Multi-Stage Elections, Sequential Elimination and Condorcet Consistency*

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Abstract

A class of sequential elimination voting eliminating candidates one-at-a-time based on repeated ballots is shown to always induce an outcome in the 'top cycle' and is thus *Condrocet consistent*, when voters behave strategically. It is also well-known that multi-stage binary agenda procedures with voters always voting over only two choices, but each choice involving one or more candidates, yield outcomes in the top cycle. Thus, multi-stage elections with one-by-one elimination and without the binary choice restriction during stage games offer new ways of inducing the Condorcet winner as the unique voting outcome, when the Condorcet winner exists. This class is important as several sequential elimination voting are essentially non-binary in nature. We also show that in the class of voting games involving one or more stages and voters typically having more than two choices, Condorcet consistency often fails. This is true of all well-known one-shot voting (plurality rule, negative voting, Borda rule, instant runoff voting) as well as some multi-stage

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voting rules (*plurality runoffs*, *exhaustive ballot* method). ***JEL*** Classification Numbers: P16, D71, C72. **Key Words:** Sequential voting, binary agendas, Condorcet winner, top cycle, weakest link voting, scoring rule, exhaustive ballot, Markov equilibrium.

1 Introduction

Any voting rule as a means of reaching collective decisions can be assessed by several alternative criteria. One such criterion is whether the voting rule can give rise to an outcome that is majority-preferred to any other candidate on binary comparisons – popularly known as the *Condorcet winner* (henceforth, CW). This property, called *Condorcet consistency*, is "widely regarded as a compelling democratic principle" (Moulin, 1988; sect. 9.4).

In this paper, we will argue that a large class of voting procedures based on repeated ballots and sequential elimination will lead uniquely to the CW being elected, if it exists, when voters behave strategically. Moreover, if there is no CW, the equilibrium in this class of voting will elect a candidate in the 'top cycle,' that is, on majority comparison the winning candidate would dominate any other candidate either directly or indirectly.

The dominant sequential (elimination) voting format that attracted researchers' attention up until now are the *binary voting* and its variants (McKelvey and Neimi, 1978; Moulin, 1980; Banks, 1985; Dutta and Pattanaik, 1985; Moulin, 1988; Dutta and Sen, 1993; Dutta, Jackson and Le Breton, 2002; Bernheim and Nataraj, 2004; Berneheim et al., 2006; etc.).[1,2] While it is well-known that binary procedures yield outcomes in the top cycle (and thus are CC), such result is not directly helpful for many sequential procedures as those are not binary. However, some careful elimination logic can still rescue the top-cycle property and extend the binary voting result to a large class of sequential voting.

The broad principles underlying sequential voting can be understood by considering one specific sequential voting that we call the *weakest link voting*: Voting occurs in rounds with all the voters simultaneously casting their votes in each successive round. In any round the candidate with minimal votes is eliminated, with any ties broken by a deterministic tie-breaking rule. Continue with this process

[1]In binary voting voters vote in each round over only two choices and a choice may include one or more than one candidate. The voting proceeds by elimination of choices through the voting rounds using majority rule. With voter choice in each round restricted to only two candidates is the well-known *sequential binary voting*.

[2]Sequential (elimination) voting in this paper differs from *sequential voting* that mostly concerns with the important issue of information aggregation, as in Dekel and Piccione (2000), Strumpf (2002), Battaglini (2005) etc.

to pick a winner.[3] The weakest link voting can be interpreted as a natural sequential extension of plurality voting, with elimination of the worst plurality loser in each round.[4] By carrying out similar eliminations in each round based on an appropriately defined elimination rule, one can extend any single-round voting to its sequential equivalent.

For sequential elimination voting to be CC, or yield a top-cycle outcome, it is sufficient (and may even be almost necessary) that any (group of) majority voters have some minimal collective influence: the voting rule must be such that by coordinating their votes in any round a majority can always ensure that any particular candidate who survived up to that round is not eliminated in that round; further, such vote coordinations by the majority must be "stable" in the sense that should the majority fail to choose some appropriate coordination of votes that may lead to the particular candidate's elimination, there will be at least one member of the majority group who will have an incentive – if his aim were to protect that candidate – to further deviate by changing his vote. We call these twin requirements, the ***majority non-elimination property***.

We show that the majority non-elimination property will be satisfied by sequential versions of most familiar single-round voting procedures (the unique exception is the sequential analogue of negative voting). To understand how majority influence works, consider for instance sequential scoring rules (which eliminate, at any round, only one candidate with the lowest total score). Clearly, for any candidate and any majority, placing the candidate at the top by every member of a majority is stable; furthermore the candidate will have a total score that is strictly higher than the average score of the remaining candidates, even if every voter outside the majority places that candidate at the bottom, if the following property holds: the

[3] The Conservative Party in Great Britain roughly follows this procedure to choose its leader: the party's parliamentary members vote in successive rounds to reduce first a small number of candidates to only two candidates, and eventually the party members vote to elect the final winner. See *http://politics.guardian.co.uk/Print/0,3858,4196604,00.html*.

Also, the last contest in 2005 to select the host city for the 2012 olympic games had the characteristics of weakest link voting (London emerged the winner after Moscow, New York, Madrid and Paris were eliminated in that order in successive votes held over four rounds). See *http://news.bbc.co.uk/sport1/hi/front_page/4655555.stm*.

[4] The weakest link voting is similar to *sequential runoff election* where alternatives are eliminated one-at-a-time based on voters submitting a full strict-order ranking, eliminating in each round the alternative with the least number of first place votes.

scores in any round for various ranks be such that the average of the two scores corresponding to the top and the bottom ranks weakly exceeds the average score for all the intermediate ranks combined (this property clearly holds for the weakest link and the sequential analogue of Borda). Thus, if this property holds the majority is able to protect the candidate from being eliminated and hence satisfies the majority non-elimination property.

The significance of our exercise could be seen in relation to the much-studied *binary agenda voting*. In contrast to only two choices from which to pick at each round in binary voting (where each choice involves one or more candidates), in our sequential elimination schemes there is virtually no exogenous restriction on how many choices might be considered but each choice consists of exactly one candidate. Thus, our sequential voting games are *complementary* to the binary agenda procedures (sequential binary voting being the only common element). In sequential voting, the sets of candidates (i.e., choices) available at later rounds evolve endogenously (rather than determined by an exogenous ordering of agendas in binary voting) through equilibrium behavior at earlier rounds; neither do the voting rules need remain the same in every round. Also, with the simple binary comparison lacking, general sequential elimination voting poses a far greater challenge as the backwards induction arguments involving iterated deletion of dominated strategies (to insure equilibrium existence and uniqueness) are no longer applicable.

In fact, applying our equilibrium solution, it can be shown that a more general one-by-one elimination based on a sequence of nested partitions of the candidates always yields an outcome in the top-cycle: first carry out one-by-one elimination of the elements in the coarsest partition, then partition the surviving element (if it consists of at least two candidates) applying the second-coarsest partition for refinement and carry out one-by-one elimination on the refined partition, and so on until a single winner gets elected having run through all the nested partitions. The nested partition approach includes both one-by-one sequential elimination voting and the binary agenda voting as special cases, and is the most general voting method, to our knowledge, with the top-cycle property when voters vote strategically.

Finally, we will also argue why *one-by-one elimination* and *repeated ballots*, both characteristics of our sequential voting, are important for Condorcet consistency – especially in the class of voting where voters must vote over more than two choices in some voting round. All one-shot voting rules and several semi-sequential voting lack one or both these characteristics and will fail to be *CC*.

Before going into the formal analysis, it might relevant to ask why do we care about the properties of voting rules in environments with full information and where we know what should be selected (perhaps, a CW)? There are two obvious defenses to such a critique: (1) the voting rules are used in practice; (2) we need to understand the basic structure of rules even if that requires studying full information environments. Our analysis should offer a useful normative guide as to how to design voting games, or modify some of the existing vote methods, to achieve the majority objective.

The next section presents the voting rules and related equilibrium solution concepts. Section 3 contains results on sequential elimination and semi-sequential voting. Finally, in section 4, we analyze single-round voting. Positive results and general statements over broad classes of voting rules appear in theorems and results on particular voting rules appear in propositions. Some proofs are relegated to an Appendix, but other proofs not included here can be found in a working paper version (Bag, Sabourian and Winter, 2008). The Appendix also contains a glossary of voting rules.

