Poisson Games, Strategic Candidacy, and Duverger's Law

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Abstract

Duverger's law predicts a long-run two-candidate stable outcome under a plurality voting system. Duverger (1954) explains the law using the waste vote argument, which emphasizes voters' tendency to abandon candidates commonly perceived to have the least support. Palfrey (1989) formalizes Duverger's argument by modeling a three-candidate voting situation as a Bayesian game, and shows that there are only two-candidate equilibrium outcomes in very large electorates. However, Palfrey (1989) does not examine non-generic cases where two or more weak candidates have the same expected equilibrium vote share, so there is no unique candidate for voters to abandon and a three-candidate equilibrium outcome is possible. We show that the incentive of candidates to strategically withdraw from elections in order to avoid extreme outcomes eliminates such non-Duvergerian equilibria. We add uncertainty about the number of voters to Palfrey's framework, and model the election situation as a two-stage game where candidates first make strategic entry decisions, followed by plurality voting, which is modeled as a Poisson game à la Myerson (2000). In this framework, the two-party tendency for both generic and non-generic cases in Palfrey (1989) are explained, as is an anomaly to Duverger's law, viz. in India there persists a strong central party and two weak parties with similar strength.

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1 Introduction

Among all important propositions in political science, Duverger's law is perhaps one of the most intriguing to economists. The law focuses on the effect of the electoral regime on the party system: *The simple-majority, single-ballot system*[1] *favours the two-party system.* [Duverger (1954), page 217.] Despite the increasing empirical support received by the law[2], and various theoretical models developed to seek foundations for the law, no model has yet succeeded in fully explaining the law. A most popular explanation is the *waste vote argument*, which emphasizes voters' tendency to abandon the party commonly perceived to have the least support. However, the waste vote argument does not apply to situations where there is no unique party to abandon. Furthermore, previous studies explaining the two-party tendency often do not account for the anomalies to the law, e.g. the persistence of three parties in India under a plurality voting system[3].

In this paper, we provide a more complete answer to both the forces underlying Duverger's law and the anomaly in India by considering the strategic behavior of both voters and candidates. Specifically, we study a two-stage game where three candidates first decide whether to participate in an election, followed by plurality voting. Candidates are distinguished by their positions on the interval [0,1]. They dislike extreme outcomes. Voters have strict and single-peaked preferences on [0,1]. The voting situation is modeled as a Poisson game à la Myerson (2000). That is, we assume the number of voters is a random variable following a Poisson distribution[4]. We completely characterize the set of limit voting equilibria for a sequence of Poisson voting games, which has the following properties: (i) In the generic case when there is a unique candidate with the least expected equilibrium vote share, no one votes for him in the realized equilibrium outcome.

[1] This is a specific case of the plurality voting system, which elects a single winner who obtains the most votes. The winner has a simple majority when there are two or fewer candidates, but not necessarily so with three or more candidates.

[2] For a survey, see Ray (1971).

[3] In Canada, there have also been three parties under plurality voting since 1921. But the third party which is the weakest at the national level is a major party at the regional level. So the Canada case is not a true anomaly to Duverger's law.

[4] Introducing voter population uncertainty has realistic flavors as in large elections a voter usually does not know the exact number of other voters in the game. The cause of the uncertainty could be that each voter has some positive probability of not turning out, or some votes are non-eligible or not counted due to technical errors, see e.g. Eguia (2007), Messner and Polborn (2005).

(ii) In the non-generic case when there are two or more weak candidates with the same expected equilibrium vote share, voters either vote for their most preferred, or for the strongest candidate. We also prove the non-existence of a three-candidate equilibrium outcome under some general conditions. In particular, in the non-generic case, so long as no candidate expects an absolute majority, and the centrist is not the strongest, then there is no equilibrium in which all three candidates participate. The intuition is that a left or a right candidate prefers to stay out in order to transfer his votes to the centrist to avoid the opposite extreme policy to win. Finally, we provide a sufficient and necessary condition for a three-candidate outcome as in India to sustain, that is, a largest central party and two smaller parties of similar strength persist under a plurality voting system.

This paper extends the framework in Palfrey (1989), which is the first paper to formalize the waste vote argument, to an environment with voter population uncertainty and strategic candidacy, and provides a theoretical explanation for the existence or non-existence of a three-candidate equilibrium outcome in the non-generic case regarded by Palfrey as empirical rarities. Our approach differs from most previous studies on Duverger's law in that candidates' participation decisions are considered as endogenous. Combining strategic candidacy and the waste vote argument, we successfully explain both generic and non-generic cases in Palfrey (1989), as well as the anomaly in India.

