi+a,e}(\xi)$ for every $(\xi, a, e) \in \widetilde{F}^{-1}(0)$. Then, from Assumptions 1 and 2, $D_\xi \widetilde{F}$ is now a continuous function on $\widetilde{F}^{-1}(0)$.

Let us consider the following set

$$\widetilde{C} := \left\{(\xi, a, e) \in \widetilde{F}^{-1}(0) :\ \operatorname{rank} D_\xi \widetilde{F}(\xi, a, e) < \dim \Xi\right\}$$

We remark that

$$\Lambda^r_\chi = \Lambda^1_\chi \setminus \Phi(\widetilde{C})$$

Then, we have to prove that $\Phi(\widetilde{C})$ is closed in Λ^1_χ and $\Phi(\widetilde{C})$ is of measure zero.

Step 1. An element (ξ, a, e) of $\widetilde{C}$ is characterized by the fact that the determinant of all the square submatrices of $D_\xi \widetilde{F}(\xi, a, e)$ of dimension $\dim \Xi$ is equal to zero. $\widetilde{C}$ is closed in $\widetilde{F}^{-1}(0)$ since the determinant is a continuous function and $D_\xi \widetilde{F}$ is continuous on $\widetilde{F}^{-1}(0)$. Then, $\Phi(\widetilde{C})$ is closed since Φ is proper.[16]

Step 2. We now show that $\Phi(\widetilde{C})$ is of measure zero in Λ^1_χ. The key lemma is the following one.

Lemma 23 *For every $\mathcal{J} \in \mathcal{P}$ such that $\mathcal{J}_3 = \emptyset$, 0 is a regular value for $\widetilde{F}_{\mathcal{J}}$.*

Then, from a consequence of Sard's Theorem (see Theorem 31 in Appendix B), for every $\mathcal{J} \in \mathcal{P}$ such that $\mathcal{J}_3 = \emptyset$, there exists a full measure subset $\Omega_{\mathcal{J}}$ of Λ^1_χ such that for each $(a,e) \in \Omega_{\mathcal{J}}$ and for each ξ^* such that $\widetilde{F}_{\mathcal{J}}(\xi^*, a, e) = 0$, $\operatorname{rank} D_\xi \widetilde{F}_{\mathcal{J}}(\xi^*, a, e) = \dim \Xi_{\mathcal{J}}$.

Now, let us consider $(a,e) \in \Phi(\widetilde{C})$. Then, there exists $\xi \in \Xi$ such that $\widetilde{F}(\xi, a, e) = 0$ and rank $D_\xi \widetilde{F}(\xi, a, e) <$ dim Ξ. Let us consider the partition

[16] The proof of the properness of Φ can be easily obtained using the same steps as in the proof of Lemma 21.

$\mathcal{J}$ associated to (ξ, a, e) as in the previous subsection. Since (a, e) belongs to Λ^1_χ, then $\mathcal{J}_3 = \emptyset$. Hence, one easily checks that $\widetilde{F}_J(\xi, a, e) = \widetilde{F}(\xi, a, e)$ on a neighborhood of (ξ, a, e). So, the partial differential with respect to ξ are the same and one concludes that $(a, e) \notin \Omega_{\mathcal{J}}$.

This prove that $\Phi(\widetilde{C})$ is included in the finite union of the complementary of $\Omega_{\mathcal{J}}$ over all $\mathcal{J} \in \mathcal{P}$ such that $\mathcal{J}_3 = \emptyset$. Since these sets have zero measure, so is $\Phi(\widetilde{C})$.

7.3 Generic regularity result in the space of economies

In this subsection, we prove Corollary 18. To show openess and density results, we follow a similar strategy to the one presented in Citanna, Kajii and Villanacci (1998).[17]

Let $\mathcal{R}$ be the set of economies $(\chi, e) \in \Theta$ such that (χ, e) is a regular economy. The density of $\mathcal{R}$ is a direct consequence of Theorem 16. Indeed, let $(\chi, e) \in \Theta$. Using Theorem 16, since an open and full measure subset is dense, one can find a sequence $(a^\nu, e^\nu)_{\nu \in \mathbb{N}} \subseteq \Lambda_\chi$ converging to $(0, e)$ such that $(\chi + a^\nu, e^\nu)$ is a regular economy for every $\nu \in \mathbb{N}$. From Remark 28 in Appendix B, the sequence $(\chi + a^\nu, e^\nu)_{\nu \in \mathbb{N}}$ converges to (χ, e), hence $\mathcal{R}$ is dense in Θ.

We now show that $\mathcal{R}$ is open. We consider the global equilibrium function $F : \Xi \times \Theta \to \mathbb{R}^{\dim \Xi}$ defined by

$$F(\xi, \chi, e) := F_{\chi, e}(\xi) \tag{12}$$

By Assumptions 1 and 2 and Remark 27 in Appendix B, F is continuous. Let us also define the restriction Π to $F^{-1}(0)$ of the projection of $\Xi \times \Theta$ onto Θ,

$$\Pi : (\xi, \chi, e) \in F^{-1}(0) \to \Pi(\xi, \chi, e) := (\chi, e) \in \Theta$$

The important property of Π is given in the following lemma.