2 The Voting Rules and Equilibrium Solutions

Voting Games

First we describe the class of voting games considered in this paper. This class is quite general.

The set of candidates is denoted as $\mathcal{K}$ with cardinality k, and the voter set is denoted as $\mathcal{N}$ with cardinality n, both k and n at least three. Throughout we assume n to be an odd number, but this can be relaxed (see footnote 8). Also for simplicity of exposition, $\mathcal{K} \cap \mathcal{N} = \emptyset$. Each voter $i \in \mathcal{N}$ has a strict, ordinal preference ordering over the candidates given by $\succ_i$. The voters have complete information about preferences.

The class of voting games we consider are as follows. Each voting rule consists of the voters/players voting in at most J rounds/stages, $J < k$. At each stage the voters simultaneously vote (i.e., take an action) and at least one candidate is removed. At the end of a maximum of J rounds voting one candidate survives who is the winner. If C is the set of candidates left at any stage $j \leq J$ with $|C| \geq 2$ ($|.|$ denoting cardinality) then a choice for voter i at that stage consists of choosing an element from an arbitrary choice set $A_i(C, j)$. Moreover, if each i chooses $a_i \in A_i(C, j)$ at this stage then we shall denote the set of eliminated candidate(s) by $e(a^j, C) \subset C$ where $a^j = (a_1, ..., a_n)$ is the profile of votes at stage j. So if the voting finishes in some $\mathcal{J} \leq J$ rounds and voters choose the sequence of votes $\{a^j\}_{j=1}^{\mathcal{J}}$, then the winning candidate is $w \notin \cup_{j=1}^{\mathcal{J}} e(a^j, C)$.

For any $j \leq J$ let $h^j = (a^1, ..., a^{j-1})$ be a complete history (description) of the *actual* voting decisions up to stage j. Define $\mathcal{H}^j$ to be the set of histories at round j and $\mathcal{H} = \bigcup_j \mathcal{H}^j$ be the set of all histories, with the convention that $\mathcal{H}^0$ refers to the initial null history. Also, let $C(h)$ be the set of remaining candidates at $h \in \mathcal{H}$.

Now a (pure) strategy for voter i is a function $s_i : \mathcal{H} \to \bigcup_{j,C} A_i(C, j)$ such that $s_i(h) \in A_i(C(h), j)$ if $h \in \mathcal{H}^j$. Also, denote the set of (pure) strategies of voter i by S_i and let $S = \times_i S_i$.

The above set of games clearly includes the weakest link voting, and more generally any sequential (elimination) voting, and any single-round voting such as plurality rule, approval voting, Borda voting and negative voting. In the case of the weakest link, the number of voting rounds J is $k-1$, the set of choices $A_i(C, j) = C$ and at each stage one candidate is eliminated so that $|e(a^j, C)| = 1$.

In general, *sequential elimination voting* rules are such that at each stage only one candidate is eliminated.

In the case of single-round voting, $J = 1$, all voters submit their strategies at the first stage and all the candidates except one are eliminated simultaneously.

Also, included in our voting games will be three other categories: one with $J > 1$ but if (and only if) at any round a candidate gets majority votes he is immediately declared the winner ending any further ballot (exhaustive ballot, for example); a second one that eliminates candidates in one or more attempts following a single ballot (so that $J = 1$, as in the case of instant runoff voting); finally, voting involving repeated ballots ($1 < J < k - 1$) and more than one candidate being eliminated in some round (such as plurality runoff voting with $J = 2$).

The equilibrium

Since the voting games we consider may have a dynamic structure, we require our equilibrium concept to be subgame-perfect. In addition, as is common in the literature on voting, we need to eliminate choices that are weakly dominated, otherwise there are a large number of trivial equilibria in which each voter's choice is immaterial. Therefore, an equilibrium in our set-up is a strategy profile for the voters that is a subgame perfect equilibrium and is such that at each stage the votes of each player is not weakly dominated given the equilibrium continuation strategies of others in future stages.

In other words, any equilibrium strategy profile $s^* \in S$ in a voting game must have the following properties. In any final stage subgame (i.e., at stage J), s^* must be a weakly undominated Nash equilibrium in the subgame. In any subgame starting with stage $J-1$, the voters' strategies must be an undominated Nash equilibrium in the subgame given that the voters play the game according to s^* in the continuation game (thus the *permissible strategies* of the other voters with respect to which the weak-domination check is carried out are consistent with the equilibrium strategy in the next stage). This backward elimination procedure continues all the way to stage 1.

Formally, for any history $h \in \mathcal{H}$, let $\Gamma(h)$ be the subgame at h and $w(s, h)$ be the candidate elected in the subgame $\Gamma(h)$ if the voters follow strategy profile s in this subgame. Also, for any strategy profile $s \in S$ and any history $h \in \mathcal{H}$, define the set of strategies for all players other than i that are consistent with s in every subgame after h by

$$\widetilde{S}_{-i}(h, s) = \{s'_{-i} \in S_{-i} \mid s'_{-i}(h, h') = s_{-i}(h, h') \text{ for all non-empty } h' \text{ s.t. } (h, h') \in \mathcal{H}\}.$$

Definition 1. *A strategy profile s^* is an equilibrium if for any history $h \in \mathcal{H}$ it satisfies the following properties in the subgame $\Gamma(h)$:*

$$\begin{aligned}
&\textit{(Nash)} \quad \textit{For any } i, \quad w(s^*, h) \succeq_i w(s_i, s^*_{-i}, h) \quad \forall s_i \in S_i, \\
&\qquad\qquad\qquad\qquad\qquad \textit{where } \succeq_i \textit{ means either } \succ_i \textit{ or } =;
\end{aligned}$$

$$\left.\begin{aligned}
&\textit{(Weak non-domination)} \quad \textit{For any } i, \quad \nexists s_i \in S_i \ \textit{s.t.} \\
&\qquad\qquad w(s_i, s_{-i}, h) \succeq_i w(s^*_i, s_{-i}, h) \quad \forall s_{-i} \in \widetilde{S}_{-i}(h, s^*) \\
&\textit{and} \qquad w(s_i, s_{-i}, h) \succ_i w(s^*_i, s_{-i}, h) \quad \textit{for some } s_{-i} \in \widetilde{S}_{-i}(h, s^*).
\end{aligned}\right\} \tag{1}$$

Notice that for any $s \in S$, at any $h \in \mathcal{H}^j$ we can define a one-shot reduced form voting game $\widehat{\Gamma}(h, s)$ in which voter i's strategy set is $A_i(C(h), j)$ and, given any profile $a^j \in A(C(h), j)$ $(= \prod_i A_i(C(h), j))$ of votes, the outcome of the game is given by $w(s, (h, a^j))$ elected. Clearly, our definition of equilibrium strategy in Definition 1 is equivalent to showing that the choices that the equilibrium strategies prescribe at any history h constitute an undominated Nash equilibrium of the one-shot reduced voting game at h. Thus, s^* *is an equilibrium if and only if* $s^*(h)$ *is an undominated Nash equilibrium of* $\widehat{\Gamma}(h, s^*)$, for all h.

Remark 1. *Our equilibrium concept is effectively a backward elimination procedure. However, note that it differs from the more familiar procedure of iterative elimination of (weakly) dominated strategies; while in the latter approach the weak-domination check is carried out in relation to the entire game, ours is only along the subgames.*[5,6] *It is well-known that iterative elimination can have very little elimination power.*[7]

Remark 2. *Note also that any trembling hand perfect equilibrium in extensive form satisfies our definition of equilibrium. This is because any trembling hand perfect equilibrium in extensive form is a subgame perfect equilibrium and excludes weakly dominated choices at different information sets. We could have alternatively started with trembling hand perfect equilibrium in extensive form as our equilibrium concept (see also our remark at the end of subsection 4.1). However, for ease of exposition we adopt the above definition of equilibrium.*

Remark 3. *In the case of single-round voting the standard equilibrium concept is* undominated Nash*. Note that our twin requirements of subgame perfection and non-domination boil down to this standard equilibrium definition for single-round voting rules.*

Next we define Markov equilibrium.

Definition 2. *An equilibrium s^* is said to be Markov if for any i and any j,*

$$s_i^*(h) = s_i^*(h') \quad \forall h, h' \in \mathcal{H}^j \text{ such that } C(h) = C(h').$$

Markov equilibrium strategies are such that at any stage onwards the strategies depend only on the candidates who have survived up to that stage and *not* on the specific history leading up to it. The justification of the Markov assumption for our sequential elimination voting can be found in Bag, Sabourian and Winter (2008).

