Optimal forest program when the carbon sequestration service of a forest has value

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Abstract

This paper develops an optimal forest program for even-aged and uneven-aged forest management when the carbon sequestration service of a forest has value. It is shown that for even-aged forest management, the optimal rotation may be longer or shorter compared with when the carbon sequestration service is not considered. For uneven-aged forest management, it is shown that the optimal sustainable forest is characterized with the optimal rotations for an even-aged forest. The optimal rotation is affected by how to value the carbon released after harvesting, which is a central concern of the issues of "harvested wood products" in the United Nations Framework Convention on Climate Change.

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1 Introduction

Carbon sequestration by forests is one of the major factors for the global carbon cycle. Forests are expected to play a significant role in mitigating human-induced climate change. The Intergovernmental Panel on Climate Change (IPCC) estimates that the global potential of biological mitigation options such as afforestation and reforestation is in the order of 100 Gt C (cumulative) by the year 2050, equivalent to about 10 to 20% of projected fossil fuel emissions during that period. (IPCC, 2001, p.25.) The IPCC also notes that the conservation and sequestration of carbon by forests, although not necessarily permanent, may allow time for other options to be further developed and implemented. The Kyoto Protocol of the United Nations Framework Convention on Climate Change (UNFCCC) recognizes that its Annex I countries can count part of the carbon sequestered by forests in their greenhouse gas reduction. Not only forests in their own countries, but also forests located out of Annex I countries can be utilized for the reduction activities under the Clean Development Mechanism. The carbon sequestration service of a forest has become an important issue in carbon markets. For forest management, an important question is how to conduct a forest practice when the carbon sequestration potential is considered.

This paper studies forest management models that incorporate the value of carbon sequestration. From the theoretical viewpoint, there are two interesting issues. First, forest management generates not only a positive effect (carbon sequestration) but also a negative effect by harvesting (carbon release). Second, carbon sequestration ability is not parallel to tree aging, however the ability begins to decline at some age. We investigate the implications of these features on an optimal forest program. Another contribution of this paper is a theoretical characterization of an optimal program for an uneven-aged forest with multiple objectives, i.e., forest production and carbon sequestration.

We begin with the so called Faustmann formula, a forest management model for an even-aged forest subject to repeated clear-cutting and planting over time. We extend the Faustmann formula to incorporate the value of carbon sequestration. This extension has already been made by Plantinga and Birdsey (1994) and van Kooten et al. (1995). While they concluded that the optimal rotation period becomes longer by considering the value of carbon sequestration, we prove that it can be shorter, too. We clarify the source of this contrariety. It turns out that the optimal rotation is affected by how to value the carbon released after harvesting, which is the central issue regarding "harvested wood products (HWP)" in the UNFCCC and its Kyoto Protocol. If the carbon in the forest products is released into the atmosphere immediately after harvesting, like a fuel wood, then the rotation becomes longer by considering carbon sequestration, whereas if the carbon release occurs slowly, like a saw wood, the rotation becomes shorter. With a numerical example based on Japanese data, we show that the shorter case could occur in the real world.

Then, we turn to uneven-aged forest management that allows a variety of ages of trees. The Mitra–Wan tree farm model (Mitra and Wan, 1985) is modified to incorporate carbon sequestration. While uneven-aged forest management has been studied by economists including Heaps (1984), Mitra and Wan (1985, 1986), Mitra et al. (1991), Wan (1989, 1994), Salo and Tahvonen (2002), and Sahashi (2002), all of those studies assumed a single purpose, typically timber production. In contrast, we study a multipurpose forest management model. We show that an interesting characteristic of an optimal steady state of the Mitra–Wan tree farm is carried over to our multipurpose model with a complication. That is, the optimal steady state consists of normal forest(s) that are sustained by cutting trees with the optimal rotation(s) of the relevant even-aged forest model. Also, we show that the number of optimal rations is at most two in general. The case of multiple optimal rotations justifies zoning of a biologically homogeneous forest when the forest provides more than one service.

This paper is organized as follows: Section 2 studies an even-aged forest model, whereas Section 3 studies an uneven-aged forest model. Section 4 concludes. The proof of selected propositions are dealt with in the Appendix.

2 Even-aged forest management

This section studies even-aged forest management. The primary aim is to show that the optimal rotation when carbon sequestration is considered may be longer or shorter than the one without it. Two points are worth addressing: (a) existing studies (Plantinga and Birdsey 1994 and van Kooten et al. 1995) concluded that the consideration of the carbon sequestration potential of a forest makes the optimal rotation longer, and (b) the condition that influences whether the rotation is longer or shorter has an interesting implication for the issues of "harvested wood products (HWP)" argued in the UNFCCC and the Kyoto Protocol.

2.1 Faustmann formula

Consider a unit of forest land on which planting and harvesting are infinitely repeated. Assume that all relevant ecological and market conditions are stationary over time. Then, the profit maximization problem for harvest revenues is formulated as follows:

$$\max_{T\geq 0} V^F(T) = \left[F(T)e^{-rT} - c\right]\left(1 + e^{-rT} + e^{-2rT} + \cdots\right) = \frac{F(T)e^{-rT} - c}{1 - e^{-rT}}, \tag{1}$$

where T is the rotation period, $F(T)$ denotes the harvest revenue when the forest is cut at age T, $c \geq 0$ is the planting cost in each rotation, which is assumed to be paid once at the time of planting, and $r > 0$ is the interest rate. The right-hand side of the problem (1) is called the Faustmann formula, which was proposed by a German forester in the 19th century, Martin Faustmann (1849). Since then, the Faustmann formula has been used as a basic model of forest economics. For a detailed history, see Samuelson (1976) and Löfgren (1983).

Applying the Faustmann formula to forest carbon sequestration, we have the following problem:

$$\max_{T\geq 0} V^C(T) = \frac{\int_0^T g(t)e^{-rt}dt - \beta G(T)e^{-rT}}{1-e^{-rT}}, \tag{2}$$

where $g(t)$ denotes the carbon sequestration rate of the forest at age t, $G(T) = \int_0^T g(t)dt$ is the amount of carbon stock accumulated in the forest at age T, and $\beta \in [0,1]$ is the discount factor applied to the carbon release by harvesting. If the harvests are immediately burned, then all carbon sequestered is released into the air at once and $\beta = 1$. If the sequestered carbon is never released, then $\beta = 0$. We express the intermediate case by $\beta \in (0,1)$.

We assume a forest that provides two useful services: forest products and carbon sequestration. The optimal use of the forest is expressed by a hybrid of the above two maximization problems (1) and (2):

$$\max_{T\geq 0} V^*(T) = V^F(T) + \alpha V^C(T), \tag{3}$$

where $\alpha > 0$ is the price of carbon sequestration. We will refer to the solutions of the problems (1), (2), and (3) as the Faustmann rotation, the carbon rotation, and the optimal rotation, and denote them by T^F, T^C, and T^*, respectively. We make the following assumptions.

A1: There is a unique Faustmann rotation $T^F \in (0,\infty)$ such that $V^F(T^F) > 0$ and $dV^F(T^F)/dT = 0$.

A2: $g(t)$ is continuously differentiable and satisfies the following conditions: (a) $g(t) > 0$ for all $t > 0$, (b) there is $T_I > 0$ such that $g'(t) \gtreqless 0$ if $t \lesseqgtr T_I$, (c) $\int_0^\infty g(t)dt < \infty$, (d) $\ln g(t)$ is strictly concave.