Lemma 24 *The projection $\Pi : F^{-1}(0) \to \Theta$ is a proper function.*

For every $h \in \mathcal{H}$, we define the following set

$$B_h := \{(\xi, a, e) \in F^{-1}(0) : \mu_h = \chi_h(x_h, x_{-h}) = 0\} \text{ and } B := \bigcup_{h \in \mathcal{H}} B_h$$

Let

$$\Theta^1 := \Theta \setminus \Pi(B) \tag{13}$$

[17] Observe that in Citanna, Kajii and Villanacci (1998), openess and density results mainly concern constrained suboptimality issues.

Θ^1 is the set of $(\chi, e) \in \Theta$ such that for every $\xi^* \in F^{-1}_{\chi,e}(0)$, $F_{\chi,e}$ is a C^1 function around ξ^*. Definition 14 implies that $\mathcal{R} \subseteq \Theta^1$.

Θ^1 is open in Θ. Indeed, B is closed in $F^{-1}(0)$ as a consequence of Remark 27 (see Appendix B), and $\Pi(B)$ is closed from the properness of Π.

Now, we prove that $\mathcal{R}$ is open in Θ^1. We denote again by F the mapping now defined on $\Xi \times \Theta^1$ instead of $\Xi \times \Theta$ and by Π the restriction to $F^{-1}(0)$ of the projection from $\Xi \times \Theta^1$ onto Θ^1. Importantly, one easily checks that from (13), now $D_\xi F(\xi, \chi, e) = D_\xi F_{\chi,e}(\xi)$ for every $(\xi, a, e) \in F^{-1}(0)$. Then, from Assumptions 1 and 2 and Remark 27 in Appendix B, $D_\xi F$ is now a continuous function on $F^{-1}(0)$.

Let us consider the set

$$C := \{(\xi, \chi, e) \in F^{-1}(0) : \operatorname{rank} D_\xi F(\xi, \chi, e) < \dim \Xi\}$$

Definition 14 implies that

$$\mathcal{R} = \Theta^1 \setminus \Pi(C)$$

C is closed due to the continuity of the determinant function and of $D_\xi F$, $\Pi(C)$ is closed due to the properness of Π. Consequently, $\mathcal{R}$ is open in Θ^1.

Appendix A

We start by a selection property of the consumption sets, which will play a fundamental role in the *properness* result used to show openess properties in the space of economies Θ (see the proof of Lemma 24). Next, we show all the lemmas stated in Section 7.

Proposition 25 *Let $h \in \mathcal{H}$, Θ_h is the projection of Θ onto $C^{0-2}(T, \mathbb{R}) \times \mathbb{R}^C_{++}$ equipped with the metric induced by the one on $C^{0-2}(T, \mathbb{R}) \times \mathbb{R}^C$.*[18] *For each $h \in \mathcal{H}$, there is a continuous function $\widehat{x}_h : \mathbb{R}^{C(H-1)}_+ \times \Theta_h \to \mathbb{R}^C_{++}$ such that for each $(z, \chi_h, e_h) \in \mathbb{R}^{C(H-1)}_+ \times \Theta_h$, $\chi_h(\widehat{x}_h(z, \chi_h, e_h), z) > 0$ and $\widehat{x}_h(z, \chi_h, e_h) \ll e_h$.*

Proof of Proposition 25. First, observe that $\mathbb{R}^{C(H-1)}_+ \times \Theta_h$ is a metric space. Second, the correspondence $\phi_h : \mathbb{R}^{C(H-1)}_+ \times \Theta_h \rightrightarrows \mathbb{R}^C$ defined by $\phi_h(z, \chi_h, e_h) := \left\{x_h \in \mathbb{R}^C_{++} : \chi_h(x_h, z) > 0 \text{ and } x_h \ll e_h\right\}$ is non-empty convex valued by (1) and Definition 15, and by point 2 of Assumption 2. We now prove that ϕ_h has open fiber, that is, for all $x_h \in \mathbb{R}^C_+$, the following set

$$\phi_h^{-1}(x_h) := \left\{(z, \chi_h, e_h) \in \mathbb{R}^{C(H-1)}_+ \times \Theta_h : \chi_h(x_h, z) > 0 \text{ and } x_h \ll e_h\right\}$$

[18] The metric on $C^{0-2}(T, \mathbb{R}) \times \mathbb{R}^C$ is given by summing the metric $\widetilde{d}$ on $C^{0-2}(T, \mathbb{R})$ (see Definition 26 in Appendix B) and the Euclidean metric on $\mathbb{R}^C$.