[5]Moulin (1979) formally analyzed the iterative elimination procedure and applied it to a significant class of voting – voting by veto, kingmaker and voting by binary choices.

[6]In our setup the two definitions may differ because at each stage our voters vote simultaneously (the game is not one of perfect information) over more than two alternatives.

[7]For example, for plurality rule Dhillon and Lockwood (2004) show that *anything* other than one's lowest-ranked candidate will survive iterative eliminations of weakly dominated strategies.

3 Multi-Stage Elections

3.1 Condorcet consistency of the weakest link

Much of our insight about sequential elimination voting can be gained by studying the *weakest link voting*, so we start with this particular voting rule and then broaden our analysis to a very general class of sequential elimination voting.

First, some notations. Given the voters' strict preference ordering over candidates, a binary comparison operator T defines a candidate x to be *majority-preferred* over another candidate y, written as xTy, if the number of voters preferring x over y exceeds the number of voters preferring y over x.[8]

Next, the CW, if it exists, is defined as a candidate $z \in \mathcal{K}$ such that zTz', for all $z' \in \mathcal{K}$. Similarly, for any set of remaining candidates $C \subseteq \mathcal{K}$ the CW with respect to C, if it exists, is a candidate $z \in C$ such that zTz' for all $z' \in C$.

We say that an equilibrium s^* of a voting rule is CC at every subgame if for every $h \in \mathcal{H}$ such that the set of remaining candidates $C(h)$ has a CW winner $z(h)$, the equilibrium strategy induces the CW with respect to $C(h)$ in the subgame defined by h (i.e. $w(s^*, h) = z(h)$ if $z(h)$ is defined for h).

Our first result is an equilibrium characterization of the weakest link game:

Theorem 1. *Any Markov equilibrium of the weakest link voting is* CC *at every subgame.*

Proof. We demonstrate this by (backward) induction on the number of remaining candidates in any subgame.

First, consider any subgame at stage $k-1$ with only two candidates, z and z'. Because sincere voting is the only Nash equilibrium that is also undominated in this final stage subgame, the CW must be the winner.

Now suppose the following induction hypothesis is true:

For every history $h \in \mathcal{H}$ such that the set of remaining candidates $C(h)$ consists of j candidates, the following holds: if $C(h)$ has a CW, z, then z will become the ultimate winner in the subgame defined by h (i.e., $w(s^*, h) = z$).

We then prove that the same holds at any history/subgame with $j+1$ remaining candidates.

[8] To relax the assumption of *odd* number of voters, extend the definition of majority preference, whenever there is a tie, by applying a tie-breaker.

Suppose not; then there exists a subgame defined by some history $\tilde{h}$ such that the set of the remaining candidates $C(\tilde{h})$ has $j+1$ candidates, $C(\tilde{h})$ has a CW, z, and some other candidate $z' \neq z$ becomes the ultimate winner in this subgame. Now since z is the CW with respect to $C(\tilde{h})$, it follows by the induction hypothesis that z is eliminated immediately at $\tilde{h}$ at stage $k-j$ (since at $\tilde{h}$ there are $j+1$ candidates, the subgame defined by $\tilde{h}$ begins in round $k-j$). Otherwise, since z is also the CW with respect to the set of candidates in the next round, by the hypothesis he will become the ultimate winner.

Next, consider those voters who prefer z over z' and their immediate vote at $\tilde{h}$ in stage $k-j$. By definition of z, these voters will form a majority. Therefore, it must be that at least one such voter, say voter i, who voted for some candidate z'' other than z. But we show that for i voting for z weakly dominates voting for z'' at this stage, given the equilibrium continuation strategies in the future stages.

To show this, first notice that if voter i chooses z'' there are two possible outcomes depending on the choices of others at this stage: either (i) z survives at this stage and, by the induction hypothesis, all the subsequent stages and becomes the ultimate winner; or (ii) z is eliminated and, by the Markov property of the equilibrium strategies, z' becomes the ultimate winner. Now if (i) is the case then if i switches his vote from z'' to z the outcome will be the same with z surviving all stages and becoming the winner. If (ii) is the case then if i switches his vote from z'' to z, either z is eliminated and the outcome will be the same with z' becoming the ultimate winner or z survives this stage, and by the induction hypothesis, all the subsequent stages and becomes the ultimate winner.

Finally, we need to show that there is a vote profile for all voters other than i such that if voter i votes for z'' then z would be eliminated and z' goes on to win whereas if he votes for z then z is not eliminated and z wins. To show this let $\overline{Z} \subset C(\tilde{h})$ be the set of remaining candidates other than z'' that are lower in the tie-breaker than z and let m be the cardinality of this set. Then, since voting at $\tilde{h}$ eliminated candidate z, it must be that $n-1 \geq m$. Otherwise, it must be that some $x \in \overline{Z}$ receives zero vote at $\tilde{h}$ and therefore is eliminated (contradiction). Now consider a vote profile for all voters other than i such that every $x \in \overline{Z}$ receives at least one vote and no other candidate receive any vote; since $n-1 \geq m$, this is feasible. Now, if i votes for z, he is not eliminated (z'' receives zero vote). On the other hand, if i votes for z'', candidate z is eliminated; this is because in this case z receives zero vote and any other candidate(s) with zero vote belong to the set

$C(\tilde{h})\backslash\{\overline{Z} \cup z''\}$ and hence must be higher up in the tie-breaker than z in the case of a tie.

Since voter i prefers z to z', the choice of z thus weakly dominates z'' for i. This implies a majority of voters would vote for z, contradicting the supposition that z is eliminated at this stage.

Since we already proved our hypothesis for subgames with two candidates, it follows by the induction step above that if there is a CW for the set C, he will be elected in any subgame with C. **Q.E.D.**

The above result is a characterization result for Markov equilibria of the weakest link voting when the set of (remaining) candidates has a CW. However, in order to ensure that the result is not vacuous one has to show that the weakest link game has a (Markov) equilibrium. This is particularly important because even if a set of candidates has a CW, there could be subgames off-the-equilibrium path without a CW among the remaining candidates and it is by no means clear that there is an equilibrium in such subgames. Thus, Theorem 1 should be viewed in combination with Theorem 2 below.[9]

Theorem 2. *Assume $n \geq 2k - 1$. Then in the weakest link game there exists a Markov equilibrium.*

The proof of this result can be found in Bag, Sabourian and Winter (2008).

Some further remarks concerning Theorem 1. First, notice that the arguments in the proof does not make any reference to the tie-breaking rule; thus the weakest link voting is CC for any arbitrary deterministic tie-breaking rule. Also, if the preferences of the voters can be represented using expected utility framework then by an analogous argument one can show that Theorem 1 holds for random tie-breaking rules.

Second, limiting the result to equilibria that are Markov could be considered a limitation of Theorem 1. However, there are two points that we like to make

[9] Since we wrote an earlier version of this paper (available under a different title: Bag, Sabourian and Winter, 2002), we came across Peress (2004) who also examines the issue of Condorcet consistency using the weakest link (that he calls multistage runoff) but under a very restrictive assumption that every subset of candidates has a CW (all candidates can be majority ranked). In particular, he does not need to consider the possibility that off-equilibrium subgames may not have a CW. This makes the required analysis in Peress (2004) much simpler. Also, his equilibrium concept seems to have similarity with ours but is not clearly defined.

with respect to the Markov restriction. First, a weaker version of the Markov property would suffice for the proof of Theorem 1. All we require to obtain the result is that the equilibrium strategies do not depend on the history through the specific configuration of votes that lead to the particular candidates' eliminations. However, the strategies can still depend on the order in which the candidates are eliminated. In fact, if we assume that the votes are not revealed between stages but only the identity of the eliminated candidate at each stage is announced, then we do not need the Markov property. Second, it could be shown that if, in choosing the strategies, players have, at least at the margin (lexicographically), a preference for simplicity (aversion to complexity) then all equilibria are Markov.[10] Basic reason is that in our sequential voting games, for any equilibrium strategy profile every set of remaining candidates occur on the equilibrium path at most once. If any player i's strategy is non-Markov, then i makes a different choice at two different subgames with the same set of remaining candidates C; but then since C occurs at most once on the equilibrium path, player i could economize on complexity by always making the same choice at every subgame with C without sacrificing payoffs.

