Asymmetric Strategies in Symmetric Tullock Contests

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Abstract

In this paper we show that, for an appropriately chosen strategy space, the equilibrium outcome of a Tullock contest with a symmetric success function is characterized by a single player winning with probability one. This equilibrium outcome fits the stylized fact that there are many uncontested elections and undefended court actions.

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1 Introduction

Tullock's (1980) theory of contests has provided insights into a wide range of strategic behavior including elections, litigation, rent-seeking and internal labour market tournaments. The central idea is to represent strategic interactions in which different agents spend effort or resources to win a particular contest or competition. A wide range of variations of the model has been considered, with a central focus on the way on which strategic choices and rent dissipation are affected by changes in the success function relating effort to the probability of succeeding in the contest.[1] In the original Tullock model, players' probability of winning is proportional to their share of total expenditure.

One fairly robust finding is that, given a symmetric success function, there always exist symmetric Nash equilibria.[2] However, as we explain below, asymmetric equilibria do not exist in these settings. In particular, there do not exist equilibria in which only one player contributes, winning with probability 1.

In reality, though, uncontested elections and undefended court actions are common.[3] A candidate with a sufficiently clear and credible determination to win may discourage others from entering the race at all. In this note, we show that, for an appropriate (asymmetric) strategy space, the unique equilibrium outcome is one in which a single player wins with probability one.

The suggested strategy space arises naturally from the analysis of Cornes and Hartley (2003). Cornes and Hartley show that best-response choices in a wide range of contests may be analyzed by focusing on shares of total expenditure rather than on contribution levels. Since shares must add to 1, the share of total expenditure cannot be used as the strategic variable for a symmetric game. However, it is natural, when thinking about asymmetric contests to suppose that at least one player might have a strategy space consisting of possible shares of

[1] See, for example, the recent survey by Konrad (2007).

[2] See, for example, Baye, Kovenock and de Vries (1994).

[3] See, for example, Squire (2000), who reports percentage rates of uncontested seats as single and double digits, respectively, for U.S. House of Representatives and State Legislatures. This phenomenon is common across many countries – see, for example, Sharman (2003) for a discussion of uncontested seats in the context of Australian elections – and a topic of intense research in political science. See also Wrighton and Squire (1997) and Squire (1989).

expenditure.

In this note, therefore, we proceed as follows. We first demonstrate that there are no Nash equilibria of the standard Tullock contest in which only one player contributes, winning with probability 1. Next, we demonstrate the existence of a unique equilibrium with an uncontested winner for an asymmetric contest in which the winner's strategy space consists of expenditure shares. Finally, we discuss examples of contests where such asymmetric equilibria might naturally arise.

2 Symmetric Contests and their Symmetric Equilibrium

Consider a Tullock contest, for example an election or an all-pay auction, in which each player $i, i = 1, ..., n$, makes a contribution p_i with probability of winning a unit prize given by the contest success function:

$$\pi_i = \frac{p_i}{\sum_j p_j} \quad . \tag{1}$$

The payoff to player i is $u_i(p_i, p_{-i}) = \pi_i - p_i$ as the value of the prize is normalized to 1.

A standard approach to this problem is to model the contest as a game in which the strategy space for player i consists of contribution levels p_i, then to consider possible Nash equilibria of the game. Commonly, the specification of the strategy space is read directly from the contest description given above, with no further discussion of players' beliefs, institutional structures and so on.

The proposition below presents a strong prediction of this model.

Proposition 1 There are no Nash equilibria of the standard Tullock contest in which only one player contributes, winning with probability 1.

P roof. Consider a candidate equilibrium in which $p_1 > 0, p_j = 0$ for $\dot{j} \neq 1$. Then player 1 can benefit by reducing her contribution. Also, if p_1 is small enough, other players can benefit by contributing. More formally, $\frac{\partial u_1}{\partial p_1} = -1$ at

$p_2 = p_3 = ... = p_n = 0$. Similarly, Player 2's best response when $p_1 > 0$ and $p_3 = ... = p_n = 0$ is such that $\frac{\partial u_2}{\partial p_2} = \frac{1}{p_1} - 1 > 0$ at $p_2 = 0$. Thus, Player 2's best reply to $p_1 > 0$ and $p_3 = ... = p_n = 0$ involves a positive effort or contribution. ■

It is not difficult to see that the finding that no single candidate wins an election with probability one in any Nash equilibrium also holds under alternative specifications of the contest success function. One example is the asymmetric case where the probability of winning is given by $\pi_i = \frac{\lambda_i p_i}{\sum\limits_j \lambda_j p_j}$, where λ_i is an effectiveness variable. Another is the power function $\pi_i = \frac{p_i^\gamma}{\sum\limits_j p_j^\gamma}$. The proof follows the same reasoning above as it is still the case that $\frac{\partial u_1}{\partial p_1} = -1$ at $p_2 = p_3 = ... = p_n = 0$.