is open in $\mathbb{R}_+^{C(H-1)} \times \Theta_h$. This follows from Remark 27 in Appendix B, which shows that the application $(\chi_h, x_h, z) \to \chi_h(x_h, z)$ is continuous on $C^{0-2}(T, \mathbb{R}) \times T$. Finally, we get the desired result since the correspondence ϕ_h satisfies the assumptions of Michael's Selection Theorem (see Florenzano, 2003). ■

Proof of Lemma 21. The proof is a direct consequence of Lemma 24 since the mapping $(\chi, a) \to \chi + a$ is continuous on $(C^{0-2}(T, \mathbb{R}))^H \times \mathbb{R}^H$ (see Remark 28 in Appendix B). ■

Proof of Lemma 22. The function $\widetilde{F}_{\mathcal{J},\bar{h}}$ is defined in (11). We have to show that for each $(\xi^*, a^*, e^*) \in \widetilde{F}_{\mathcal{J},\bar{h}}^{-1}(0)$, the Jacobian matrix $D_{\xi,a,e}\widetilde{F}_{\mathcal{J},\bar{h}}(\xi^*, a^*, e^*)$ has full row rank.

Let $\Delta := ((\Delta x_h, \Delta\lambda_h, \Delta\mu_h)_{h\in\mathcal{H}}, \Delta p^\setminus, \Delta w) \in \mathbb{R}^{(C+2)H} \times \mathbb{R}^{C-1} \times \mathbb{R}$. It is enough to show that $\Delta D_{\xi,a,e}\widetilde{F}_{\mathcal{J},\bar{h}}(\xi^*, a^*, e^*) = 0$ implies $\Delta = 0$. To prove it, we consider the computation of the partial Jacobian matrix with respect to the following variables

$$((x_h, \lambda_h, e_h)_{h\in\mathcal{H}}, (a_{h'})_{h'\in\mathcal{J}_2}, a_{\bar{h}}, p^\setminus)$$

The partial system $\Delta D_{\xi,a,e}\widetilde{F}_{\mathcal{J},\bar{h}}(\xi^*, a^*, e^*) = 0$ is written in detail below.

$$\begin{cases} \sum_{h\in\mathcal{H}} \Delta x_h D^2_{x_k x_h} u_h(x^*_h, x^*_{-h}) + \sum_{h'\in\mathcal{J}_2} \mu^*_{h'} \Delta x_{h'} D^2_{x_k x_{h'}} \chi_{h'}(x^*_{h'}, x^*_{-h'}) + \\ \sum_{h'\in\mathcal{J}_2} \Delta\mu_{h'} D_{x_k}\chi_{h'}(x^*_{h'}, x^*_{-h'}) - \Delta\lambda_k p^* + \Delta p^\setminus [I_{C-1}|0] + \\ \Delta w D_{x_k}\chi_{\bar{h}}(x^*_{\bar{h}}, x^*_{-\bar{h}}) = 0, \ \forall\, k \in \mathcal{H} \\ -\Delta x_h \cdot p^* = 0, \ \forall\, h \in \mathcal{H} \\ \Delta x_h \cdot D_{x_h}\chi_h(x^*_h, x^*_{-h}) + \Delta\mu_h = 0, \ \forall\, h \in \mathcal{J}_1 \cup \mathcal{J}_3 \\ \Delta x_{h'} \cdot D_{x_{h'}}\chi_{h'}(x^*_{h'}, x^*_{-h'}) = 0, \ \forall\, h' \in \mathcal{J}_2 \\ \Delta\lambda_h p^* - \Delta p^\setminus [I_{C-1}|0] = 0, \ \forall\, h \in \mathcal{H} \\ \Delta\mu_{h'} = 0, \ \forall\, h' \in \mathcal{J}_2 \\ \Delta w = 0 \\ \sum_{h\in\mathcal{H}} \lambda^*_h \Delta x^\setminus_h + \sum_{h\in\mathcal{H}} \Delta\lambda_h (x^{*\setminus}_h - e^{*\setminus}_h) = 0 \end{cases} \tag{14}$$

Since $p^{*C} = 1$, we get $\Delta\lambda_h = 0$ for each $h \in \mathcal{H}$ and $\Delta p^\setminus = 0$. Then, the relevant equations of the above system become

$$\begin{cases} \forall\, k \in \mathcal{H}, \\ \sum_{h\in\mathcal{H}} \Delta x_h D^2_{x_k x_h} u_h(x^*_h, x^*_{-h}) + \sum_{h'\in\mathcal{J}_2} \mu^*_{h'} \Delta x_{h'} D^2_{x_k x_{h'}} \chi_{h'}(x^*_{h'}, x^*_{-h'}) = 0 \\ -\Delta x_h \cdot p^* = 0,\ \forall\, h \in \mathcal{H} \\ \Delta x_h \cdot D_{x_h}\chi_h(x^*_h, x^*_{-h}) + \Delta\mu_h = 0,\ \forall\, h \in \mathcal{J}_1 \cup \mathcal{J}_3 \\ \Delta x_{h'} \cdot D_{x_{h'}}\chi_{h'}(x^*_{h'}, x^*_{-h'}) = 0,\ \forall\, h' \in \mathcal{J}_2 \\ \sum_{h\in\mathcal{H}} \lambda^*_h \Delta x_h = 0 \end{cases} \tag{15}$$