By assumption A1, we assume that the Faustmann rotation is unique and finite because we want to compare T^* with T^F. If $T^F = \infty$, then the answer is trivial. The inequality $V^F(T^F) > 0$ ensures that the problems (1) and (3) are meaningful. The assumption that $V^F(T)$ is differentiable at $T = T^F$ is not essential, but with the assumption we have sharp results. Assumption A2(b) implies the shape of $g(t)$ is unimodal. Assumptions A2(b) and A2(c) together imply that the graph of carbon accumulation $G(T)$

is S-shaped with the inflection point $T = T_I$ and there is a carrying capacity (biological equilibrium) $\lim_{T\to\infty} G(T) = \int_0^\infty g(t)dt$. The log-concavity in A2(d) is a common feature of plant growth functions employed in biology and forestry. An example is the Chapman-Richards growth function:

$$G(T) = A\left[1 - b\exp(-kT)\right]^{\frac{1}{1-m}}, \quad A, k > 0, m \geq 0, b/(1-m) > 0,$$

which includes Logistic, Mitcherlich, and Gompertz functions as special cases. See Vanclay (1994, Chapter 6).

2.2 Optimal rotation

Under these assumptions, we examine the relative locations of the Faustmann rotation T^F and the optimal rotation T^*. As shown below, the relationship of these rotations depends on the locations of the Faustmann rotation T^F and the carbon rotation T^C. The following lemma gives useful information on the carbon rotation. Let:

$$\tilde{\Phi} = \lim_{T\to\infty} \frac{\int_0^T g(t)e^{-rt}dt}{\int_0^T g(t)dt} \in (0, 1), \tag{4}$$

which is the limit of the ratio of the present value of sequestered carbon in a new rotation and the value of accumulated carbon in the current rotation. Also, define T_L by:

$$T_L = \max\{T \geq 0 | g(T)(1 - e^{-rT}) - rG(T) = 0\}. \tag{5}$$

It is easily verified that $T_L \in [0, \infty)$, and $T_L > 0$ if and only if $\lim_{T\to 0} g'(T)/g(T) > r$.

Lemma 1 *Under assumptions A1 and A2, (a) if the carbon release factor β is greater than or equal to $\tilde{\Phi}$, then $dV^C(T)/dT > 0$ for all $T > 0$ and $T^C = \infty$. (b) If $\beta \in [0, \tilde{\Phi})$, then there is a finite and unique carbon rotation T^C such that $dV^C(T)/dT \gtrless 0$ if $T \lessgtr T^C$. Furthermore, (c) the carbon rotation T^C as* *a function of* β *is continuously differentiable at* $\beta \in [0, \tilde{\Phi})$ *satisfying* $\partial T^C/\partial\beta > 0$, $\lim_{\beta\to\tilde{\Phi}} T^C(\beta) = \infty$, *and* $\lim_{\beta\to 0} T^C(\beta) > \max\{T_I, T_L\}$.

Proof. See the Appendix. ■

To understand the results of Lemma 1, notice that forest management has both positive and negative effects on carbon sequestration. The positive effect is to sequester carbon. This effect is best performed when the rotation is chosen so as to maximize $\int_0^T g(t)e^{-rt}dt/(1-e^{-rT})$. The negative effect is carbon release. This effect is minimized if a forest is never harvested. The carbon release factor β determines how to weigh these effects. As β increases, the negative effect exceeds the positive effect and the carbon rotation becomes longer, eventually infinite. We will see this with numerical examples later (Figure 1 below).

Using Lemma 1, we have:

Proposition 1 *Under assumptions A1 and A2,* $T^* \lesseqgtr T^F$ *if and only if* $T^C \lesseqgtr T^F$.

Proof. We only prove the statement: $T^* < T^F$ if and only if $T^C < T^F$. The case with the opposite inequality is proved by the symmetric argument. Sufficiency: Assume $T^C < T^F$. Because $T^C < T^F < \infty$, Lemma 1 implies that $\beta < \tilde{\Phi}$ and $dV^C(T)/dT < 0$ on (T^C, ∞). Then, for any $T > T^F$, $V^*(T) < V^F(T^F) + \alpha V^C(T^F) = V^*(T^F)$. Therefore, $T^F \geq T^*$. However, $T^F \neq T^*$, because $dV^*(T^F)/dT = dV^F(T^F)/dT + \alpha dV^C(T^F)/dT < 0$.

Necessity: We prove the following statement: if $T^C \geq T^F$, then $T^F \leq T^*$. If $T^C \geq T^F$, by Lemma 1, $dV^C(T)/dT > 0$ holds for all $T \in (0, T^F)$. This implies that:

$$V^*(T) = V^F(T) + \alpha V^C(T) < V^F(T^F) + \alpha V^C(T^F) \text{ for all } T \in (0, T^F),$$

and thus $T^* \geq T^F$ must hold. ■

Proposition 1 indicates that depending on the locations of T^C and T^F, the optimal rotation T^* can be shorter or longer than the Faustmann rotation T^F. The following proposition gives a sufficient condition for $T^* < T^F$.

Proposition 2 *Under assumptions A1 and A2, if* $T^F > \lim_{\beta \to 0} T^C(\beta)$, *then there is* $\tilde{\beta}$ *such that* $T^* < T^F$ *when* $\beta < \tilde{\beta}$.

Proof. This follows from the fact that the carbon rotation $T^C(\beta)$ as a function of β is continuous on $[0, \tilde{\Phi})$ by Lemma 1 (c). ■

By Proposition 2, we have a different result from Plantinga and Birdsey (1994) and van Kooten et al. (1995), both of whom concluded that incorporation of carbon sequestration into the Faustmann formula always makes the optimal rotation longer. The model employed by Plantinga and Birdsey (1994) is more general than ours. In their model, the carbon release factor β can vary depending on the cutting age. While this assumption is more realistic because the usage of the harvests can vary over the vintage of trees, they did not examine the implication. Instead, they applied their model to the data for three major US forest types and made the conclusion. Van Kooten et al. (1995) employed a model that is a special case of our model. They assumed no planting costs ($c = 0$) and harvest revenue proportional to the accumulated carbon ($F(T) = PG(T)$ with constant $P > 0$). This is the case that a forest is naturally regenerated and the price of harvests P is independent of the vintage of the trees. With these assumptions, we can show that independent of the values of parameters β and r, the optimal rotation is always longer than the Faustmann rotation. Therefore, the result by van Kooten et al. (1995) is true for a forest management that satisfies these assumptions.

Proposition 3 *In addition to assumptions A1 and A2, assume* $c = 0$ *and* $F(T) = G(T)$. *Then,* $T^F < T^*$.

Proof. Because $F(T) = G(T)$, $F(T)$ is continuously differentiable and the first order condition for the Faustmann problem (1) is given by:

$$
\begin{aligned}
0 = \frac{dV^F(T^F)}{dT^F} &= G'(T^F) - r\left(G(T^F) + \frac{G(T^F)e^{-rT^F}}{1-e^{-rT^F}}\right) \\
&= \frac{r}{1-e^{-rT^F}}\left[r^{-1}g(T^F)\left(1-e^{-rT^F}\right) - G(T^F)\right].
\end{aligned} \tag{6}
$$

Note that with these additional assumptions, the value function of the carbon problem V^C is written as:

$$
V^C(T) = \frac{\int_0^T g(t)e^{-rt}dt}{1-e^{-rT}} - \beta V^F(T). \tag{7}
$$

Evaluate dV^C/dT at $T = T^F$ to obtain:

$$
\begin{aligned}
\frac{dV^C(T^F)}{dT} &= \frac{d}{dT}\left(\frac{\int_0^{T^F} g(t)e^{-rt}dt}{1-e^{-rT^F}} - \beta V^F(T^F)\right) \\
&= \frac{re^{-rT^F}}{\left(1-e^{-rT^F}\right)^2}\left(r^{-1}\left(1-e^{-rT^F}\right)g(T^F) - \int_0^{T^F} g(t)e^{-rt}dt\right) \\
&= \frac{re^{-rT^F}}{\left(1-e^{-rT^F}\right)^2}\left(G(T^F) - \int_0^{T^F} g(t)e^{-rt}dt\right) \\
&= \frac{re^{-rT^F}}{\left(1-e^{-rT^F}\right)^2}\left(\int_0^{T^F} g(t)dt - \int_0^{T^F} g(t)e^{-rt}dt\right) > 0.
\end{aligned} \tag{8}
$$