A common feature in Palfrey (1989) and subsequent papers using Palfrey's framework to explain Duverger's law is that they show the existence of a two-party equilibrium under plurality voting, but do not address the issue whether/when there is an equilibrium in which a strong party and two equally weak parties, or three parties with equal strength coexist. Feddersen (1992) shows that in any pure strategy equilibrium of a finite plurality voting game, votes are entirely concentrated on two candidates. Messner and Polborn (2005) adds uncertainty about the number of actual votes to Palfrey's model, and uses trembling hand perfection to obtain a two-party equilibrium. Myerson (1998, 2000) introduce Poisson games to model uncertainty about the number of players, which has a natural application to the waste vote argument. However, Myerson mainly uses it to study how different voting rules affect the set of voting equilibria, see e.g. Myerson (2002). In an early paper, Riker (1976) explains both Duverger's law and the persistence of three parties in India by employing an argument of disillusional voting. However, Riker's logic again fails when it comes to the non-generic case.

In our model, we assume that candidates are averse to extreme outcomes, and use the citizen-candidate approach developed in Osborne and Slivinski (1996), Besley and Coate (1997) to refine away the three-candidate equilibria in Palfrey's non-generic case. Extreme aversion seems to be a natural assumption about politicians who care about policies. Our work is related to Riviere (2000), which studies a three-stage game with a party formation stage and a candidate entry stage before the voting subgame. Riviere (2000) shows that when entry costs are shared among party members, there is a central party and a left or a right party in the majority of cases. Some key assumptions in Riviere (2000) but absent in our model are large entry costs, and uncertainty about the median voter's position which is resolved before the voting subgame. The waste vote argument is irrelevant in Riviere's setting since voting is sincere in equilibrium, and Palfrey's non-generic case is left untouched.

The remainder of the paper is structured as follows. Section 2 describes the model. Section 3 characterizes the set of limit voting equilibria. Section 4 studies equilibria in the full game, and the implications of strategic candidacy on Duverger's law. Section 5 concludes. The appendix contains most of the technical proofs.

2 The Model

We model a three-candidate election situation as a two-stage game. The policy space is the unit interval $[0,1]$, and a generic policy is denoted by x.

Stage 1. Entry game

We call the first stage an *entry game*. The players in an entry game are candidates. The set of candidates is $K = \{1,2,3\}$. Each candidate k has a unique most preferred policy, which we denote by x_k. We assume that x_k are three distinct points on the interval $[0,1]$ with $x_1 < x_2 < x_3$. We call candidates $1,2,3$ a leftist, a centrist, and a rightist, respectively. Note that $x_2 = \theta x_1 + (1-\theta)x_3$, for some $\theta \in (0,1)$. When we do not wish to distinguish candidates by their positions, we refer to them as i, j, h. Each candidate's preference is defined on [0,1], which has a VNM utility representation $u_k : [0, 1] \to \mathbb{R}$, where $u_k(x)$ is candidate k's payoff if the policy x is implemented. We assume that u_k is strictly concave on [0,1], that is,

$$u_k(\theta x + (1-\theta)y) > \theta u_k(x) + (1-\theta)u_k(y), \ \forall x, y \in [0,1] \text{ with } x \neq y, \ \forall \theta \in (0,1)$$

The interpretation of this assumption is that candidates are averse to extreme outcomes.

The strategies available to each candidate are *to participate (I)* and *not to participate (O)*. If a candidate participates, he proposes his most preferred policy and commits to implement the policy after being elected. That is, policy is not a choice variable of candidates. We denote candidate k's strategy space by $S_k = \{I, O\}$. Let $S = S_1 \times S_2 \times S_3$. An entry profile in S is denoted by $\mathbf{s} = (s_1, s_2, s_3)$. $\mathbf{s}_{-k}$ refers to a profile that excludes the strategy of k.

Stage 2. Poisson voting game

Each entry profile $\mathbf{s}$ induces a second stage game, which we call a *voting subgame*, and denote by $\Gamma_{\mathbf{s}}$. The players in a voting subgame are voters. To model uncertainty about the number of voters in the game, we follow Myerson (2000) and assume that this number is a Poisson random variable with some mean n. We refer to this type of game as a *Poisson voting game*. Each voter has a strict preference ordering $\succ$ on $[0, 1]$ which is single peaked with respect to the usual greater than ($>$) relation on the real line. The single-peakedness assumption is standard in spatial voting models. The VNM utility representation of a single peaked preference, $v : [0, 1] \to \mathbb{R}$, satisfies the following condition:

There exists a policy $\hat{x} \in [0,1]$ such that if $\hat{x} > z > y$, or $y > z > \hat{x}$, then $v(z) > v(y)$.