Observe that from $\widetilde{F}^{h.1}_{\mathcal{J}}(\xi^*, a^*, e^*) = 0$ and the above system, we get

$$(\Delta x_h)_{h\in\mathcal{H}} \in \prod_{h\in\mathcal{H}} \operatorname{Ker} D_{x_h} u_h(x^*_h, x^*_{-h})$$

Indeed, $D_{x_h} u_h(x^*_h, x^*_{-h}) \cdot \Delta x_h = \lambda^*_h p^* \cdot \Delta x_h = 0$ for each $h \in \mathcal{J}_1 \cup \mathcal{J}_3$, and $D_{x_{h'}} u_{h'}(x^*_{h'}, x^*_{-h'}) \cdot \Delta x_{h'} = \lambda^*_{h'} p^* \cdot \Delta x_{h'} - \mu^*_{h'} D_{x_{h'}} \chi_{h'}(x^*_{h'}, x^*_{-h'}, e^*_{h'}) \cdot \Delta x_{h'} = 0$ for each $h' \in \mathcal{J}_2$. Now, for each $h \in \mathcal{H}$ define

$$v_h := \lambda^*_h \Delta x_h \tag{16}$$

From equations in (15) and the above conditions, it follows that the vector $(x^*_h, v_h)_{h\in\mathcal{H}} \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CH}$ satisfies the following conditions

$$\sum_{h\in\mathcal{H}} v_h = 0 \text{ and } (v_h)_{h\in\mathcal{H}} \in \prod_{h\in\mathcal{H}} \operatorname{Ker} D_{x_h} u_h(x^*_h, x^*_{-h})$$

and

$$v_{h'} \in \operatorname{Ker} D_{x_{h'}} \chi_{h'}(x^*_{h'}, x^*_{-h'}) \text{ for each } h' \in \mathcal{J}_2 \tag{17}$$

Now, observe that the first equation of system (15) implies that for each $k \in \mathcal{H}$

$$\sum_{h\in\mathcal{H}} \Delta x_h D^2_{x_k x_h} u_h(x^*_h, x^*_{-h})(v_k) = -\sum_{h'\in\mathcal{J}_2} \mu^*_{h'} \Delta x_{h'} D^2_{x_k x_{h'}} \chi_{h'}(x^*_{h'}, x^*_{-h'})(v_k)$$

Since $\lambda^*_h \neq 0$ for all $h \in \mathcal{H}$, then it follows by (16) that for each $k \in \mathcal{H}$

$$\sum_{h\in\mathcal{H}} \frac{v_h}{\lambda^*_h} D^2_{x_k x_h} u_h(x^*_h, x^*_{-h})(v_k) = -\sum_{h'\in\mathcal{H}_2} \mu^*_{h'} \frac{v_{h'}}{\lambda^*_{h'}} D^2_{x_k x_{h'}} \chi_{h'}(x^*_{h'}, x^*_{-h'})(v_k)$$

Summing up $k \in \mathcal{H}$, we get

$$\sum_{h\in\mathcal{H}} \frac{v_h}{\lambda^*_h} \sum_{k\in\mathcal{H}} D^2_{x_k x_h} u_h(x^*_h, x^*_{-h})(v_k) = -\sum_{h'\in\mathcal{J}_2} \frac{\mu^*_{h'}}{\lambda^*_{h'}} v_{h'} \sum_{k\in\mathcal{H}} D^2_{x_k x_{h'}} \chi_{h'}(x^*_{h'}, x^*_{-h'})(v_k)$$

Since $\lambda_{h'}^* > 0$ and $\mu_{h'}^* > 0$ for each $h' \in \mathcal{J}_2$, then from the above condition, (17) and point 2 of Assumption 10 we have that

$$\sum_{h\in\mathcal{H}} \frac{1}{\lambda_h^*} v_h \sum_{k\in\mathcal{H}} D^2_{x_k x_h} u_h(x_h^*, x_{-h}^*)(v_k) \geq 0$$

Therefore, since $\lambda_h^* > 0$ for all $h \in \mathcal{H}$, point 1 of Assumption 10 implies that $v_h = 0$ for each $h \in \mathcal{H}$. By (16), we get $\Delta x_h = 0$ for all $h \in \mathcal{H}$. Then, by system (15), we have that $\Delta\mu_h = 0$ for each $h \in \mathcal{J}_1 \cup \mathcal{J}_3$. Thus, $\Delta = 0$. ■

Proof of Lemma 23. The function $\widetilde{F}_{\mathcal{J}}$ is defined in (7). We have to show that for each $(\xi^*, a^*, e^*) \in \widetilde{F}_{\mathcal{J}}^{-1}(0)$, the Jacobian matrix $D_{\xi,a,e}\widetilde{F}_{\mathcal{J}}(\xi^*, a^*, e^*)$ has full row rank.