Here the second line uses $dV^F(T^F)/dT = 0$, and the third line uses the last equation in (6). Inequality (8) implies $T^C > T^F$. Therefore, $T^* > T^F$ by Proposition 1. ∎

Another important case where the optimal rotation is always longer than the Faustmann rotation is the case of $\beta = 1$, which corresponds to the "IPCC recommended default approach" adopted in the Kyoto Protocol of the UNFCCC. In this approach, all carbon dioxide emissions and removals associated with harvesting and the oxidation of wood products are accounted for in the year of harvesting (removal), although IPCC recognizes the possible delay between harvesting and carbon release and notes that if data permits, the stock changes of carbon in harvested wood products can be reported in national greenhouse gas inventories. See IPCC (2003).

Proposition 4 *Assume A1 and A2. If $\beta = 1$, then $T^F < T^*$.*

Proof. Because $\beta = 1 > \tilde{\Phi}$, Lemma 1 (a) implies $T^C = \infty$. By Proposition 1, we have $T^F < T^*$. ■

2.3 Numerical example

We have shown by Proposition 2 that the optimal rotation can be shorter than the Faustmann rotation in theory. In this subsection, corresponding to Plantinga and Birdsey's (1994) study based on actual data, we apply our model to the data for two major species of trees planted in Japan: *Sugi* (Japanese cedar, *Cryptomeria japonica*) and *Hinoki* (Japanese cypress, *Chamaecyparis obtusa*). The aim is to show that in contrast to the conclusion of Plantinga and Birdsey (1994), the optimal rotation could be shorter than the Faustmann rotation in the real world. Japanese forestry is featured by labor intensive, high cost silviculture ($c \neq 0$) and progressive timber prices in the age of trees ($F(T) \neq G(T)$).[1] Therefore, the assumptions of van Kooten et al. (1995) mentioned in Proposition 3 do not apply to Japanese forestry.

We take the data from the study by Akao (1993). He used the yield table for the forest land with the medium site index in Wakayama Prefecture located in the western part of Japan to estimate the growth function for growing stocks:

$$G(t) = A\left(1 - b\exp\left(-k\left(t+d\right)/5\right)\right)^{\frac{1}{1-m}}, \tag{9}$$

with the parameter values: $A = 723.8$, $m = 0.2259$, $k = 0.1732$, $b = 1.388$, $d = 9.463$ for Japanese

[1] See, for example, Akao (2003).

cedar, and $A = 487.7$, $m = 0.1297$, $k = 0.1693$, $b = 1.369$, $d = 9.270$ for Japanese cypress. Considering age-dependent prices as well as the site-specific conditions, Akao (1993) calculated the Faustmann rotations T^F for 644 forest units. When the annual interest rate is 0.03, most of the cutting ages are about 85 years for Japanese cedar and about 55 years for Japanese cypress.[2]

Assume that the sequestered carbon is proportional to the growing stock. By Proposition 1, if the carbon rotation is shorter than the Faustmann rotation ($T^C < T^F$), then $T^* < T^F$. The carbon rotation T^C depends on the carbon release factor β. Figure 1 depicts the graphs of the carbon release factor and the carbon rotation with interest rate $r = 0.03$. From these, we can see that in general, if $\beta < 0.45$, then $T^C < T^F$ for both species. Therefore, if about more than half of the sequestered carbon is not released into the atmosphere, the optimal rotation is shorter than the Faustmann rotation. Alternatively, assume that any sequestered carbon eventually goes back to the atmosphere but the reflux occurs exponentially with a constant decay rate b. Then, the negative present value of carbon release at the harvest point in time is given by $[\alpha b/(b + r)]\, G(T)$. With the carbon release factor $\beta = b/(b + r) = 0.45$ and the interest rate $r = 0.03$, the exponential decay rate is calculated as $b = 0.025$. The corresponding half-life is about 28 years. Therefore, if it takes more than 3 decades for half of the sequestered carbon to be released into the atmosphere, the optimal rotation is shorter than the Faustmann rotation. IPCC (2003, Appendix 3a.1) surveyed studies of the half life of harvested wood products. According to the results, the half lives of saw wood, veneer, plywood and structural panels, nonstructural panels, and paper are 35, 30, 20, and 2 years, respectively. Because Japanese cedar and cypress are used as a saw wood, we conclude that $T^* < T^F$ could occur in forests planted with these species.

[2] We use the term "cutting age" rather than "rotation," because for some forest units regeneration is expected to be unprofitable.

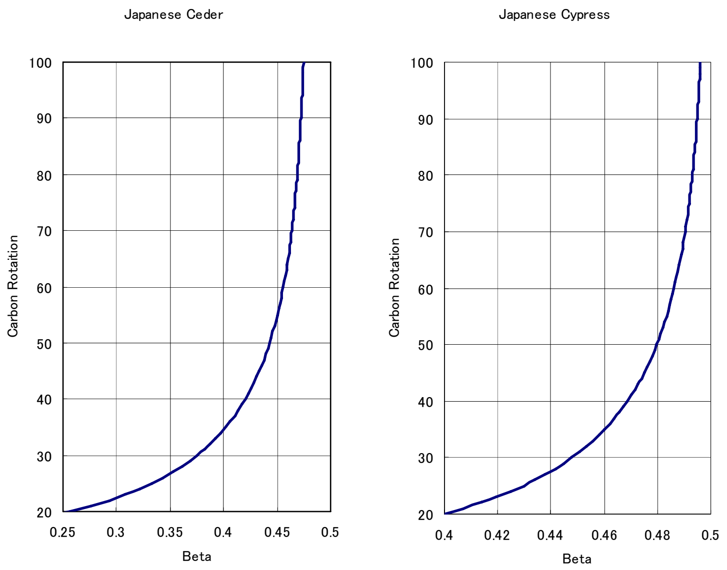

Figure 1: Carbon release factor and carbon rotations

2.4 Comparative statics

We show comparative statics results on α and β. The results give useful information on forest practice when the value of carbon sequestration rises relative to the price of forest products or when the rate of release of carbon from forest products into the atmosphere changes. Denote the optimal rotation by $T^*(\alpha, \beta)$ as a function of α and β. We assume that $T^*(\alpha, \beta)$ is unique for each (α, β). Formally:

A3: $V^*(T)$ is twice differentiable and there is a unique optimal rotation $T^*(\alpha, \beta)$, satisfying the first order condition $\partial V^*(T^*(\alpha, \beta))/\partial T = 0$ and the second order condition $\partial^2 V^*(T^*(\alpha, \beta))/\partial T^2 < 0$, respectively.