We restrict voters' preferences to the set $\{x_1, x_2, x_3\}$. Note that restricted preferences are also strict and single peaked. We normalize utilities for each voter such that if $x_i \succ x_j \succ x_h$, then $v(x_i) = 1$, $v(x_h) = 0$, and $v(x_j) \in (0, 1)$. Denote by $\mathscr{P}_K$ the set of four strict, single peaked preference orderings on $\{x_1, x_2, x_3\}$. We define the type space to be $T = \mathscr{P}_K \times (0, 1)$. That is, each voter has a type $t \in T$ that specifies an ordering from $\mathscr{P}_K$ and a utility number for her second most preferred candidate. We assume that each voter's type is independently drawn from T according to some probability distribution λ. $\lambda(t)$ is independent of n. A voter's type is her private information, but λ is common knowledge among all voters and candidates.

Following an entry profile $\mathbf{s}$, each voter casts her vote for one participating candidate. The common action space for all voters is $A = \{a_1, a_2, a_3\}$, where a_k refers to the action of voting for candidate k. We assume that there is no abstention. Also, for technical convenience, we assume that candidates do not vote. Relaxing this assumption does not affect our results. Following Myerson (2000), we use distributional strategies to describe voters' strategic behavior[5]. This is described as follows. Let $\Delta(A)$ be the set of probability distributions on A. $\sigma : T \to \Delta(A)$ is a measurable function satisfying $\sigma_t(a_k) \geq 0,\ \forall k \in K$ and $\sum_{k \in K} \sigma_t(a_k) = 1,\ \forall t \in T$, where $\sigma_t(a_k)$ is the probability that a type-t voter uses action a_k. A *distributional strategy* $\boldsymbol{\pi}$ is a probability distribution on the set $A \times T$ with the marginal distribution on T equal to λ, and the marginal distribution on A given by $\pi_{a_k} = \int_T \sigma_t(a_k)\ d\lambda(t)$. Here, π_{a_k} is the probability that a randomly selected voter uses action a_k. For conciseness, henceforth we write π_{a_k} as π_k.

The voting game uses a single ballot plurality rule. That is, each voter can vote for only one candidate, and the candidate with the most votes wins the election. When multiple candidates tie for the first place, each candidate wins with equal probability (we call this a uniform tie-breaking rule). We summarize the number of votes that each candidate gets by a vector in $\mathbb{Z}_+^3$, $\mathbf{m} = (m(k))_{k \in K}$, where $m(k)$ is the number of voters using action a_k. We call $\mathbf{m}$ a *voting profile.* Note that Poisson distribution on voter population implies that the expected voting profile induced by $\boldsymbol{\pi}$ is $n\boldsymbol{\pi} = (n\pi_k)_{k \in K}$, where $n\pi_k$ is the expected number of voters who use action a_k.

A voter's payoff is given by the function $V : \mathbb{Z}_+^3 \times A \times T \to \mathbb{R}$. $V(\mathbf{m}, a_k, t)$ is a type-t voter's payoff when she uses action a_k and the realized voting profile is $\mathbf{m}$. We assume that V is bounded and continuous in t on T. A type-t voter's expected payoff in a Poisson game with mean n when she uses action a_k and all other voters follow the distributional strategy $\boldsymbol{\pi}$ is given by $EV_t(a_k, \boldsymbol{\pi}) = \sum_{\mathbf{m} \in \mathbb{Z}_+^3} \mathbb{P}(\mathbf{m} \mid n\boldsymbol{\pi}) V(\mathbf{m}, a_k, t)$, where $\mathbb{P}(\mathbf{m} \mid n\boldsymbol{\pi})$ is

[5] This is to avoid measurability problems associated with defining mixed strategies when there is a continuum of types, see Milgrom and Weber (1985).

the probability that $\mathbf{m}$ is the realized voting profile given $\boldsymbol{\pi}$ and n. Here the expected utility calculation uses an *environmental equivalence* property of Poisson games, which basically says that if the number of voters follows a Poisson distribution with mean n, then any voter already in the game would assess the number of other voters in the game using the same Poisson distribution (for details, see Myerson (1998)).