Let $\Delta := ((\Delta x_h, \Delta\lambda_h, \Delta\mu_h)_{h\in\mathcal{H}}, \Delta p^\backslash) \in \mathbb{R}^{(C+2)H} \times \mathbb{R}^{C-1}$. It is enough to show that $\Delta D_{\xi,a,e}\widetilde{F}_{\mathcal{J}}(\xi^*, a^*, e^*) = 0$ implies $\Delta = 0$. To prove it, we consider the computation of the partial Jacobian matrix with respect to the following variables

$$((x_h, \lambda_h, e_h)_{h\in\mathcal{H}}, (a_{h'})_{h'\in\mathcal{J}_2}, p^\backslash)$$

and the corresponding partial system. Then, the proof follows the same steps as in the proof of Lemma 22. Indeed, note that there is a slight difference between this partial system and the one given in (14): now $\mathcal{J}_3 = \emptyset$. Then, we have one variable less, i.e., Δw, and one equation less, i.e., $\Delta w = 0$. ■

Proof of Lemma 24. We show that any sequence $(\xi^\nu, \chi^\nu, e^\nu)_{\nu\in\mathbb{N}} \subseteq F^{-1}(0)$, up to a subsequence, converges to an element of $F^{-1}(0)$, knowing that $(\chi^\nu, e^\nu)_{\nu\in\mathbb{N}} \subseteq \Theta$ converges to $(\chi^*, e^*) \in \Theta$.

We recall that $\xi^\nu = (x^\nu, \lambda^\nu, \mu^\nu, p^{\nu\backslash})$.

• $(x^\nu)_{v\in\mathbb{N}}$*, up to a subsequence, converges to* $x^* \in \mathbb{R}^{CH}_{++}$.

$(x^\nu)_{v\in\mathbb{N}} \subseteq \mathbb{R}^{CH}_{++}$. From $F^M(\xi^\nu, \chi^\nu, e^\nu) = 0$ and $F^{k.2}(\xi^\nu, \chi^\nu, e^\nu) = 0$, $x_k^\nu = \sum_{h\in\mathcal{H}} e_h^\nu - \sum_{h\neq k} x_h^\nu \leq \sum_{h\in\mathcal{H}} e_h^\nu$ for each $k \in \mathcal{H}$. Then, $(x^\nu)_{\nu\in\mathbb{N}}$ is bounded from above by an element of $\mathbb{R}^{CH}_{++}$, since for each $h \in \mathcal{H}$, $(e_h^\nu)_{\nu\in\mathbb{N}}$ converges to $e_h^* \in E_{\chi_h^*} \subseteq \mathbb{R}^C_{++}$. Then, $(x^\nu)_{\nu\in\mathbb{N}}$, up to a subsequence, converges to $x^* \geq 0$.

Now, we prove that $x_h^* \gg 0$ for each $h \in \mathcal{H}$.

By $F^{h.1}(\xi^\nu, \chi^\nu, e^\nu) = 0$, $F^{h.2}(\xi^\nu, \chi^\nu, e^\nu) = 0$ and $F^{h.3}(\xi^\nu, \chi^\nu, e^\nu) = 0$, it follows that $u_h(x_h^\nu, x_{-h}^\nu) \geq u_h(\hat{x}_h(x_{-h}^\nu, \chi_h^\nu, e_h^\nu), x_{-h}^\nu)$ for every $\nu \in \mathbb{N}$, where $\hat{x}_h$ is the continuous selection function given by Proposition 25. Define $\mathbf{1} := (1, ..., 1) \in \mathbb{R}^C_{++}$, from point 2 of Assumption 1 we have that for each $\varepsilon > 0$, $u_h(x_h^\nu + \varepsilon\mathbf{1}, x_{-h}^\nu) \geq u_h(\hat{x}_h(x_{-h}^\nu, \chi_h^v, e^\nu), x_{-h}^\nu)$ for every $\nu \in \mathbb{N}$. So taking the limit on ν, since $(\chi_h^\nu, e_h^\nu)_{\nu\in\mathbb{N}}$ converges to $(\chi_h^*, e_h^*) \in \Theta_h$, and u_h and $\hat{x}_h$ are continuous functions (see point 1 of Assumption 1 and Proposition 25), then $u_h(x_h^* + \varepsilon\mathbf{1}, x_{-h}^*) \geq u_h(\hat{x}_h(x_{-h}^*, \chi_h^*, e_h^*), x_{-h}^*) := \underline{u}_h$ for each $\varepsilon > 0$. By point 4 of Assumption 1, $x_h^* \in \mathbb{R}_{++}^C$ since x_h^* belongs to the set $cl_{\mathbb{R}^C}\{x_h \in \mathbb{R}_{++}^C : u_h(x_h, x_{-h}^*) \geq \underline{u}_h\}$.

- $(\lambda^\nu, \mu^\nu)_{v\in\mathbb{N}}$, *up to a subsequence, converges to* $(\lambda^*, \mu^*) \in \mathbb{R}_+^H \times \mathbb{R}_+^H$.