Finally, as discussed after the equilibrium definition, since every trembling hand perfect equilibrium in extensive form satisfies our equilibrium concept, it follows that every Markov trembling hand perfect equilibrium in extensive form of the weakest link voting is CC at every subgame.

3.2 Top cycle consistency and a general sequential elimination procedure

Next we extend the Condorcet consistency result in Theorem 1 by introducing the broader concept of *'top cycle'* (which always exists and is same as the CW when the latter exists) and by considering a general sequential elimination procedure.

Arbitrary voter preferences including no CW

For any set of candidates $C \subseteq \mathcal{K}$, the *top cycle* with respect to C is defined as

$$\begin{aligned} \mathcal{TC}(C) &= \{x \in C : \forall y \in C, y \neq x, \text{ either } xTy \\ \text{or there exist } x_1, x_2, \dots, x_\tau &\in C \text{ candidates such that } xTx_1T\dots Tx_\tau Ty\} \end{aligned}$$

[10]Properties of Markov equilibrium in general dynamic games have been studied by Chatterjee and Sabourian (2000), Sabourian (2004), and Gale and Sabourian (2005).

where, as before, T is the binary operator representing majority preference. We also refer to $\mathcal{TC}(\mathcal{K})$ simply by the *top cycle*.

Sequential elimination with majority property

A general *sequential* process of elimination is one where in each round only one candidate is eliminated. In these games, as mentioned before, players vote in $k-1$ rounds, the set of votes for voter i at round $j < k$ when C is the set of remaining candidates is $A_i(C, j)$, and one candidate $e(a^j, C)$ is eliminated at each round j.

An important aspect of this procedure would be the decisive role that any group of majority voters can play: at any round a majority of voters can ensure that any particular candidate is *not* eliminated. We now specify this important property for the set of sequential (elimination) voting games as follows.

Majority non-elimination* (MNE) *property: *For any stage $j < k$, any set of remaining candidates C, any $c \in C$, and any set of majority voters $\phi \subseteq \mathcal{N}$, there exists a set of strategy profiles $\mathcal{D}^c_\phi(C, j) \subseteq \Pi_{i\in\phi} A_i(C, j)$ for the majority ϕ such that the following two conditions hold:*

[i] **(Majority protection)** *If all members of ϕ choose some profile $a_\phi \in \mathcal{D}^c_\phi(C, j)$ then c is not eliminated, i.e.,*

$$e(a_\phi, a_{-\phi}, C) \neq c, \; \forall a_{-\phi} \in \Pi_{\ell\notin\phi} A_\ell(C, j).$$

[ii] **(Protection stability)** *For any profile $a_\phi \notin \mathcal{D}^c_\phi(C, j)$ such that $e(a_\phi, a_{-\phi}, C) = c$ for some $a_{-\phi} \in \Pi_{\ell\notin\phi} A_\ell(C, j)$, there exists some member of the majority $i \in \phi$ and an action $a^c_i \in A_i(C, j)$ such that*

$$\forall a'_{-i} \in A_{-i}(C, j) \;\text{ if }\; e(a_i, a'_{-i}, C) \neq c \;\text{ then }\; e(a^c_i, a'_{-i}, C) \neq c \qquad (2)$$

$$\textit{and} \;\; \exists a'_{-i} \in A_{-i}(C, j) \;\text{ s.t. }\; e(a_i, a'_{-i}, C) = c \;\text{ and }\; e(a^c_i, a'_{-i}, C) \neq c. \qquad (3)$$

That is, a_i is "inferior" to a^c_i in protecting c.

All sequential voting rules satisfying these two non-elimination conditions constitute the family $\mathcal{F}$. ||

Note that $\{\mathcal{D}^c_\phi(C, j)\}$ are sets of actions/votes for non-elimination of any candidate c. For instance, if each stage of the sequential voting involves voters ranking the candidates, one can think of $\{\mathcal{D}^c_\phi(C, j)\}$ as all actions by the majority that place c at the top of their ranking; then the two conditions in the **MNE**-property require that [i] if a majority of voters place c at the top then c cannot be eliminated, and [ii] if a majority fails to place c at the top and c is eliminated then there is some voter from that majority who will have an action that is (weakly) better than his particular action in the 'failed majority action profile' in protecting c. Later we will verify that ***sequential extensions*** of approval voting and a class of ***scoring voting rules*** (that includes plurality and Borda rules as special cases) plus Copeland and Simpson rules (see Moulin, 1988, ch. 9 for the last three voting rules) fall under the family $\mathcal{F}$. Further, it can be checked that the important class of ***sequential binary voting*** comes under $\mathcal{F}$.[11]

Theorem 3. *In all Markov equilibria of any sequential voting rule in the family $\mathcal{F}$, candidate w is the winner in any subgame with remaining candidates C only if $w \in \mathcal{TC}(C)$.*

For binary voting trees (footnote 1), McKelvey and Niemi (1978) also obtain outcomes in the top cycle. McKelvey and Niemi's equilibrium, that they call multistage sophisticated solution, is as follows: presenting the voting game as a tree of binary choices and treating each decision node with its specific binary choices as a constituent game, McKelvey and Niemi solve recursively the various constituent stage games backwards using elimination of weakly dominated strategies. Our sequential, one-by-one elimination voting family is inherently different from the class of binary voting games of McKelvey and Niemi (with the exception of sequential binary voting). In particular, in our framework with more than two remaining candidates voters usually have more than two choices, making the backwards induction type reasoning of binary voting (based on iterative deletion of dominated strategies) problematic.[12] Furthermore, binary voting may involve multiple candidates being

[11] Note that our sequential voting is quite general – voters can submit a (weak or strict) ranking, or the preference submission may even be more abstract than a simple ranking of candidates.

[12] In binary voting, every decision node involves two choices. As a result working backwards each voter has a unique dominant choice at each stage and the game can be solved uniquely through iterative deletion of dominated strategies: At the final decision nodes, with only two choices sincere voting is the unique dominant choice and thus one can associate each final decision node with its "sophisticated equivalent" (Shepsle and Weingast, 1984) – the candidate that wins conditional on reaching that particular subgame; iterating back up the tree, by the same reasoning, voters again have two choices over two sophisticated equivalents and voting sincerely over these choices is dominant. Under our general sequential elimination scheme, working backwards and iteratively deleting dominated strategies does not typically yield a unique choice at each stage because the

eliminated in a single stage, including selecting a winner even in the first stage.[13]

Finally, two technical remarks. First, the top-cycle result in Theorem 3 does not require strategies to be Nash as part of the equilibrium definition; we impose the Nash requirement mainly to make the equilibrium definition consistent with the non-sequential voting games of section 5 and a related negative result in Theorem 4. Second, as in Theorem 1, any Markov trembling hand perfect equilibrium in extensive form will be in the top cycle and is CC when a CW exists.

The scope of $\mathcal{F}$. To fully appreciate Theorem 3, it is important that we elaborate the scope of the voting family $\mathcal{F}$. First consider *scoring rules.*

Definition 3. (***Scoring voting rules*** [***Moulin, 1988, ch.9***]) *Fix a nondecreasing sequence of real numbers $\varsigma_1 \leq \varsigma_2 \leq \ldots \leq \varsigma_k$ with $\varsigma_1 < \varsigma_k$. Voters rank the candidates, thus giving ς_1 score to the one ranked last, ς_2 to the one ranked next to last, and so on. A candidate with a maximal total score is elected.*

Definition 4. (***Sequential scoring rule***) *A sequential scoring rule is the sequential, one-by-one elimination analogue of scoring rules:*

- *At any stage and for any set of remaining $J \leq k$ candidates, fix a nondecreasing sequence of real numbers $\varsigma_1 \leq \varsigma_2 \leq \cdots \leq \varsigma_J$ with $\varsigma_1 < \varsigma_J$.*
- *At the particular stage, voters rank the candidates according to the above sequence, and the candidate receiving the lowest total score is eliminated.*

Proposition 1. *Any sequential scoring rule belongs to the family $\mathcal{F}$, if at each stage the scores associated with different ranks are such that*

$$\frac{1}{2}(\varsigma_1 + \varsigma_J) \geq \frac{1}{(J-2)} \sum_{j=2}^{J-1} \varsigma_j. \tag{4}$$

choice is not necessarily between two alternatives (sophisticated equivalents). In the case of the weakest link voting with three candidates, for instance, there are three final decision nodes, each involving a pairwise vote, so identifying the sophisticated equivalents is not a problem; but then in the previous (first) stage there is a three-way choice over the sophisticated equivalents and an individual voter's best vote choice at this stage depends on the choices of others.