A candidate's payoff is given by the function $U : S \times \mathbb{Z}^3_+ \times K \to \mathbb{R}$. $U(\mathbf{s}, \mathbf{m}, k)$ is candidate k's payoff when the entry profile is $\mathbf{s}$ and the realized voting profile in the induced subgame $\Gamma_{\mathbf{s}}$ is $\mathbf{m}$. A candidate calculates his expected payoff based on his anticipation of the voting behavior $\boldsymbol{\pi}$ in $\Gamma_{\mathbf{s}}$, which is given by $EU_k(\mathbf{s}, \boldsymbol{\pi}) = \sum_{\mathbf{m} \in \mathbb{Z}^3_+} \mathbb{P}(\mathbf{m} \mid n\boldsymbol{\pi}) U(\mathbf{s}, \mathbf{m}, k)$. We also assume that if a candidate expects to receive zero vote, then he prefers not to enter, see e.g. Eguia (2007).

A *voting equilibrium* of a Poisson voting game with mean n is a distributional strategy $\boldsymbol{\pi}$ such that each marginal distribution on A, $\pi_k = \int_T \sigma_t(a_k)\, d\lambda(t)$, satisfies the following condition:

$$\text{For all } t \in T \text{ with } \sigma_t(a_k) > 0,\ a_k \in \operatorname*{arg\,max}_{a_i \in A}\ EV_t(a_i, \boldsymbol{\pi}).$$

That is, all the probabilities of choosing a_k come from types for whom a_k maximizes their expected payoffs when all other voters follow the distributional strategy $\boldsymbol{\pi}$. We say a_k is a weakly dominated action for a type-t voter if there exists another action $a_i \in A$ such that $V(\mathbf{m}, a_k, t) \leq V(\mathbf{m}, a_i, t)$, for all $\mathbf{m} \in \mathbb{Z}^3_+$, and with strict inequality for at least one $\mathbf{m}$. We assume that no voter uses a weakly dominated action in equilibrium.

An entry profile $\mathbf{s}$ is an *entry equilibrium* if for all $k \in K$, $EU_k(\mathbf{s}, \boldsymbol{\pi}) \geq EU_k(\mathbf{s}', \boldsymbol{\pi}')$, where $\mathbf{s}' = (s'_k, \mathbf{s}_{-k})$ with $s'_k \neq s_k$, and $\boldsymbol{\pi}'$ is the anticipated voting behavior in $\Gamma_{\mathbf{s}'}$. That is, no candidate obtains a higher expected payoff by unilaterally deviating, anticipating voting outcomes with and without his presence.

An equilibrium in the full game consists of an entry equilibrium $\mathbf{s}$, a voting equilibrium in each voting subgame induced by $\mathbf{s}$ and $(s'_k, \mathbf{s}_{-k})$, for all $k \in K$. We restrict our attention to equilibria that are subgame perfect.

3 Equilibria in the Poisson Voting Game

In this section, we provide a full characterization of the limit equilibria in large Poisson voting games. A key step is to derive limit relative probabilities of pivot events. Palfrey (1989) proves the waste vote argument by showing that voters abandon the weakest candidate because the probability of this candidate contending for the first place is infinitesimal relative to the probability of a stronger candidate doing so. Hence, conditional on a vote being pivotal, it has almost zero probability of swinging the voting outcome in favor of the weakest candidate. Palfrey (1989) directly computes the probabilities that a tie for most votes occurs between: (i) the weakest candidate and a strong one; (ii) two strong candidates, and shows that the ratio of these two probabilities goes to zero as the size of the electorate becomes very large. Although Palfrey's approach is straightforward, the computations involved are rather complicated. Myerson (2000) provides an alternative, and considerably simple way of solving problems of the same nature in a Poisson game. Instead of directly computing the probabilities of some events, Myerson calculates the rates at which these probabilities go to zero. This is equivalent to showing whether the relative probability of some event is zero in the limit. In particular, Myerson (2000) shows that this calculation can be reduced to a simple maximization problem. We use this technique here to prove the waste vote argument with voter population uncertainty.

Consider a sequence of Poisson voting games $(\Gamma^n)_{n=1}^{\infty}$ following an entry profile $\mathbf{s}$ where all three candidates participate (for conciseness we suppress the subscript $\mathbf{s}$). We associate each Γ^n with a voting equilibrium $\boldsymbol{\pi}^n$ if an equilibrium exists[6]. We say that the sequence $(\boldsymbol{\pi}^n)$ converges to $\boldsymbol{\pi}$ if there is a subsequence of $(\boldsymbol{\pi}^n)$ (still denote it by $(\boldsymbol{\pi}^n)$) such that (π_k^n) converges to π_k, for each $k \in K$, as $n \to \infty$. Since each (π_k^n) is a bounded sequence of reals, it has a convergent subsequence. Thus, in the rest of the paper, we assume that $\lim_{n\to\infty} \boldsymbol{\pi}^n$ exists.