The total derivative of the first order condition is:

$$\frac{\partial^2 V^*(T^*(\alpha, \beta))}{\partial T^2} dT^* + \frac{\partial V^C(T^*(\alpha, \beta))}{\partial T} d\alpha + \alpha \frac{\partial^2 V^C(T^*(\alpha, \beta))}{\partial T \partial \beta} d\beta = 0. \tag{10}$$

Because $\partial^2 V^*(T^*(\alpha, \beta))/\partial T^2 < 0$ by assumption A3, the signs of $\partial T^*/\partial \alpha$ and $\partial T^*/\partial \beta$ coincide with the signs of $\partial V^C(T^*(\alpha, \beta))/\partial T$ and $\partial^2 V^C(T^*(\alpha, \beta))/\partial T \partial \beta$, respectively. Then, we have:

Proposition 5 *Under assumptions A1–A3, (a) the optimal rotation T^* approaches T^C (possibly, infinity) as α increases. (b) $\partial T^*/\partial\beta \lesseqgtr 0$ if $T^* \lesseqgtr T_L$, where T_L is defined by (5) above Lemma 1.*

Proof. (a) By Lemma 1, $V^C(T)$ is strictly increasing or unimodal. Furthermore, as mentioned below (10), $\partial V^C(T^*(\alpha,\beta))/\partial T \lesseqgtr 0 \Rightarrow \partial T^*/\partial\alpha \lesseqgtr 0$. Therefore, as α increases, T^* approaches the age of the peak of $V^C(T)$ (possibly, infinity). (b) Notice that the sign of $\partial^2 V^C(T^*(\alpha,\beta))/\partial T\partial\beta$ coincides with the sign of $\Psi(T)$, where $\Psi(T)$ is defined by (41) in the Appendix. Then, the statement follows from Sublemma 5 in the Appendix. ■

The result for α is quite intuitive. That is, as the value of carbon rises, the forest function of carbon sequestration is more valuable than forest products and the optimal rotation approaches the carbon rotation. To understand the result for carbon release factor β, notice that the present value of the cost of carbon release is given by $\beta G(T)e^{-rT}/(1-e^{-rT})$. This cost is maximized at T_L. Therefore, as β goes up, the optimal rotation tends to move away from T_L. The above numerical example with interest rate $r = 0.03$, $T_L = 8$ for Japanese cedar and $T_L = 0$ for Japanese cypress, indicates that $\partial T^*/\partial\beta > 0$ is plausible for most forests planted with these species. Furthermore, $\partial T^*/\partial\beta > 0$ always holds for the model by van Kooten et al. (1995). Under the assumptions in Proposition 3, T_L is the Faustmann rotation. Proposition 3 proves $T^* > T^F = T_L$. Then, $\partial T^*/\partial\beta > 0$ follows from Proposition 5.

3 Uneven-aged forest management

This section studies an uneven-aged forest version of the carbon forest problem. Proposition 4.1 of Mitra et al. (1991) suggests that when a forest program is formulated as a linear control problem, the optimal control is to apply the Faustmann formula to each tree in the forest. Therefore, it is adequate to analyze an even-aged forest model. However, the assumption of linearity may not be very plausible when one considers forest management across a broad area such as a water basin, or a state. On that scale, as Dasgupta (1982, chapter 9) pointed out, it could be a policy target to smooth out time variations of timber supply and employment in forest industries. This indicates that an appropriate objective function for forest management should be nonlinear.

3.1 Mitra–Wan tree farm with multiple objectives

In this study, we extend the nonlinear forest model developed by Mitra and Wan (1985) to allow multiple objectives, i.e., forest production and carbon sequestration. Following Mitra and Wan (1985), we use a discrete time model, by which we can avoid some mathematical burden.[3] The notation in the previous section is modified for the discrete time case. F_i denotes harvest revenue from a forest at the age $i \in \mathbb{Z}_+$ on a unit of land. G_i denotes the amount of the carbon stock. The associated carbon sequestration rate g_i is defined by $g_i = G_{i+1} - G_i$ with $G_0 = 0$. We assume that:

A4: There is a positive integer N such that $F_i = F_N$, $G_i = G_N$ for all $i \geq N$.

This assumption implies that a forest reaches a biological equilibrium in a finite time. With this assumption, we regard all trees at ages more than N as the ones at the age N.[4] We define the following $N + 1$ dimensional vectors:

$$F = (c, F_1, ..., F_N), \tag{11}$$

$$G = (0, G_1, ..., G_N), \tag{12}$$

$$\text{and } g = (g_0, ..., g_{N-1}, 0). \tag{13}$$

The following matrices and vectors are used to express the state and the transition of an uneven-aged

[3] A continuous time model for an uneven-aged forest is studied by Heaps (1984) as an optimal control problem with delay. The continuous time counterpart of our model must treat a control of a functional differential equation, which is rather technical. See Aniţa et al. (1998) and Feichtinger et al. (2003).

[4] Formally, we just need a state space with a finite dimension. Thus, we can alternatively assume, for example, that the value of F_i begins to decline at a certain age as Mitra and Wan (1985) assumed.

forest.

$$A = \underbrace{\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & 0 \\ 0 & & 1 & 1 \end{bmatrix}}_{(N+1)\times(N+1) \text{ matrix}}, \quad B = \underbrace{\begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}}_{(N+1)\times(N+1) \text{ matrix}}, \quad \nu = \underbrace{(1, ..., 1)}_{N+1 \text{ vector}}. \tag{14}$$

Consider a unit land that is utilized for forestry. The states of the forest at the beginning and the end of the period $t \in \mathbb{Z}_+$ are denoted by the vectors:

$$x_{t-1} = (x_{t-1,0}, x_{t-1,1}, ..., x_{t-1,i}, ..., x_{t-1,N}) \in \mathbb{R}_+^{N+1},$$
$$\text{and } x_t = (x_{t,0}, x_{t,1}, ..., x_{t,i}, ..., x_{t,N}) \in \mathbb{R}_+^{N+1},$$

respectively. The first element $x_{t,0}$ is the area just planted at the end of the period t. The state space of the forest resources is given by:

$$X = \{x_t \in \mathbb{R}_+^{N+1} | \nu x_t \leq 1\},$$

for each $t \in \mathbb{Z}_+$. The area $1 - \nu x_t \geq 0$ is the land that is not utilized. The set of feasible input-output pairs (x_{t-1}, x_t) in period t is defined by:

$$D = \{(x_{t-1}, x_t) \in X^2 | B\left(Ax_{t-1} - x_t\right) \geq 0\}. \tag{15}$$

Given an initial state $x_0 \in X$, if a forest program $\{x_t : t \in \mathbb{Z}_+\}$ satisfies $(x_{t-1}, x_t) \in D$ for each $t \in \mathbb{N}$, then the forest program is feasible.

For an input-output pair $(x_{t-1}, x_t) \in D$ in the period t, the area on which trees at the age i are cut is given by $x_{t-1,i-1} - x_{t,i}$. The associated harvest revenue is $F_i\left(x_{t-1,i-1} - x_{t,i}\right)$. The planting cost is given by $cx_{t,0}$. Therefore, the net harvest revenue π_t associated with the pair (x_{t-1}, x_t) is given by:

$$\pi_t = F\left(Ax_{t-1} - x_t\right). \tag{16}$$

Similarly, the amount of released carbon because of harvesting is counted as $\beta G\left(Ax_{t-1} - x_t\right)$. Here, $\beta \in [0, 1]$ is the carbon release factor as before. The amount of sequestered carbon in period t is given by gx_{t-1}. Therefore the net carbon sequestration in period t is given by:

$$\zeta_t = gx_{t-1} - \beta G\left(Ax_{t-1} - x_t\right). \tag{17}$$

Let $w(\pi, \zeta)$ be the one period utility function. We assume that:

A5: $w : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuously differentiable, strictly increasing, and strictly concave function.

As a reduced form of the utility function $w(\pi, \zeta)$, we use the function $u :\ D \to R$ defined by:

$$u(x, y) = w[F(Ax - y), gx - \beta G(Ax - y)]. \tag{18}$$

It is easily verified that the reduced form utility function u is a continuously differentiable and concave function.