It is enough to show that $(\lambda_h^\nu p^\nu, \mu_h^\nu)_{\nu\in\mathbb{N}}$ is bounded for each $h \in \mathcal{H}$. Then, $(\lambda_h^\nu p^\nu, \mu_h^\nu)_{\nu\in\mathbb{N}} \subseteq \mathbb{R}_{++}^C \times \mathbb{R}_+$, up to a subsequence, converges to $(\pi_h^*, \mu_h^*) \in \mathbb{R}_+^C \times \mathbb{R}_+$, and $\lambda_h^* = \pi_h^{*C}$ since $p^{vC} = 1$ for each $\nu \in \mathbb{N}$.

Suppose otherwise that there is a subsequence of $(\lambda_h^\nu p^\nu, \mu_h^\nu)_{\nu\in\mathbb{N}}$ (that without loss of generality we continue to denote with $(\lambda_h^\nu p^\nu, \mu_h^\nu)_{v\in\mathbb{N}}$) such that $\|(\lambda_h^\nu p^\nu, \mu_h^\nu)\| \to +\infty$. Consider the sequence $\left(\frac{(\lambda_h^\nu p^\nu, \mu_h^\nu)}{\|(\lambda_h^\nu p^\nu, \mu_h^\nu)\|}\right)_{\nu\in\mathbb{N}}$ in the sphere, a compact set. Then, up to a subsequence $\left(\frac{(\lambda_h^\nu p^\nu, \mu_h^\nu)}{\|(\lambda_h^\nu p^\nu, \mu_h^\nu)\|}\right) \to (\pi_h, \mu_h) \neq 0$. $\pi_h \geq 0$ and $\mu_h \geq 0$, since $\lambda_h^\nu p^\nu \gg 0$ and $\mu_h^\nu \geq 0$ for each $\nu \in \mathbb{N}$. By $F^{h.1}(\xi^\nu, \chi^\nu, e^\nu) = 0$ for each $\nu \in \mathbb{N}$, we get $\lambda_h^\nu p^\nu = D_{x_h} u_h(x_h^\nu, x_{-h}^\nu) + \mu_h^\nu D_{x_h} \chi_h^\nu(x_h^\nu, x_{-h}^\nu)$. Now, divide both sides by $\|(\lambda_h^\nu p^\nu, \mu_h^\nu)\|$ and take the limits. From point 1 of Assumption 1 and Remark 27, we get

$$\pi_h = \mu_h D_{x_h} \chi_h^*(x_h^*, x_{-h}^*)$$

Then, $\mu_h > 0$. Otherwise we get $(\pi_h, \mu_h) = 0$. From point 4 of Assumption 2, we have that $D_{x_h} \chi_h^*(x_h^*, x_{-h}^*) \neq 0$. Then, $\pi_h \neq 0$. From Kuhn-Tucker necessary and sufficient conditions, we have that

$$\begin{aligned} \pi_h \cdot x_h^* = \min_{x_h \in \mathbb{R}_{++}^C} \quad & \pi_h \cdot x_h \\ \text{subject to } & \chi_h^*(x_h, x_{-h}^*) \geq 0 \end{aligned} \tag{18}$$

By $F^{h.2}(\xi^\nu, \chi^\nu, e^\nu) = 0$, we get $\lambda_h^\nu p^\nu \cdot x_h^\nu = \lambda_h^\nu p^\nu \cdot e_h^\nu$ for each $\nu \in \mathbb{N}$. Now, divide both sides by $\|(\lambda_h^\nu p^\nu, \mu_h^\nu)\|$ and take the limits. We get $\pi_h \cdot x_h^* = \pi_h \cdot e_h^*$. By (1), there is $\widetilde{x}_h \in \mathbb{R}_{++}^C$ such that $\chi_h^*(\widetilde{x}_h, x_{-h}^*) > 0$ and $\pi_h \cdot \widetilde{x}_h < \pi_h \cdot e_h^* = \pi_h \cdot x_h^*$ which contradict (18).

- $(p^{\nu\setminus})_{\nu\in\mathbb{N}}$, *up to a subsequence, converges to* $p^{*\setminus} \in \mathbb{R}_{++}^{C-1}$.

Taking the limit, from Remark 27, points 1 and 2 of Assumption 1, and points 1 and 5 of Assumption 2, we get $\lambda_k^* = D_{x_k^C} u_k(x_k^*, x_{-k}^*) + \mu_k^* D_{x_k^C} \chi_k^*(x_k^*, x_{-k}^*) > 0$ for some $k = h(C) \in \mathcal{H}$. From the previous step, $(\lambda_k^\nu p^{v\setminus})_{\nu\in\mathbb{N}}$ admits a subsequence converging to $\pi_k^{*\setminus} \geq 0$. Then, $(p^{v\setminus})_{\nu\in\mathbb{N}}$, up to a subsequence, converges to $p^{*\setminus} \geq 0$, since $\lambda_k^* > 0$. Now, suppose that there is $c \neq C$, such that $p^{*c} = 0$. Taking the limit, from Remark 27, points 1 and 2 of Assumption

1, and points 1 and 5 of Assumption 2, for some $k' = h(c) \in \mathcal{H}$ we get $0 < D_{x^c_{k'}} u_{k'}(x^*_{k'}, x^*_{-k'}) + \mu^*_{k'} D_{x^c_{k'}} \chi^*_{k'}(x^*_{k'}, x^*_{-k'}) = \lambda^*_{k'} p^{*c} = 0$, which is a contradiction.