[13]McKelvey and Niemi do not require the Markov assumption because of the binary nature of choices at every decision node. The equilibrium in any continuation game of their binary voting following elimination of the CW is essentially unique. For more than two choices possible (as in our case), the uniqueness can be guaranteed only by assuming the Markov property.

Condition (4) implies that if any majority voters place a candidate c at the top and the remaining voters place c at the bottom then the resulting total score of c can never be the lowest (exceeds the average score of the other candidates). Therefore, this condition ensures that c is not eliminated, irrespective of what others do, and thus the set of actions by a majority that place a candidate at the top satisfies majority protection and hence the **MNE**-property (protection stability is also satisfied by any strategy that does not place c at the top because it is then always possible to protect c better by improving its ranking). In fact, the **MNE**-property cannot be guaranteed with sequential scoring if condition (4) fails.

Both plurality and Borda rules satisfy (4), so the corresponding sequential extensions – the weakest link and sequential Borda rules – satisfy the **MNE**-property. However, the negative voting with $\varsigma_1 = 0$ and $\varsigma_j = 1$ for all $j > 1$ would fail (4). Moreover, one can show that its sequential extension – *sequential veto* rule (in each stage each voter vetoes one candidate and the one receiving the maximum number of vetoes is eliminated) – fails the **MNE**-property. This is because a majority of voters may not always be able to guarantee non-elimination of a candidate c by giving it the maximum point, 1. The only way to ensure non-elimination of c is for the majority to coordinate to veto some other candidate(s) other than c; but this may violate *protection stability* because strategies that do not coordinate on vetoing some other candidate(s) need not be inferior in protecting the particular c.

Three other one-shot rules (not part of scoring rules) – approval voting, Copeland rule and Simpson rule – have similar sequential extensions with the Condorcet consistency property, as summarized below.[14]

[14]In approval voting, a voter is allowed to approve or disapprove any number of candidates (point 1 to indicate approval of a candidate and point 0 to denote disapproval) except that the voter cannot approve all or disapprove all the candidates. The candidate with maximal votes wins (Brams and Fishburn, 1978; Myerson, 2002).

Copeland and Simpson rules (Moulin 1988, ch. 9) are based on voters submitting only strict order rankings (so that $J = k$). For Copeland rule, candidate a, compared with another candidate b, is assigned a score $+1$ if a majority prefers a to b, -1 if a majority prefers b to a, and 0 if it is a tie. Summing up the scores over all $b, b \neq a$, yields the Copeland score of a. A candidate with the highest such score, called a *Copeland winner*, is elected. For Simpson rule, for candidate a denote by $N(a, b)$ the number of voters preferring a to another candidate b. The Simpson score of a is the minimum of $N(a, b)$ over all $b, b \neq a$. A candidate with the highest such score, called a *Simpson winner*, is elected.

Sequential extensions of the above voting rules would eliminate, at any round, the candidate that receives the lowest score, applying a tie-breaker wherever necessary.

Proposition 2. *The sequential extensions of approval, Copeland and Simpson voting rules belong to the family $\mathcal{F}$.*

The proof of Proposition 1 appears in the Appendix. Proposition 2 proof is very similar and omitted (see also footnote 19).

3.3 Generalizing binary voting and the family $\mathcal{F}$

The binary voting procedure of McKelvey and Niemi (1978) and our sequential one-by-one elimination voting raise the possibility of constructing an even more general sequential one-by-one elimination voting. Below we present one such method.

Nested partition voting: Let υ be any voting rule in the family $\mathcal{F}$ defined in section 3.2. Consider any sequence of nested partitions $B_1, B_2, ..., B_k$ of the set of candidates $\mathcal{K}$, where B_{i+1} is a refinement of the partition B_i and where B_k is the singleton partition. The voting goes through k stages at most. The first stage involves voting over the partition B_1. Suppose B_1 is partitioned into sets $S_1, ..., S_r$. In stage 1 voters vote on $S_1, ..., S_r$, treating the sets as candidates and using the rule υ. Now, suppose $S \in B_1$ wins in the first stage and let $T_1, ..., T_m$ be the partition of S induced by B_2. In the second stage, voters use the rule υ on $T_1, ..., T_m$ (as candidates). Continue in this manner until at some stage the partition consists of singleton sets. The rule υ will now determine the winner in the standard sense.

If B_1 is the singleton partition, the nested partition voting is same as the rule υ. If B_{j+1} partitions the elements of B_j into two sets only (for all j), then the above voting is same as the binary voting of McKelvey and Niemi (1978).

For the Markov equilibrium defined in section 2, using arguments similar to Theorem 3 proof it can be shown that the equilibrium of the nested partition voting game would always belong to the top cycle.

3.4 Semi-sequential voting

Next we consider three specific voting rules with certain traits of sequential voting that fail some of its other characteristics. With the help of these, we like to demonstrate why *one-by-one elimination* and *repeated ballots* – two defining characteristics of sequential voting – are both important for Condorcet consistency.

Proposition 3. *The plurality runoff rule and the exhaustive ballot method are not CC.*

Proposition 4. *The one-shot weakest link voting (with voters submitting their entire weakest link strategies once-for-all in a single round followed by one-by-one eliminations) is not CC.*

The proofs, based on counter-examples, appear in Bag, Sabourian and Winter (2008).

Note that the plurality runoff rule and the exhaustive ballot method, used in various political appointments, share features of weakest link voting in that both use multiple ballots but fail *one-by-one elimination*. On the other hand, the one-shot version of the weakest link voting eliminates candidates one-by-one but fails *repeated ballots* (and likewise for the instant runoff voting without the majority top-rank trigger, to be mentioned in Proposition 5).

An intuition on why elimination of more than one candidate in some round may lead to a non-Condorcet outcome would be instructive. The basic idea is that with one-by-one elimination, when the CW is eliminated in some voting round the (off-equilibrium) outcome is unique in the induction argument. When more than one candidate are eliminated, following the CW's elimination the outcome is not necessarily unique – it depends on who else is being eliminated along with the CW; as a result, in this case, the voters may not vote for the CW in order to influence the final outcome in the case when the CW is eliminated.

4 Single-round voting mechanisms

Given our observations in Propositions 3 and 4, it might be expected that one-shot voting would also fail Condorcet consistency under strategic voting. In this section, we formally address this question.

First define a class of single-round voting rules.

For any set of candidates $\mathcal{K}$ with cardinality k (as defined in section 3), the set of strategies for a voter is to rank the k candidates in J different categories for some J such that $1 < J \leq k$ subject to some bounds on the number of candidates in each category. Denote the minimum and the maximum number of candidates in each category $j \leq J$ by $m(j)$ and $M(j)$, respectively. Let Λ be the set of all such J rankings over $\mathcal{K}$. Thus, the strategy for voter i, denoted by $R_i \in \Lambda$, is a profile $(X_1, ..., X_J)$ with J components such that it partitions the set $\mathcal{K}$ into J non-empty cells $X_1, ..., X_J$ and $m(j) \leq |X_j| \leq M(j)$. Since it is a partition it must be the case that $\sum_j m(j) \leq k$. From R_i we can also specify for each $x, y \in \mathcal{K}$ whether x is ranked strictly above y, denoted by $x\, P_i\, y$, or y is ranked strictly above x, denoted by $y\, P_i\, x$, or x is ranked the same as y (in the same category), denoted by $x\, I_i\, y$. For any set of n voters, the one-shot voting game also specifies the winning candidate as a function of the submitted strategies of the n voters given by an outcome function $f^n : \Lambda^n \to \mathcal{K}$. A voting rule with k candidates is then defined by the number of categories, the bounds on the size of each category and the outcome function. We refer to such a one-shot voting rule by $v(k) = (J, \{m(j)\}_{j\leq J}, \{M(j)\}_{j\leq J}, \{f^n\}_{n\in\mathbb{N}})$, where $\mathbb{N}$ is the set of odd numbers (as elsewhere, this restriction is made for simplicity).