The optimal forest program is a solution of the following problem:

$$\max_{\{x_t : t \in \mathbb{Z}_+\}} \sum_{t=1}^{\infty} \rho^{t-1} u(x_{t-1}, x_t) \tag{19}$$

$$\text{subject to } (x_{t-1}, x_t) \in D \text{ all } t \in \mathbb{Z}_+,\ x_0 = x \in X \text{ given,}$$

where $\rho \in (0, 1)$ is the discount factor. The existence of the solution is ensured by Theorem 4.6 in Stokey and Lucas (1989). Our model (19) extends Mitra and Wan's (1985) tree farm model by incorporating the multiple objectives of forest management. As other new features, we allow a positive planting cost c and allow the possibility that never harvesting can be optimal.

3.2 Normal forest

A normal forest is a forest that has equal amounts of forest land in each of the age classes below the rotation age. Letting the rotation age be $m \in \{1, 2, ..., N\}$, the normal forest with rotation m is defined by:

$$\tilde{x}(m) = (x_0^m, x_1^m, ..., x_i^m, ..., x_N^m), \text{ where } x_i^m = \begin{cases} 1/m, & \text{if } i < m, \\ 0, & \text{if } i \geq m. \end{cases} \quad . \tag{20}$$

We also include, as a member of normal forests, the "never harvesting program":

$$\tilde{x}(\infty) = (0, ..., 0, 1).$$

We refer to a forest x_{ss} such that $(x_{ss}, x_{ss}) \in D$ as a sustainable forest. Denote the set of sustainable forests by $\Delta = \{x \in X |\ (x, x) \in D\}$. Any sustainable forest $x \in \Delta$ is expressed as a convex combination of normal forests, $\{\tilde{x}(1), ..., \tilde{x}(N), \tilde{x}(\infty)\}$, and the null vector $0 \in \mathbb{R}^{N+1}$. That is, letting $\gamma_i = x_{i-1} - x_i$ for $i = \{1, ..., N\}$ and $\gamma_\infty = x_N$, where x_i is the ith element of x, $x \in \Delta$ can be decomposed as:

$$x = \sum_{i \in \{1,...,N,\infty\}} \gamma_i \tilde{x}(i),\ \ \gamma_i \geq 0 \text{ all } i, \quad \sum_{i \in \{1,...,N,\infty\}} \gamma_i \leq 1. \tag{21}$$

3.3 optimal sustainable forest

We refer to an optimal steady state in the forest management problem (19) as an optimal sustainable forest. This subsection relates an optimal sustainable forest to the solution of the even-aged forest model (3) studied in the previous section. The discrete time counterpart of (3) is given by:

$$V^{**}(\alpha) = \max_{i \in \{1,...,N,\infty\}} V^*(i;\alpha), \quad V^*(i;\alpha) = \frac{(\rho^i F_i - c) + \alpha \left(\sum_{t=1}^{i} \rho^t g_{t-1} - \rho^i \beta G_i \right)}{1-\rho^i}, \tag{22}$$

where the symbol ∞ implies "never harvesting," the revenue of which is $\alpha \sum_{t=1}^{N} \rho^t g_t - c$. We assume:

A6: There is $i \in \{1, ..., N\}$ such that $\rho^i F_i - c > 0$.

With this assumption, a counterpart of assumption A1, $V^{**}(\alpha)$ is always positive and thus the solution to (22) is meaningful. Let:

$$\mathcal{T} : \mathbb{R}_+ \twoheadrightarrow \{1, 2, ..., N, \infty\} \tag{23}$$

be the correspondence that corresponds $\alpha \in \mathbb{R}_+$ to the set of solutions of the problem (22). We denote by $\#\mathcal{T}(\alpha)$ the number of solutions in the set $\mathcal{T}(\alpha)$. Furthermore, we denote the elements of $\mathcal{T}(\alpha)$ by $T_i(\alpha)$ $(i = 1, ..., \#\mathcal{T}(\alpha))$.

As mentioned above, if the one period welfare function w is linear, then each tree is treated to maximize the relevant Faustmann formula in an optimal forest program. For a sustainable forest, we formally state this fact and relate it to the duality approach developed by Weitzman (1973). Define function $l : X \times X \times \mathbb{R}_+ \to \mathbb{R}$ by:

$$l(x, y, \alpha) = F(Ax - y) + \alpha \left[gx - \beta G(Ax - y) \right], \tag{24}$$

and let us call the following problem the linear forest program:

$$\max_{\{x_t : t \in \mathbb{Z}_+\}} \sum_{t=1}^{\infty} \rho^{t-1} l(x_{t-1}, x_t, \alpha) \tag{25}$$

subject to $(x_{t-1}, x_t) \in D$ all $t \in \mathbb{Z}_+$, $x_0 = x \in X$ and $\alpha \in \mathbb{R}_+$ given.

Lemma 2 *Assume assumptions A4–A6. Fix α and take arbitrarily a sustainable forest $x_{ss} \in \Delta$ such that:*

$$x_{ss} = \sum_{i=1}^{\#\mathcal{T}(\alpha)} \gamma_i \tilde{x}(T_i(\alpha)), \sum_{i=1}^{\#\mathcal{T}(\alpha)} \gamma_i = 1, \gamma_i \geq 0 \text{ all } i = 1, ..., \#\mathcal{T}(\alpha).$$

(a) The continuation of x_{ss} (i.e., $x_t = x_{ss}$ for all $t \in \mathbb{Z}_+$) is a solution to the linear forest problem (25) starting from $x_0 = x_{ss}$. (b) There is a support price $p(\alpha) \in \mathbb{R}^{n+1}$ to the optimal sustainable forest x_{ss} such that for all $(x, y) \in D$, it holds that:

$$l(x_{ss}, x_{ss}, \alpha) - p(\alpha)x_{ss} + \rho p(\alpha)x_{ss} \geq l(x, y, \alpha) - p(\alpha)x + \rho p(\alpha)y. \tag{26}$$

Proof. For (a), apply Proposition 4.1 in Mitra et al. (1991). (b) follows from Theorem 7.1 in McKenzie (1986). ■

Remark: By applying Mitra and Wan's (1985) method, we can explicitly find a support price $p(\alpha)$, although it is not necessarily unique. Define $p = (p_0, ..., p_N)$ by:

$$\begin{aligned} p_0 &= \rho^{-1}c \\ p_i &= \rho^{-i-1}\left[\left(1-\rho^i\right)V^{**}(\alpha) + c - \alpha\sum_{t=1}^{i}\rho^t g_{t-1}\right] \text{ for } i \in \{1, ..., N\}. \end{aligned}$$

The candidate for the support price p has the following properties: First, by (22):

$$\rho p_i \geq F_i + \alpha\beta G_i \tag{27}$$

for $i \in \{1, ..., N\}$ with equality if $i \in \{0\} \cup \mathcal{T}(\alpha)$. Second:

$$\rho p_i - p_{i-1} = \rho^{-1}(1-\rho)V^{**}(\alpha) - \alpha g_{i-1} \text{ for all } i \in \{1, ..., N\}. \tag{28}$$

Finally, by the definition of $V^{**}(\alpha)$:

$$\rho^{-1}V^{**}(\alpha) + p_N = \rho^{-N-1}\left[V^{**}(\alpha) - \left(\alpha\sum_{t=1}^{N}\rho^t g_{t-1} - c\right)\right] \geq 0, \tag{29}$$

with equality if $\{\infty\} \in \mathcal{T}(\alpha)$, i.e., if never harvesting is a solution of the relevant Faustmann formula (22).