In the standard Tullock contest, the unique (symmetric) Nash equilibrium is such that

$$p_i = \frac{n-1}{n^2} = p \text{ for } i = 1, ..., n. \tag{2}$$

To see this, note that $\frac{n-1}{n^2}$ is the solution to $\frac{\partial u_1}{\partial p_1} \mid_{p_2=p_3=...=p_n} = \frac{1}{p_1+(n-1)p} - \frac{p_1}{(p_1+(n-1)p)^2} - 1 = 0$. That is, under a strategy space where players choose a contribution level p_i, the prediction is that all players will make positive and identical contributions.

3 Strategically Asymmetric Contests

We show next that by considering a natural modification of the standard symmetric contest, where one of the participants can commit to a probability of winning, we derive an equilibrium where this participant emerges as an uncontested winner. We refer to this class of contests as 'strategically asymmetric' to distinguish it from contests where the success function or the payoffs are asymmetric.

We model this strategically asymmetric game by assuming that player 1's strategy space is given by a number $s_1^*, 0 \leq s_1^* \leq 1$, interpreted as a share of total expenditure, while for $j \neq 1$, the strategy spaces consist of contribution levels p_j as in the standard Tullock contest. That is, having chosen s_1^*, and conditional on the strategies p_j of the other players, player 1 is required to contribute p_1^* such that

$$s_1^* = \frac{p_1^*}{\sum_j p_j}, \tag{3}$$

provided $p_j > 0$ for some $j > 1$.

To complete the game description it is necessary to specify the outcome for cases where equation (3) is ambiguous or incomplete. First, if $p_j = 0$, for all $j > 1$, then player one contributes zero and receives the prize with probability s_1^*, while the remaining players each contribute zero and receive the prize with probability $(1 - s_1^*) / (n - 1)$. Second, if player 1 chooses $s_1^* = 1$ and $p_j > 0$, for some $j > 1$, the contribution of player 1 is $M > 1$, with suitably large values of M ensuring that p_1 is near 1. The condition $M > 1$ implies that the net return for players 1 and j is negative in this case.

The following proposition characterizes the unique Nash equilibrium of this game.

Proposition 2 The asymmetric game described above has a unique pure-strategy Nash equilibrium in which player 1 chooses $s_1^* = 1$, and all players $j > 1$ choose $p_j = 0$. In this equilibrium player 1 contributes zero and receives the prize with probability 1.

P roof. First we check that $(s_1^* = 1, p_2 = p_3 = ... = p_n = 0)$ is a Nash equilibrium. To see this, note that when player 1 chooses $s_1^* = 1$ and players 3 to n choose $p_3 = ... = p_n = 0$, Player 2's best reply is to set $p_2 = 0$, since otherwise player 1 will contribute $M > 1$. This argument of course applies equally to all players $j > 1$. It is also clear that when $p_2 = p_3 = ... = p_n = 0$, Player 1's (weakly) best reply is to set $s_1^* = 1$.

Second, to show uniqueness it suffices to show that $s_1 < 1$ cannot be part of equilibrium play. Consider a candidate equilibrium $(\tilde{s}_1 < 1, p_2, p_3, ..., p_n)$. Players 2 to n now play a symmetric contest where the number of players is

$n-1$ and the value of the prize is equal to $(1-\tilde{s}_1)$.Hence, the best reply is that given by modifying equation 2 appropriately to yield the contributions:

$$\overline{p}_j = \frac{n-2}{(n-1)^2}(1-\tilde{s}_1). \tag{4}$$

Now, to test the candidate equilibrium, it is necessary to derive the best-reply choice s_1^* for player 1. We have

$$u_1(s_1^*, \overline{p}_{j,j\geq 2}) = s_1^* - p_1^*.$$

We can rewrite (3) as:

$$p_1^* = s_1^*[p_1^* + (n-1)\overline{p}_j].$$

Replacing (4) into the equation above and solving for p_1^* yields:

$$p_1^* = \frac{(1-\tilde{s}_1)}{(1-s_1^*)}\frac{n-2}{n-1}.$$

Therefore

$$u_1(s_1^*, \overline{p}_{j,j\geq 2}) = s_1^* - \frac{(1-\tilde{s}_1)}{(1-s_1^*)}\frac{n-2}{n-1}.$$

The first-order condition on s_1^* is:

$$\frac{\partial u_1}{\partial s_1^*} = 1 - \frac{(1-\tilde{s}_1)}{(1-s_1^*)}\frac{n-2}{n-1} = 0$$

so

$$\begin{aligned}\frac{\partial u_1}{\partial s_1^*}_{|s_1^*=\tilde{s}_1} &= 1 - \frac{n-2}{n-1} \\ &= \frac{1}{n-1} > 0.\end{aligned}$$