Rankings Λ can accommodate strict order submissions (Borda, Copeland and Simpson rules), or standard voting rules that ask for submission of candidates of a particular rank (plurality rule, negative voting). Further, it can accommodate *approval voting*, which asks voters to partition candidates into 1's and 0's. Thus, our one-shot voting game is the most comprehensive (one-shot) generalization of Moulin's (1988) **scoring rules**.[15]

We shall see that, for any fixed number of candidates k, all single-round voting games satisfying two intuitive properties, called scale invariance and responsiveness, are not CC in strategic voting. This will be a strong assertion because the lack of Condorcet consistency is demonstrated for any fixed k. (The number of voters can of course vary). The meaning of scale invariance is rather straightforward.

Definition 5. *A voting rule $v(k)$ with k candidates is scale invariant if replicating the set of voters with their submitted strategies by any multiple will not alter the winner.*

Responsiveness is about voter pivotalness. Roughly it requires that for each voter and any pair of candidates, there is a scenario at which the voter is pivotal in determining the winner between the two candidates. Before defining responsiveness, we need to define sincere behavior and Condorcet consistency (in sincere voting) in the above class of voting games.

We say that a strategy $R_i = (X_1, ..., X_J) \in \Lambda$ submitted by voter i is *sincere* if $X_1 = \{c^1, ..., c^{m(1)}\}$, $X_2 = \{c^{m(1)+1}, ..., c^{m(1)+m(2)}\}$,...,$X_J = \{c^{\sum_{j<J} m(j)+1}, ..., c^k\}$, when the true preference ranking of voter i is $c^1 \succ_i \ldots \succ_i c^k$.[16]

[15] Copeland and Simpson rules and approval voting are *not* part of scoring rules.

[16] This definition is a generalization of the standard definition of sincere behavior when $J = k$.

Finally, for any k, a voting rule $v(k)$ is said to be CC under sincere voting if for any number of voters and any preference profile over k candidates that admits a CW, the voting rule $v(k)$ selects the CW whenever the voters' strategies are sincere.

Definition 6. *A voting rule $v(k)$ with k candidates is responsive if it satisfies the following two conditions for each voter i:*

1. For any pair of candidates x and y and any two strategies R_i and R'_i such that $x\, P_i\, y$ and $y\, P'_i\, x$ there exists a profile of strategies R_{-i} by the remaining voters such that (R_i, R_{-i}) elects x as the winner, and (R'_i, R_{-i}) elects y as the winner.

2. If the voting rule is CC under sincere voting, then (i) the submissions are strict $(J = k)$;[17] and (ii) for any three candidates $X = \{x, y, z\}$, there exists a candidate z in X such that the following holds: for any pair of strategies $R_i = (X_1, X_2, X_3, ..., X_k)$ and $R'_i = (X_2, X_1, X_3, ..., X_k)$ such that $(X_1, X_2, X_3) = (x, y, z)$, there exists a profile of strategies R_{-i} by the remaining voters such that (R_i, R_{-i}) elects x as the winner, and (R'_i, R_{-i}) elects z as the winner.

Condition 2 is required only for Copeland and Simpson rules. For all other one-shot voting, only condition 1 is needed.

Theorem 4. *For any single-round voting rule $v(k)$ with k candidates, if $v(k)$ satisfies responsiveness and scale invariance then $v(k)$ is not CC.*

Proposition 5. *Suppose $n \geq k - 1$. Then standard one-shot voting rules, in particular, all scoring rules (including plurality rule, negative voting, Borda rule), approval voting, the two variants of Instant runoff voting (with and without the majority top-rank trigger), Copeland rule and Simpson rule will all satisfy responsiveness and scale invariance conditions of Theorem 4. Hence none of these voting rules will be CC.*

For Theorem 4 proof, see the Appendix. Proposition 5 proof can be found in Bag, Sabourian and Winter (2008).

While Condorcet consistency failing for specific one-shot voting rule(s) is not surprising (Dhillon and Lockwood, 2004; De Sinopoli et. al., 2006), to our knowledge there is nothing to suggest that Condorcet consistency (under strategic voting)

[17] All voting rules that are CC under sincere voting are based on strict rankings (Moulin, 1988).

should fail for the entire class of one-shot voting. On the contrary, significant positive results in the implementation literature would have led one to believe otherwise. In this respect, failure of Condorcet consistency for one-shot voting rules should be formally noted.[18]

Theorem 4 and Proposition 5 are at the opposite spectrum of the sequential one-by-one elimination voting methods. Single-round voting fails both one-by-one elimination and repeated ballot tests (required for Condorcet consistency). The wasted vote phenomenon, due to miscoordination of voter strategies, may thus lead to an outcome that is inferior to some other alternative for a majority of voters, with little bite from the refinement due to weakly non-dominated strategies.

Appendix

Proof of Proposition 1: Fix a stage with the set of remaining candidates C having the cardinality J. Also, fix a candidate $c \in C$ and a majority ϕ.

For any voter i, let $\mathcal{D}_i^c(C, j)$ be the set of all strategies that place c at the top (with no other restriction on the positions of other candidates).[19] Also, let $\mathcal{D}_\phi^c(C, j) = \Pi_{i \in \phi} \mathcal{D}_i^c(C, j)$.

First we verify condition *[i]*. Fix any $a \in A(\mathcal{C}, j)$ such that $a_\phi \in \mathcal{D}_\phi^c(C, j)$. We need to show that $e(a, C) \neq c$.

For any $x \in C$ and any $a' \in A(\mathcal{C}, j)$, denote the total score of candidate x at this stage when action profile a' is chosen by $TS(x, a')$.

Next, define θ_{top} to be the total score of a candidate if he receives the highest score, ς_J, from a majority of $(n+1)/2$ voters and gets the lowest score, ς_1, from the remaining $n - (n+1)/2$ voters:

$$\theta_{\text{top}} = \frac{(n+1)}{2} \varsigma_J + [n - \frac{(n+1)}{2}] \varsigma_1.$$

[18]Note that the Condorcet map is *Maskin monotonic* (1999) on the restricted domain of preferences where Condorcet winner exists (and will be Maskin monotonic even in unrestricted domains if one defines social choice rule to select all outcomes when Condorcet winner fails to exist). Since the Condorcet map also satisfies 'no veto power,' it is Nash implementable if one considers arbitrary, rather than just one-shot voting, mechanisms.

[19]Proposition 2 proof, omitted, will follow a similar argument as in the rest of Proposition 1 proof. For sequential extension of approval voting, $\mathcal{D}_i^c(C, j)$ will consist of the unique strategy of voter i approving only candidate c and disapproving all the remaining candidates. For sequential extensions of Copeland and Simpson rules – given that these rules are based on strict order submissions – $\mathcal{D}_i^c(C, j)$ will place only candidate c at the top.

Since a is such that the majority ϕ place c at the top, it follows that $TS(c,a) \geq \theta_{\text{top}}$. Therefore, the average score that the other candidates receive when a is chosen cannot exceed

$$\theta_{\text{min}} = \frac{n[\varsigma_J + \ldots + \varsigma_1] - \theta_{\text{top}}}{J-1}.$$

But then there must exist a candidate $d \in C$ such that $TS(d,a) \leq \theta_{\text{min}}$. Now to complete verification of condition *[i]*, it suffices to show that $\theta_{\text{top}} - \theta_{\text{min}} > 0$. Note that

$$(J-1)(\theta_{\text{top}} - \theta_{\text{min}}) = J \cdot \theta_{\text{top}} - n \sum_{\ell=1}^{J} \varsigma_\ell$$

$$= \frac{(J-2)n+J}{2}\varsigma_J + \frac{(J-2)n-J}{2}\varsigma_1 - n\sum_{\ell=2}^{J-1}\varsigma_\ell.$$

Therefore, $\theta_{\text{top}} - \theta_{\text{min}} > 0 \Leftrightarrow \frac{1}{2}(\varsigma_J + \varsigma_1) + \frac{J}{2n(J-2)}(\varsigma_J - \varsigma_1) > \frac{1}{(J-2)}\sum_{\ell=2}^{J-1}\varsigma_\ell.$

But since, by assumption, $\frac{1}{2}(\varsigma_J + \varsigma_1) \geq \frac{1}{(J-2)}\sum_{\ell=2}^{J-1}\varsigma_\ell$ and $\varsigma_J > \varsigma_1$, condition *[i]* must hold. $||$

Next, we verify condition *[ii]*. Fix $a \in A(C,j)$ such that $a_\phi \notin \mathcal{D}^c_\phi(C,j)$ and $e(a,C) = c$. For any i, let m^i be a candidate to whom i attaches the highest score ς_J: $a_i(m^i) = \varsigma_J$.