Then, for any $(x, y) \in D$, we have:

$$\begin{aligned}
l(x, y, \alpha) - px + \rho py &= (F - \alpha\beta G)(Ax - y) + \alpha gx - p(\alpha)x + \rho p(\alpha)y \\
&\leq \rho p(\alpha)(Ax - y) + \alpha gx - p(\alpha)x + \rho p(\alpha)y. \\
&= [\rho p(\alpha)A + \alpha g - p(\alpha)]\, x \\
&= \rho^{-1}(1-\rho)\, V^{**}(\alpha)(\nu x - x_N) + (\rho - 1)\, p_N x_N \\
&= \rho^{-1}(1-\rho)\, V^{**}(\alpha)\nu x - (1-\rho)\left(\rho^{-1}V^{**}(\alpha) + p_N\right) \\
&\leq \rho^{-1}(1-\rho)\, V^{**}(\alpha)\nu x \\
&\leq \rho^{-1}(1-\rho)\, V^{**}(\alpha).
\end{aligned}$$

Here, the second line uses (27), the forth line uses (28), and the sixth line uses (29). The inequality in the last line follows from $V^{**}(\alpha) > 0$, a consequence of assumption A6. All above inequalities hold with equality if each tree is planted and cut following the relevant Faustmann formula (22). Therefore, p is a support price for the linear forest program (25).

Now, using Lemma 2, we have the main result of this section.

Proposition 6 *Assume assumptions A4–A6. (a) The optimal forest program (19) has an optimal* *sustainable forest $x_{ss}^* \in \Delta$. (b) Let:*

$$\pi^* = F(Ax_{ss}^* - x_{ss}^*),\ \zeta^* = gx_{ss}^* - \beta G(Ax_{ss}^* - x_{ss}^*),\ \textit{and}\ \alpha^* = \frac{\partial w\left(\pi^*, \zeta^*\right)/\partial \zeta}{\partial w\left(\pi^*, \zeta^*\right)/\partial \pi}. \tag{30}$$

Then, there are some numbers γ_i $(i = 1, ..., \#\mathcal{T}(\alpha^*))$ such that $\sum_{i=1}^{\#\mathcal{T}(\alpha^*)} \gamma_i = 1$ and $\gamma_i \geq 0$ all i, and x_{ss}^* satisfies:

$$x_{ss}^* = \sum_{i=1}^{\#\mathcal{T}(\alpha^*)} \gamma_i \tilde{x}(T_i(\alpha^*)). \tag{31}$$

Proof. The proof is divided into two steps. Step 1 shows that there is a sustainable forest $x_{ss}^* \in \Delta$ that satisfies (30) and (31). Step 2 verifies that $x_{ss}^* \in \Delta$ is indeed an optimal sustainable forest in the optimal forest program (19).

Step 1: Define function $\alpha : \Delta \to \mathbb{R}_{++}$ by:

$$\alpha(x) = \frac{\partial w[F(Ax - x), gx - \beta G(Ax - x)]/\partial \zeta}{\partial w[F(Ax - x), gx - \beta G(Ax - x)]/\partial \pi}.$$

Furthermore, let $M : 2^{\{1,2,...,N,\infty\}} \twoheadrightarrow \Delta$ correspond to a set of rotations $J \subset \{1, 2, ..., N, \infty\}$ for a subset of sustainable forests:

$$\left\{x \in \Delta | x = \textstyle\sum_{i \in J} \gamma_i \tilde{x}(i),\ \gamma_i \geq 0,\ \textstyle\sum_{i \in J} \gamma_i \leq 1\right\},$$

where $\tilde{x}(i)$ is the normal forest with rotation i defined in (20). Then, consider the correspondence $\Phi = M \circ \mathcal{T} \circ \alpha : \Delta \twoheadrightarrow \Delta$, where the correspondence $\mathcal{T}$ is defined in (23). We can show that $\Phi : \Delta \twoheadrightarrow \Delta$ is upper hemicontinuous with nonempty convex compact values: Obviously, α and M are continuous. $\mathcal{T}$ is upper hemicontinuous by Berge's theorem of maximum. Therefore, Φ is upper hemicontinuous. Other properties are easily verified. Therefore, Kakutani's fixed point theorem can be applied to ensure the existence of x_{ss}^*. That is, there is $x_{ss}^* \in \Delta$ such that (a) $\alpha^* = \alpha(x_{ss}^*)$, and (b) $x_{ss}^* \in M \circ \mathcal{T}(\alpha^*) \subset \Delta$.

Step 2: Take arbitrarily $(x, y) \in D$ and let:

$$\pi = F(Ax - x) \text{ and } \zeta = gx - \beta G(Ax - x).$$

Then, we have:

$$\begin{aligned} u(x_{ss}^*, x_{ss}^*) - u(x, y) &= w(\pi^*, \zeta^*) - w(\pi, \zeta) \\ &\geq \left(\partial w\left(\pi^*, \zeta^*\right)/\partial \pi\right)\left[l\left(x_{ss}^*, x_{ss}^*, \alpha^*\right) - l(x, y, \alpha^*)\right] \\ &\geq \left(\partial w\left(\pi^*, \zeta^*\right)/\partial \pi\right)\left[(p(\alpha^*)x_{ss}^* - \rho p(\alpha^*)x_{ss}^*) - (p(\alpha^*)x - \rho p(\alpha^*)y)\right], \end{aligned}$$

where the inequality in the second line follows from the concavity of w and the inequality in the last line follows from Lemma 2. Note that x_{ss}^* is an optimal sustainable forest in the linear forest program (25) when $\alpha = \alpha^*$. Therefore, there is a support price for x_{ss}^*:

$$q = \left(\partial w\left(\pi^*, \zeta^*\right)/\partial \pi\right) p(\alpha^*),$$

and

$$u(x_{ss}^*, x_{ss}^*) - qx_{ss}^* + \rho q x_{ss}^* \geq u(x, y) - qx + \rho q y, \text{ all } (x, y) \in D \tag{32}$$

holds. Because the state space X is compact, the transversality condition $\lim_{t\to\infty} \rho^t q x_t = 0$ holds for any feasible path $\{x_t | t \in \mathbb{Z}_+\}$. Therefore, the inequality (32) is sufficient for x_{ss}^* to be an optimal sustainable forest by Weitzman's Theorem (Weitzman, 1973). ■

Proposition 6 shows that the optimal sustainable forest coincides with an optimal steady state of the linear forest program (25) and, therefore, consists of the normal forests with the rotation(s) which are the solution(s) of the relevant Faustmann formula (22). This result has been obtained by Mitra and

Wan (1985, 1986) for an uneven-aged forest model with a single objective. However, a complication arises from the extension to a multipurpose model. That is, the appropriate shadow price of carbon sequestration α^* is endogenously determined by the fixed point calculation, as shown in the proof of Proposition 6.

Remark: The stability of an optimal sustainable forest is an interesting topic. Mitra and Wan (1985) and Wan (1989, 1994) illustrated the existence of an optimal path that does not converge to an optimal sustainable forest. Mitra et al. (1991, Proposition 5.2) proved for their model that all optimal paths, except for the case that the initial state coincides with an optimal sustainable forest, do not converge regardless of the magnitude of the discount factor $\rho < 1$, which is a novel result in capital theory (also see Wan, 1993 and Nishimura and Yano, 1995). However, one can show that if an optimal sustainable forest is unique under the parameter values considered, then it is generically true that corresponding to a neighborhood of the optimal sustainable forest, there is the lower bound of the discount factor such that any optimal path converges to the neighborhood when the discount factor is greater than the lower bound. This so called neighborhood turnpike theorem can be obtained from a straightforward application of Proposition 5.3 in Mitra et al. (1991). Stronger results are provided by Salo and Tahvonen (2002) and Sahashi (2002), who proved for their models that when the model is extended to allow the forest land size to be endogenously determined, an optimal sustainable forest is asymptotically stable when the discount factor is close to one.