Therefore, Player 1 is better off by increasing the probability of winning. Iterating the best reply relationship yields,

$$s_1^* = \frac{1}{n-1} + \frac{n-2}{n-1}\tilde{s}_1,$$

which satisfies $s_1^* = \tilde{s}_1$ if and only if $s_1^* = \tilde{s}_1 = 1$. ■

As with proposition 1, it is easy to see that $(s_1^* = 1, p_2 = p_3 = ... = p_n = 0)$ is a Nash equilibrium for alternative specifications of the success function. As regards uniqueness, the approach of Proposition 2 may be extended to the general case where the aggregate equilibrium contribution of players $j \neq 1$ may be written as $\theta(1 - \tilde{s}_1)$ for some $\theta < 1$, and the required contribution for player 1 to achieve given π_1^* is $p_1^*(s_1^*, \theta(1 - \tilde{s}_1))$. Provided that, for all $\tilde{s}_1 < 1$,

$$\frac{\partial s_1^*}{\partial p_1^*}_{|s_1^* = \tilde{s}_1} = p_1^*(s_1^*, \theta(1 - \tilde{s}_1)) < 1,$$

the given equilibrium is unique.

The uniqueness result obtained in Proposition 2 can be compared with the uniqueness result obtained for the standard Tullock contest. For example, Szidarovszky and Okuguchi (1997) show uniqueness of equilibrium for a general success function given by $\frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)}$, where f_i is twice differentiable and $f_i'(x_i) > 0, f_i''(x_i) < 0$, and $f_i(0) = 0$. In contrast, Baye, Kovenock and de Vries (1994, 1999) have proved the existence of a mixed strategy Nash equilibrium when $f_i(x_i) = x_i^r$ and $r > 2$.

4 Discussion

As noted above, uncontested elections and undefended court actions are common, suggesting that contests may often take the strategic form described in this note. It is of interest to consider how such a strategy space might work in practice. That is, how can player 1 ensure that opponents treat his expenditure share s_1^* (and thus his probability of winning $\pi_1^* = s_1^*$) as given, rather than, as in the Nash equilibrium of the standard game, treating his absolute contribution p_1 as given.

In many contests, effort is associated with specific resources, such as legal services in the case of litigation or campaign advertising in the case of elections. In the case of litigation, suppose that player 1 retains the services of an expensive law firm, with instructions to defend any action taken by another player. Then any other player considering expending resources knows that player 1 will spend more than they will, and that increasing outlays, for example, by fighting longer court actions, will not change shares of total expenditures. Similarly, a participant in an election may retain campaign consultants, and signal a willingness to match expenditures on advertising. Provided this signal is credible, opponents will accept that they cannot reduce player 1's expenditure share or winning probability by increasing their own expenditure.

An equally important question relates to the identity of player 1. In the standard Tullock problem, as modified above, the players are symmetric in all respects except that player 1 has a different strategy space, and therefore obtains a more favorable outcome. This asymmetry raises the question of how it is determined that player 1, and not some other player, has this advantage. This question could be addressed in several different ways.

In some cases, such as that of incumbency in political contests, there is a natural reason for regarding one outcome, the incumbent's re-election, as salient, and for specifying the strategy space accordingly. In many systems, it seems reasonable for potential candidates to treat the incumbent as being able to determine the probability of her own re-election.

Alternatively, if the contest success function is asymmetric, or if some players value the prize more highly than others it seems reasonable to treat players with an advantage as leaders, determining their own probability of winning, and other players as followers. For example, a liberal candidate will normally be at a disadvantage in a conservative district, and may therefore choose to put in a substantial effort only if it appears that, for some reason, the conservative candidate has acted to reduce their own probability of winning. Through the determination of the strategy space, small asymmetries in the contest success function may have a large impact on the equilibrium outcome.[4]

Although the issue of incumbency advantage has been discussed in the contest literature, this paper makes a distinct point; namely that the determination

[4] The issue of incumbency advantage has been discussed in the contest literature. For example, Konrad (2002) considers a two-player contest where an incumbent fights with an entrant. The advantage is modelled by a sucess functions that allows the incumbent to spend less than the entrant and still win the contest.

of the strategy space may be treated as the first stage in a two-stage game.[5] Success in establishing an expenditure share strategy may reflect a mixture of inherent advantages, strategic choices and stochastic factors.

5 Concluding comments

Menezes and Quiggin (2007) argue that the determination of the strategy space has received insufficient attention in the literature on contests. By showing how the commonly observed outcome of uncontested victory may be derived as an equilibrium solution for a Tullock contest with an appropriately chosen strategy space, this paper supports that claim.

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