• $\lambda^* \in \mathbb{R}^H_{++}$.

Otherwise, suppose that $\lambda^*_h = 0$ for some $h \in \mathcal{H}$. By $F^{h.1}(\xi^\nu, \chi^\nu, e^\nu) = 0$, we get $\lambda^\nu_h p^\nu = D_{x_h} u_h(x^\nu_h, x^\nu_{-h}) + \mu^\nu_h D_{x_h} \chi^\nu_h(x^\nu_h, x^\nu_{-h})$ for each $\nu \in \mathbb{N}$. Taking the limit, from Remark 27 and point 1 of Assumption 1, we get $0 = \lambda^*_h p^* = D_{x_h} u_h(x^*_h, x^*_{-h}) + \mu^*_h D_{x_h} \chi^*_h(x^*_h, x^*_{-h})$. By point 2 of Assumption 1 and point 4 of Assumption 2, we get $0 < D_{x^c_h} u_h(x^*_h, x^*_{-h}) + \mu^*_h D_{x^c_h} \chi^*_h(x^*_h, x^*_{-h}) = \lambda^*_h p^{*c} = 0$ for some good c, which is a contradiction. ■

Appendix B

Topology of the C^{0-2} uniform convergence on compacta

Let $T := \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+}$. We are interested on continuous functions defined on T which are C^2 in the interior of T (see point 1 of Assumption 2). Then, first, define the following set

$$C^{0-2}(T, \mathbb{R}) := \{f \in C^0(T, \mathbb{R}) : f_{|\mathrm{Int}\, T} \in C^2(\mathrm{Int}\, T, \mathbb{R})\} \quad (19)$$

where $\mathrm{Int}\, T$ denotes the interior of T and $f_{|\mathrm{Int}\, T}$ denotes the restriction of f to $\mathrm{Int}\, T$. The topology on $C^{0-2}(T, \mathbb{R})$ of the C^{0-2} *uniform convergence on compacta* is a “combination” of the topology on $C^0(T, \mathbb{R})$ of the C^0 uniform convergence on compacta and of the topology on $C^2(\mathrm{Int}\, T, \mathbb{R})$ of the C^2 uniform convergence on compacta.

Definition 26 *The topology on $C^{0-2}(T, \mathbb{R})$ of the C^{0-2} uniform convergence on compacta is the topology generated by the metric $\widetilde{d}$ defined by*

$$\forall\, (f, g) \in C^{0-2}(T, \mathbb{R}),\ \widetilde{d}(f, g) := d_0(f, g) + d_2(f_{|\mathrm{Int}\, T}, g_{|\mathrm{Int}\, T})$$

where d_2 is the metric d given in Allen (1981), p. 281, and d_0 is defined in an analogous way: let $\{T_n\}$ be a sequence of compact subsets of T such that $\bigcup_{n=1}^\infty T_n = T$, $d_0(f, g) := \sum_{n=1}^\infty \frac{1}{2^n} \min\{\|f - g\|_{0,T_n}, 1\}$ for f and g in $C^0(T, \mathbb{R})$, where $\|\cdot\|_{0,T_n}$ is defined by $\|w\|_{0,T_n} := \sup\limits_{x \in T_n} |w(x)|$ for $w \in C^0(T_n, \mathbb{R})$.

Remark 27 *Observe that, by definition $f_n \xrightarrow{\widetilde{d}} \bar{f}$ in $C^{0-2}(T, \mathbb{R})$ if and only if*

$$f_n \xrightarrow{d_0} \bar{f} \text{ in } C^0(T, \mathbb{R}) \text{ and } f_{n|\mathrm{Int}\, T} \xrightarrow{d_2} \bar{f}_{|\mathrm{Int}\, T} \text{ in } C^2(\mathrm{Int}\, T, \mathbb{R})$$

That is, $f_n \xrightarrow{\widetilde{d}} \bar{f}$ in $C^{0-2}(T, \mathbb{R})$ if and only if $(f_n)_{n \in \mathbb{N}}$ converges uniformly to $\bar{f}$ on any compact set included in T, and $(f_{n|\mathrm{Int}\, T})_{n \in \mathbb{N}}$, $(Df_{n|\mathrm{Int}\, T})_{n \in \mathbb{N}}$ and $(D^2 f_{n|\mathrm{Int}\, T})_{n \in \mathbb{N}}$ converge uniformly to $\bar{f}_{|\mathrm{Int}\, T}$, $D\bar{f}_{|\mathrm{Int}\, T}$ and $D^2 \bar{f}_{|\mathrm{Int}\, T}$ respectively, on any compact set included in $\mathrm{Int}\, T$.