3.4 Characterization of an optimal sustainable forest

In this subsection, we examine characteristics of the optimal sustainable forest x^*_{ss}. The duality equation (32) in the proof of Proposition 6 allows us to interpret x^*_{ss} as the solution to the following problem:

$$\max_x u(x, x) \text{ subject to } z - (1 - \rho)qx \geq 0, x \in \Delta, \tag{33}$$

where $z = (1-\rho)qx^*_{ss}$. Note that the associated Lagrange multiplier is equal to one and the Lagrangian satisfies:

$$u(x^*_{ss}, x^*_{ss}) = u(x^*_{ss}, x^*_{ss}) + [z - (1-\rho)qx^*_{ss}] \geq u(x, x) + [z - (1-\rho)qx] \geq u(x, x)$$

for all $x \in \Delta$. We want to transform the problem (33) to the maximization problem in terms of forest products and sequestered carbon, π and ζ. The objective function $u(x, x)$ is replaced with $w(\pi, \zeta)$. Correspondingly, the constraint is written as:

$$\{(\pi, \zeta) | \pi = F(A - I)x,\ \zeta = gx - G(A - I)x, z - (1-\rho)qx \geq 0, x \in \Delta\}, \tag{34}$$

where I is the $N+1$-dimensional unit matrix. Recall that the set of sustainable forests Δ is given by:

$$\Delta = \{x \in X | x = \sum_{i \in \{1,...,N,\infty\}} \gamma_i \tilde{x}(i),\ \gamma_i \geq 0 \text{ all } i, \text{ and } \sum_{i \in \{1,...,N,\infty\}} \gamma_i \leq 1\},$$

where $\tilde{x}(i)$ is the normal forest with rotation i defined in (20). Define the vertex on the $\pi - \zeta$ plain associated with the normal forest $\tilde{x}(i)$ by:

$$(\pi(i), \zeta(i)) = (F(A - I)\tilde{x}(i), g\tilde{x}(i) - G(A - I)\tilde{x}(i)).$$

Furthermore, define a set of rotations $\mathcal{S}$ by:

$$\mathcal{S} = \{i \in \{1, ..., N, \infty\} | z - (1-\rho)q\tilde{x}(i) \geq 0\}. \tag{35}$$

Then, the constraint (34) is rewritten as:

$$\{(\pi,\zeta)|\pi = \textstyle\sum_{i\in\mathcal{S}}\gamma_i\pi(i),\ \zeta = \sum_{i\in\mathcal{S}}\gamma_i\zeta(i), \gamma_i \geq 0 \text{ all } i, \text{ and} \sum_{i\in\mathcal{S}}\gamma_i \leq 1\}. \tag{36}$$

Therefore, we have the problem equivalent to (33):

$$\max_{(\pi,\zeta)} w(\pi,\zeta) \text{ subject to (36).}$$

Figure 2 illustrates the solution. The white vertices exemplify that the associated $\tilde{x}(i)$ does not satisfy the constraint in (35), i.e., $i \notin \mathcal{S}$. For the black vertices, the associated rotations belong to $\mathcal{S}$. As shown in Figure 2, the number of optimal rotations to sustain an optimal sustainable forest x_{ss}^* is generically one or two. For the case with two optimal rotations (Figure 2(b)), the forest is divided into two zones covered with two different normal forests. Because the objectives are multiple, it may appear natural to zone a forest into two areas. However, zoning is not always necessary as shown in Figure 2 (a).

4 Concluding remarks

This paper studied forest management models when a forest provides forest products and the carbon sequestration service. The optimal rotation may be longer or shorter than the one without consideration of carbon sequestration potential. An intuitive explanation of this result is that concerned with the global warming, a forest has two functions: to sequester the atmospheric carbon and to postpone sequestered carbon release. If the latter function is relatively more important, then the optimal rotation becomes longer by considering the carbon sequestration potential and vice versa. Formally, whether the optimal rotation becomes shorter or longer depends on how the negative effect of carbon release at harvest is valued. How to value the carbon in harvested wood products and its socioeconomic and

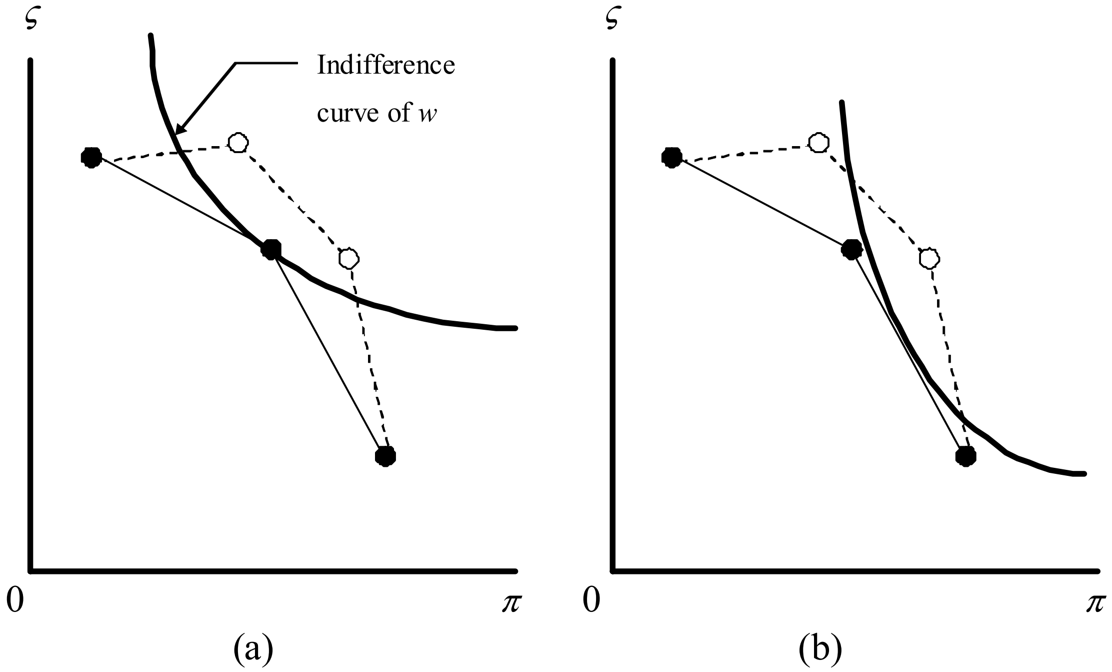

Figure 2: Optimal stationary forest on the π-ζ plain

environmental implications has been an important issue in the UNFCCC and its Kyoto Protocol. The results obtained in this study shed a new light on this issue. Our comparative statics results reveal the effects of the carbon release factor on the optimal rotation of even-aged forest management. The results are applied to an optimal sustainable forest of uneven-aged forest management, indicating the long term pattern of forest resources across a broad area such as a water basin or a state.