Without loss of generality denote the set of voters in the ϕ-majority by $\{1, 2, \ldots, |\phi|\}$. Next, consider the sequence of vote profiles, $a^{(0)}, a^{(1)}, \ldots, a^{(|\phi|)}$, defined as follows: $a^{(0)} = a$ and for any i and ℓ such that $1 \leq i, \ell \leq |\phi|$,

$$a^{(i)}_\ell(x) = \begin{cases} \varsigma_J & \text{if } x = c \text{ and } \ell \leq i \\ a_\ell(c) & \text{if } x = m^\ell \text{ and } \ell \leq i \\ a_\ell(x) & \text{otherwise.} \end{cases}$$

Note that $a^{(|\phi|)}$ is such that $a^{(|\phi|)}_i(c) = \varsigma_J$ for all $i \in \phi$. Therefore, $a^{(|\phi|)}_\phi \in \mathcal{D}^c_\phi(C,j)$ and hence, by condition *[i]*, $e(a^{(|\phi|)}, C) \neq c$. Moreover, by assumption $e(a^{(0)}, C) = c$. Therefore, there exists some i, $1 \leq i \leq |\phi|$, such that $e(a^{(i-1)}, C) = c$ and $e(a^{(i)}, C) \neq c$. Furthermore, by the definition of the sequence $a^{(0)}, a^{(1)}, \ldots, a^{(|\phi|)}$ we have that $a^{(i-1)}_i = a_i$ and $a^{(i-1)}_{-i} = a^{(i)}_{-i}$. Therefore, we have $e(a_i, a^{(i-1)}_{-i}, C) = c$ and $e(a^{(i)}_i, a^{(i-1)}_{-i}, C) \neq c$ verifying (3) in condition *[ii]*.

To verify (2), for $a_i (= a^{(i-1)}_i)$ and $a^{(i)}_i$ note that $a^{(i)}_i(c) = \varsigma_J$, $a^{(i)}_i(m^i) = a_i(c)$ and $a^{(i)}_i(x) = a_i(x)$ for all $x \neq c$. Thus, for any $a_{-i} \in A_{-i}(C,j)$ we have $TS(c, a^{(i)}_i, a_{-i}) \geq$

$TS(c, a_i, a_{-i})$, $TS(m^i, a_i^{(i)}, a_{-i}) \leq TS(m^i, a_i, a_{-i})$ and $TS(x, a_i^{(i)}, a_{-i}) = TS(c, a_i, a_{-i})$ for all $x \neq c$. But this implies that if $e(a_i, a_{-i}, C) \neq c$ then $e(a_i^{(i)}, a_{-i}, C) \neq c$ for all $a_{-i} \in A_{-i}(C, j)$, hence verifying (2). **Q.E.D.**

Proof of Theorem 3: We use induction on the number of remaining candidates.

Assume the following hypothesis: *Theorem 3 is true for any subgame with j candidates.*

We want to show that the result is also true for any subgame with $j+1$ remaining candidates. Suppose not. Then there is a subgame Γ at stage $k-j$ with remaining candidates C of cardinality $j+1$ such that w is the ultimate winner and $w \notin \mathcal{TC}(C)$. This implies there exists some $y \in C$ such that

$$y \; T \; w \;\; \underline{\text{and}} \text{ it is } \textit{not} \text{ the case that } \; w \; T \; x_1 \; T \; x_2 \; T \; \ldots T \; x_\ell \; T \; y, \tag{5}$$

for some $x_1, x_2, ..., x_\ell \in C$.

Next we establish two intermediate claims.

Claim 1: y must be the first eliminated candidate in the subgame Γ.

If not, let $y' \neq y$ be the candidate eliminated at this stage. Then in this subgame the remaining candidate set is $C \setminus y'$ and w wins, which implies by hypothesis $w \in \mathcal{TC}(C \setminus y')$. But then there will be a (direct or an indirect) chain such that $w \; T \; x_1 \; T \; x_2 \; T \; \ldots T \; x_\ell \; T \; y$, contradicting (5). ||

Claim 2: y is the ultimate winner in any subgame at stage $k-j$ with remaining candidates C if y is not the first eliminated candidate in this subgame.

Let $a \neq y$ be the candidate that is eliminated first. Denote the winner after (a is eliminated) by $\hat{w}$.

First note that, $y \in C \setminus a$ implies $w \notin \mathcal{TC}(C \setminus a)$, hence by hypothesis $\hat{w} \neq w$.

Now suppose Claim 2 is false; then $\hat{w} \neq y$. Since $\hat{w} \in \mathcal{TC}(C \setminus a)$, it must be that $\hat{w}$ is majority-preferred over y directly or indirectly. Also, since by Claim 1 and hypothesis $w \in \mathcal{TC}(C \setminus y)$, and $\hat{w} \neq w$ (established above), therefore w is majority-preferred over $\hat{w}$ directly or indirectly. These together imply that w is majority-preferred over y indirectly, contradicting (5). So Claim 2 must be true. ||

Now in the subgame Γ with remaining candidates C, consider any voter i such that $y \succ_i w$; there will be a majority of such voters because $y \, T \, w$. Denote these majority voters by ϕ. By condition *[i]* of the **MNE**-property, there exist vote profiles $a_\phi \in \mathcal{D}_\phi^y(C, k-j)$ such that $e(a_\phi, a_{-\phi}, C) \neq y$, $\forall a_{-\phi} \in \Pi_{\ell \notin \phi} A_\ell(C, k-j)$.

Then since by Claim 1 y must be the first eliminated candidate in the subgame Γ, it must be that the majority ϕ chose some vote profile $\tilde{a}_\phi \notin \mathcal{D}^y_\phi(C, k-j)$. But then by condition *[ii]* of the **MNE**-property, there is some voter $i \in \phi$ whose vote choice $\tilde{a}_i$ (corresponding to the profile $\tilde{a}_\phi$) is "inferior" to some other vote choice a^y_i (as defined in condition *[ii]* of the **MNE**-property in section 4.2) in protecting y. But then we have a contradiction because for such i voting for a^y_i weakly dominates voting for $\tilde{a}_i$ at this stage $k-j$, given the equilibrium continuation strategies in the future stages.

To show the last claim, first note that if i votes for $\tilde{a}_i$, there are two possible outcomes depending on the choices of others at this stage: ***[1]*** y survives and becomes the ultimate winner, by Claim 2; ***[2]*** y is immediately eliminated in which case by Claim 1 and the Markov property of the equilibrium strategies, w becomes the ultimate winner.

Now if ***[1]*** is the case and i switches his vote from $\tilde{a}_i$ to $\tilde{a}^y_i$ then y would still survive this stage (by (2) in condition *[ii]* of the **MNE**-property) and, by Claim 2, become the ultimate winner.

If ***[2]*** is the case and i switches from $\tilde{a}_i$ to $\tilde{a}^y_i$ then either y is immediately eliminated that ensures, by Claim 1 and the Markov property of the equilibrium strategies, that w is the winner, or y survives and becomes the ultimate winner (by Claim 2).

Finally, the switch from $\tilde{a}_i$ to $\tilde{a}^y_i$ will ensure y surviving stage $k-j$ and becoming the ultimate winner in some situation (whereas with $\tilde{a}_i$ the winner would have been w) because, by (3) in condition *[ii]* of the **MNE**-property, there is some $a'_{-i} \in A_{-i}(C, k-j)$ such that $e(\tilde{a}_i, a'_{-i}, C) = y$ (and w wins by Claim 1 and the Markov property of the equilibrium strategies), and yet $e(\tilde{a}^y_i, a'_{-i}, C) \neq y$ that would result in y winning (by Claim 2).

Thus, $w \notin \mathcal{TC}(C)$ cannot be the winner, establishing the hypothesis for any subgame with $j+1$ candidates.

By a similar argument as above, it is easy to check that the hypothesis is true for $j = 2$, hence by induction Theorem 3 proof is now complete. **Q.E.D.**

Proof of Theorem 4: Given any voting rule satisfying the two assumptions of responsiveness and invariance, we need to show that for some preference profile admitting a *CW*, there exists a Nash equilibrium with weakly undominated strategies such that a non-Condorcet winner is elected.