Consequentially, the mapping $(f, x) \to f(x)$ *is continuous on* $C^{0-2}(T, \mathbb{R}) \times T$ *if and only if*

(1) the mapping $(f, x) \to f(x)$ *is continuous on* $C^0(T, \mathbb{R}) \times T$*, and*
(2) the mappings $(f_{|\mathrm{Int}\,T}, x) \to f_{|\mathrm{Int}\,T}(x)$, $(Df_{|\mathrm{Int}\,T}, x) \to Df_{|\mathrm{Int}\,T}(x)$, *and* $(D^2 f_{|\mathrm{Int}\,T}, x) \to D^2 f_{|\mathrm{Int}\,T}(x)$ *are continuous on* $C^2(\mathrm{Int}\,T, \mathbb{R}) \times \mathrm{Int}\,T$.

Since (1) *holds true by definition of topology of the* C^0 *uniform convergence on compacta and* (2) *holds true by definition of topology of the* C^2 *uniform convergence on compacta, then the mapping* $(f, x) \to f(x)$ *is continuous on* $C^{0-2}(T, \mathbb{R}) \times T$.

Remark 28 *One easily checks that the mapping* $(f, a) \to f + a$ *is continuous on* $C^{0-2}(T, \mathbb{R}) \times \mathbb{R}$.

Finally, we remark that the topology of the C^0 uniform convergence on any compact set included in T is uniquely used to show Proposition 25 which plays a fundamental role in the proof of Lemma 24.

Regular values and transversality

The theory of general economic equilibrium from a differentiable prospective is based on results from differential topology. Following are the ones used in our analysis. These results, as well as generalizations on these issues, can be found for instance in Guillemin and Pollack (1974), Hirsch (1976), Mas-Colell (1985) and Villanacci et al. (2002).

Theorem 29 *(Regular Value Theorem) Let* M, N *be* C^r *manifolds of dimensions* m *and* n*, respectively. Let* $f : M \to N$ *be a* C^r *function. Assume* $r > \max\{m - n, 0\}$. *If* $y \in N$ *is a regular value for* f*, then*

(1) if $m < n$, $f^{-1}(y) = \emptyset$,
(2) if $m \geq n$*, either* $f^{-1}(y) = \emptyset$*, or* $f^{-1}(y)$ *is an* $(m - n)$*-dimensional submanifold of* M.

Corollary 30 *Let* M, N *be* C^r *manifolds of the same dimension. Let* $f : M \to N$ *be a* C^r *function. Assume* $r \geq 1$. *Let* $y \in N$ *a regular value for* f *such that* $f^{-1}(y)$ *is non-empty and compact. Then,* $f^{-1}(y)$ *is a finite subset of* M.

The following results is a consequence of Sard's Theorem for manifolds.

Theorem 31 *(Transversality Theorem) Let* M, Ω *and* N *be* C^r *manifolds of dimensions* m, p *and* n*, respectively. Let* $f : M \times \Omega \to N$ *be a* C^r *function. Assume* $r > \max\{m - n, 0\}$. *If* $y \in N$ *is a regular value for* f*, then there exists a full measure subset* Ω^* *of* Ω *such that for any* $\omega \in \Omega^*$, $y \in N$ *is a regular* *value for f_ω, where*

$$f_\omega : \xi \in M \to f_\omega(\xi) := f(\xi, \omega) \in N$$

Definition 32 *Let (X, d) and (Y, d') be two metric spaces. A function $\pi : X \to Y$ is proper if it is continuous and one among the following conditions holds true.*

(1) π is closed and $\pi^{-1}(y)$ is compact for each $y \in Y$,
(2) if K is a compact subset of Y, then $\pi^{-1}(K)$ is a compact subset of X,
(3) if $(x^n)_{n\in\mathbb{N}}$ is a sequence in X such that $(\pi(x^n))_{n\in\mathbb{N}}$ converges in Y, then $(x^n)_{n\in\mathbb{N}}$ has a converging subsequence in X.

The above conditions are equivalent.

Theorem 33 *(Implicit Function Theorem) Let M, N be C^r manifolds of the same dimension. Assume $r \geq 1$. Let (X, τ) be a topological space, and $f : M \times X \to N$ be a continuous function such that $D_\xi f(\xi, x)$ exists and it is continuous on $M \times X$. If $f(\xi, x) = 0$ and $D_\xi f(\xi, x)$ is onto, then there exist an open neighborhood I of x in X, an open neighborhood U of ξ in M and a continuous function $g : I \to U$ such that $g(x) = \xi$ and $f(\xi', x') = 0$ holds for $(\xi', x') \in U \times I$ if and only if $\xi' = g(x')$.*

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