Appendix: Proof of Lemma 1

To prove the Lemma, we prepare a series of sublemmas. Define:

$$\Phi(T;\beta) = r^{-1}\left(1-e^{-rT}\right)(1-\beta)g(T) + \beta G(T) - \int_0^T g(t)e^{-rt}dt. \tag{37}$$

Sublemma 1 *The sign of $dV^C(T)/dT$ is the same as the sign of $\Phi(T;\beta)$.*

Proof. This follows from:

$$\frac{dV^C(T)}{dT} = \frac{re^{-rT}}{(1-e^{-rT})^2}\Phi(T;\beta). \tag{38}$$

■

Let

$$k = -\lim_{t\to\infty} g'(t)/g(t) \in (0,\infty]. \tag{39}$$

Sublemma 2 *The graph of $(T, \Phi(T;\beta))$ is either strictly increasing or unimodal: If $\beta \geq k/(r+k)$, then $\partial\Phi(T;\beta)/\partial T > 0$ for all T. If $\beta < k/(r+k)$, then there exists $T^o > 0$ such that $T^o > T_I$ and $\partial\Phi(T;\beta)/\partial T \gtreqless 0$ if $T \lesseqgtr T^o$.*

Proof. The statement follows from:

$$\frac{\partial\Phi(T;\beta)}{\partial T} = \frac{1-e^{-rT}}{r}(1-\beta)g(T)\left[\frac{g'(T)}{g(T)} + \frac{r\beta}{1-\beta}\right]. \tag{40}$$

Note that $\Phi(0;\beta) = 0$ and $\partial\Phi(T;\beta)/\partial T > 0$ for all $T \in [0, T_I]$ because $g'(T) \geq 0$ on the interval. Also, note that $g'(T)/g(T)$ is strictly decreasing and converges to $-k$ as $T \to \infty$ by assumption A2(d). Finally, note that $k \leq r\beta/(1-\beta)$ is equivalent to $\beta \geq k/(r+k)$. ■

Define $\tilde{\Phi}$ as in (4):

$$\tilde{\Phi} = \lim_{T\to\infty} \frac{\int_0^T g(t)e^{-rt}dt}{\int_0^T g(t)dt} \in (0,1).$$

Sublemma 3 *(a) If the carbon release factor β is greater than or equal to $\tilde{\Phi}$, then $dV^C(T)/dT > 0$ for all $T > 0$ and $T^C = \infty$. (b) If $\beta \in [0, \tilde{\Phi})$, then there is a unique carbon rotation $T^C \in (T_I, \infty)$ such that $dV^C(T)/dT \gtreqless 0$ if $T \lesseqgtr T^C$.*

Proof. From Sublemma 2, if $\lim_{T\to\infty}\Phi < 0$, then there is $T^C \in (T_I,\infty)$ such that $\Phi(T;\beta) \gtreqless 0$ if $T \lesseqgtr T^C$. Therefore, $dV^C(T)/dT \gtreqless 0$ if $T \lesseqgtr T^C$ by Sublemma 1. Similarly, if $\lim_{T\to\infty}\Phi \geq 0$, then $dV^C(T)/dT > 0$ and therefore, $T^C = \infty$. Notice that $\lim_{T\to\infty}\Phi \geq 0$ is equivalent to $\beta \geq \tilde{\Phi}$. ■

Define function $\Psi(T)$ by:

$$\Psi(T) = \frac{\partial \Phi(T;\beta)}{\partial \beta} = -g(T)(1-e^{-rT}) + rG(T). \tag{41}$$

Sublemma 4 *(a) The carbon rotation T^C as a function of β is continuously differentiable on $\beta \in [0,\tilde{\Phi})$. (b) The sign of $\partial T^C/\partial \beta$ is the same as the sign of $\Psi(T^C)$ for all $\beta \in [0,\tilde{\Phi})$.*

Proof. Fix $\beta \in [0,\tilde{\Phi})$. Then, there is unique $T^C(> T_I)$ and $d^2V^C(T^C)/dT^2 < 0$ holds by Sublemma 3. By the total derivative of the first order condition:

$$\frac{dV^C(T^C)}{dT} = \frac{re^{-rT^C}}{\left(1-e^{-rT^C}\right)^2}\Phi(T^C;\beta) = 0,$$

we have:

$$\frac{d^2V^C(T^C)}{dT^2}dT^C + \frac{re^{-rT^C}}{\left(1-e^{-rT^C}\right)^2}\frac{\partial \Phi(T^C;\beta)}{\partial \beta}d\beta = 0.$$

Since $d^2V^C(T^C)/dT^2 \neq 0$, we have the statement (a) by the implicit function theorem. Also, $\partial T^C/\partial \beta$ *satisfies:*

$$\frac{\partial T^C}{\partial \beta} = -\frac{\left[re^{-rT^C}/\left(1-e^{-rT^C}\right)^2\right]\Psi(T^C)}{d^2V^C(T^C)/dT^2}.$$

Since $d^2V^C(T^C)/dT^2 < 0$, we have the statement (b). ■

Define T_L by:

$$T_L = \max\{T \geq 0 | \Psi(T) = 0\}.$$

Note that this definition is equivalent to (5) in the main text.

Sublemma 5 *Assume* $\beta \in [0,\tilde{\Phi})$. $\Psi(T) \lesseqgtr 0$ *if and only if* $T \lesseqgtr T_L$.

Proof. Note that $\Psi(0) = 0$, $\lim_{T\to\infty}\Psi(T) > 0$, and

$$\Psi'(T) = g(T)(1-e^{-rT})\left(r - g'(T)/g(T)\right).$$

Also, note that $g'(T)/g(T)$ is strictly decreasing by assumption A2(d). Therefore, if $\lim_{t\to 0} g'(t)/g(t) \leq r$, then $T_L = 0$, and $\Psi(T) > 0$ for all $T > 0$. If $\lim_{t\to 0} g'(t)/g(t) > r$, then $T_L > 0$, and $\Psi(T) \lesseqgtr 0$ if and only if $T \lesseqgtr T_L$. ■

Sublemma 6 *Assume* $\beta \in [0, \tilde{\Phi})$. $\Psi(T^C) > 0$ and $T^C > T_L$.

Proof. If $T_L = 0$, then $T^C > T_I > 0 = T_L$ by Sublemma 3(b) and $\Psi(T^C) > 0$ by Sublemma 5. For the case of $T_L > 0$, T_L satisfies:

$$\Psi(T_L) = -g(T_L)(1 - e^{-rT_L}) + rG(T_L) = 0.$$

Using the fact that T^C satisfies:

$$0 = \Phi(T^C; \beta) = -r^{-1}(1-\beta)\Psi(T^C) + G(T^C) - \int_0^{T^C} g(t)e^{-rt}dt,$$

we have:

$$r^{-1}(1-\beta)\Psi(T^C) = G(T^C) - \int_0^{T^C} g(t)e^{-rt}dt > 0.$$

Therefore, $0 < \Psi(T^C)$. This implies $T^C > T_L$ by Sublemma 5. ■

Proof of Lemma 1 Lemma 1 (a) and (b) follow from Sublemma 3. As for (c), by Sublemma 4, the carbon rotation as a function of β, $T^C(\beta)$, is continuously differentiable at $\beta \in [0, \tilde{\Phi})$. Sublemmas 4 and 6 imply $\partial T^C/\partial\beta > 0$. $\lim_{\beta\to 0} T^C(\beta) > \max\{T_I, T_L\}$ follows from Sublemmas 3 and 6. Finally, since $\partial T^C/\partial\beta > 0$, we can consider the inverse function of $T^C(\beta)$. Denote it by denote by $\beta(T^C)$. Since the first order condition $\Phi(T^C; \beta(T^C)) = 0$ holds, where Φ is defined in (37), we have:

$$\beta(T^C) = \frac{\int_0^{T^C} g(t)e^{-rt}dt - r^{-1}\left(1 - e^{-rT^C}\right)g(T^C)}{G(T^C) - r^{-1}\left(1 - e^{-rT^C}\right)g(T^C)} \to \tilde{\Phi},$$

as $T^C \to \infty$. This implies $\lim_{\beta\to\tilde{\Phi}} T^C(\beta) = \infty$

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