First we show that *sincere* submission of one's ranking is never a weakly dominated strategy. Without any loss of generality assume that voter i has the preference relation $c^1 \succ_i c^2 \succ_i ... \succ_i c^k$. Suppose $R_i = (X_1, ..., X_J)$ is sincere and R_i is dominated by $R'_i = (X'_1, ..., X'_J)$. Note that for any $\tau \leq J$, $|X_\tau| = m(\tau)$ and $|X'_\tau| \geq m(\tau)$. Let j be the first cell such that $X'_j \neq X_j$. If $|X_j| < |X'_j|$ then for some $r > j$ it must be that $|X_r| > |X'_r|$, but this is not possible. So $|X_j| = |X'_j|$, hence there exist some $x \in X_j$ and $y \in X'_j$ such that $x \in X'_\ell$ for some $\ell > j$ and $y \in X_r$ for some $r > j$. Hence $x \, P_i \, y$ and $y \, P'_i \, x$. But then by condition 1 in Definition 6 there exists R_{-i} such that (R_i, R_{-i}) results in x winning, and (R'_i, R_{-i}) results in y winning, thus contradicting that R_i is dominated by R'_i.

Now consider two separate cases.

Case A: The voting rule is not CC with respect to sincere voting.

Consider any specific preference profile $(\succ_1, ..., \succ_n)$ and the sincere strategy profile $R_\mathcal{N} \equiv (R_i)_{i \in \mathcal{N}}$ for which Condorcet consistency is violated in sincere voting. By the above argument, each i submitting R_i is weakly undominated. Replicate this voting game sufficiently large with every voter with preference ordering $\succ_i$ submitting R_i (so that the *scale invariance* of Definition 6 applies) such that unilateral deviation does not alter the non-Condorcet outcome and hence constitute a Nash equilibrium in undominated strategies.

Case B: The voting rule is CC in sincere voting.

Consider the first three candidates c^1, c^2 and c^3. Without any loss of generality assume that c^3 is the candidate among the first three candidates that satisfies the property in condition 2 in Definition 6 (i.e. c^3 is in the role of candidate z in condition 2). Next, let $\kappa = \max\{\kappa' \mid 3\kappa' \leq n\}$, where n is the number of voters. Suppose that the true preference profile of the voters is such that the set of voters can be partitioned into three sets $\mathcal{S}^1, \mathcal{S}^2$ and $\mathcal{S}^3$ as follows: The set $\mathcal{S}^1$ consists of $n - 2\kappa$ voters and each $i \in \mathcal{S}^1$ has preferences given by $c^1 \succ_i c^2 \succ_i c^3 \succ_i ... \succ_i c^k$; the set $\mathcal{S}^2$ consists of κ voters and each $i \in \mathcal{S}^2$ has preferences given by $c^2 \succ_i c^1 \succ_i c^3 \succ_i c^4 \succ_i ... \succ_i c^k$; the set $\mathcal{S}^3$ consists of κ voters and each $i \in \mathcal{S}^3$ has preferences given by $c^3 \succ_i c^1 \succ_i c^2 \succ_i c^4 \succ_i ... \succ_i c^k$. Then note that c^1 is the CW and c^2 is the CW among all candidates other than c^1. Also, denote the set of voters that prefer c^2 to c^3 by $\mathcal{S} = \mathcal{S}^1 \cup \mathcal{S}^2$; clearly, $\mathcal{S}$ forms a majority.

Now since the voting rule is CC in sincere voting, by condition 2 in Definition 6, $J = k$ (all rankings are strict). Next consider for any $i \in \mathcal{S}$ the strategy $R_i = (c^2, c^1, c^3, ..., c^k)$. First we show that for any $i \in \mathcal{S}$, R_i is not weakly dominated.

Suppose not; then for some $i \in \mathcal{S}$, R_i is weakly dominated by another strategy $R'_i = (X'_1, ..., X'_k)$. Now since R_i is sincere if $i \in \mathcal{S}^2$ and voting sincerely is not weakly dominated, it follows that $i \in \mathcal{S}^1$ and $c^1 \succ_i c^2 \succ_i c^3 \succ_i ... \succ_i c^k$. Using this, we next establish in several steps that $X'_\tau = c^\tau$ for all $\tau \leq k$.

Step 1: We claim that $X'_1 \neq c^\tau$ for any $\tau > 2$. Suppose not; then by condition 1 in Definition 6 there exists R_{-i} such that R_i results in c^2 winning, and R'_i results in c^τ for some $\tau > 2$ winning, thus contradicting that R_i is dominated by R'_i.

Step 2: We claim that $X'_1 = c^1$. Suppose not; then by the previous step $X'_1 = c^2$. But since $R_i = (c^2, c^1, c^3, ..., c^k)$ and $c^1 \succ_i c^2 \succ_i c^3 \succ_i ... \succ_i c^k$, it must then be that $X'_2 = c^1$. Otherwise $X'_2 = c^\tau$ for some $\tau > 2$, and by condition 1 there exists some R_{-i} such that (R_i, R_{-i}) elects c^1 whereas (R'_i, R_{-i}) elects c^τ, contradicting that R_i is dominated by R'_i. That is, from $X'_1 = c^2$ follows $X'_2 = c^1$, and continuing with a similar reasoning using induction yields $R'_i = R_i$. But this is a contradiction.

Step 3: We claim that for $X'_j = c^j$ for all $j \leq J$. Since by the previous step the claim is true for $j = 1$, by induction, it suffices to show that for any $j \leq J$, if $X'_{j'} = c^{j'}$ for all $j' < j$, then $X'_j = c^j$. To show this suppose contrary to the claim that $X'_{j'} = c^{j'}$ for all $j' < j$ and $X'_j \neq c^j$. Then $X'_j = c^\tau$ for some $\tau > j$. This implies, by condition 1 in Definition 6, that there exists R_{-i} such that R_i results in either c^1 (if $j = 2$) or c^j (if $j > 2$) winning, and R'_i results in c^τ. Since i prefers both c^1 and c^j to c^τ ($\tau > j$), this contradicts R_i being dominated by R'_i.

Now since $R_i = (c^2, c^1, c^3, ..., c^k)$ and $R'_i = (c^1, c^2, c^3, ..., c^k)$, by condition 2 in Definition 6, there exists a strategy profile R_{-i} such that (R'_i, R_{-i}) elects c^3 whereas (R_i, R_{-i}) elects c^2. Since c^3 is worse than c^2 in i's true ranking, R'_i cannot weakly dominate R_i. But this is a contradiction. Hence R_i is not weakly dominated.

Now consider any strategy combination $R_\mathcal{N}$ in which every $i \in \mathcal{S}$ submits the strategy R_i, and the rest of the voters submit any undominated strategies (e.g. they vote sincerely by submitting $(c^3, c^1, c^2, c^4, ..., c^k)$). We next show that such a profile results in c^2 being elected. Consider any preference profile $\succ' = (\succ'_1, ..., \succ'_n)$ such that $c^2 \succ'_i c^1 \succ'_i c^3 \succ'_i ... \succ'_i c^k$ for every $i \in \mathcal{S}$ and $R_{i'}$ is sincere with respect to $\succ'_{i'}$ for every $i' \in \mathcal{N} \backslash \mathcal{S}$. Clearly, $R_\mathcal{N}$ is sincere with respect to $\succ'$. Moreover, since $\succ'$ is such that c^2 is the most preferred for every $i \in \mathcal{S}$ and the set $\mathcal{S}$ constitutes a majority, it follows that c^2 is the CW with respect to $\succ'$. Hence, since the voting rule is, by assumption, CC in sincere voting and $R_\mathcal{N}$ is sincere with respect to $\succ'$, it follows that c^2 must be elected when the voters submit $R_\mathcal{N}$.

Now assume that $n > 5$ and $R_\mathcal{N}$ is chosen. Then no individual voter can affect the outcome because for any single deviation there are at least $n - 2\kappa + \kappa - 1 = n - \kappa - 1 \geq 2\kappa - 1$ voters (the numbers of $\mathcal{S}_1$ and $\mathcal{S}_2$ minus 1) who put c^2 first. Since $2\kappa - 1$ forms a majority if $n > 5$ and the voting rule is CC with respect to sincere voting, it follows that c^2 is still elected if any single voter deviates. Thus the strategy profile $R_{\mathcal{N}}$ is a Nash equilibrium with undominated strategies, yielding the candidate c^2. But c^1 is the CW with respect to the true preferences. **Q.E.D.**

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