[6]The existence of an equilibrium in Γ^n can be guaranteed if the type space is compact, that is, if we allow voters to have weak preferences, so $T = \mathscr{P}_K \times [0, 1]$. By equipping T with a proper metric, we can make T a compact metric space and then apply Theorem 0 in Myerson (2000). However, the existence result is not necessary for our purpose.

3.1 Pivot probabilities in the large Poisson game

In this subsection, we examine the magnitudes of pivot events in large Poisson games. We first explain what we mean by an event, a pivot event, and magnitude of an event. In our model, an *event* is a collection of voting profiles, which is a subset of $\mathbb{Z}^3_+$. For any event $M \subset \mathbb{Z}^3_+$, we define the probability of M by $\mathbb{P}(M \mid n\boldsymbol{\pi}) = \sum_{\mathbf{m}\in M} \mathbb{P}(\mathbf{m} \mid n\boldsymbol{\pi})$. Sometimes we abbreviate $\mathbb{P}(\cdot \mid n\boldsymbol{\pi}^n)$ as $\mathbb{P}(\cdot)$ when $n, \boldsymbol{\pi}$ are clear from the context. The *magnitude* of M is defined as $\mu_M = \lim_{n\to\infty} \ln(\mathbb{P}(M \mid n\boldsymbol{\pi}^n))/n$, provided that the limit exists. We say that M is a *pivot event* if for each $\mathbf{m} \in M$, an additional vote for different candidates yields different voting outcomes. We distinguish two types of pivot events.

- A *close race between i and j* is a pivot event where i and j get the top two highest number of votes, respectively[7]. Notice it must be $|m(i) - m(j)| \leq 1$ before a pivot vote is cast. Denote by Λ_{ij} a close race between i, j, and μ_{ij} its magnitude.
- An *ij-sensible* event is a pivot event where a vote for i and for j yield different outcomes. Denote by Ω_{ij} the set of all ij-sensible events, which is listed in Table 1.

We make two remarks on these pivot events before analyzing their magnitudes. First, it is not necessary that an ij-sensible event is a close race between i and j. For example, in Table 1, the close race is between i and h for events ii., iii., iv., viii., and ix., and is between j and h for events ii., iv., v., viii., and x. More specifically, an ij-sensible event concerns how an action from a pivot voter affects the voting outcome (this can be seen clearly from Table 1), whereas a close race is about who will be the final winner. If a close race is between i and j, then regardless of the action of a pivot voter, h is never going to win. Second, an ij-sensible event is particularly relevant to a voter whose strategic decision is between i and j in a three-candidate election. From this voter's point of view, an ij-sensible event can be regarded as generated by all other voters who follow a distributional strategy $\boldsymbol{\pi}^n$, and her vote for i or for j swings the voting outcome.

The key to a voter's expected utility calculation is probabilities of pivot events that have the largest magnitude. This is because the probabilities of smaller-magnitude events

[7] A close race among three candidates can be defined in a similar way, but it suffices to look at a close race between two candidates for our purpose.

become infinitesimal in very large elections (for a formal statement, see Lemma 3.1), hence the impact on expected utilities is negligible. Before going through the main result of this subsection which identifies the pivot events with the largest magnitude in a large Poisson voting game (Proposition 3.4), we first state three lemmas. Lemma 3.1 relates the magnitudes of two events to the limit ratio of their probabilities. Lemma 3.2 is a restatement of Theorem 1 in Myerson (2000), which provides a key step in computing the magnitude of any event in a large Poisson game. Lemma 3.3 computes the magnitude of some event generated by a sequence of voting equilibria $\boldsymbol{\pi}^n$.

Lemma 3.1. *Let* $\{A^n\}, \{B^n\}$ *be two sequences of events with finite magnitudes* μ_{A^n}, μ_{B^n}*. Then* $\lim_{n\to\infty} \mathbb{P}(A^n \mid n\boldsymbol{\pi}^n)/\mathbb{P}(B^n \mid n\boldsymbol{\pi}^n) = 0$ *if* $\mu_{A^n} < \mu_{B^n}$*.*

Proof. The rate that $\mathbb{P}(A^n \mid n\boldsymbol{\pi}^n)$ goes to 0 is $e^{n\mu_{A^n}}$. If $\mu_{A^n} < \mu_{B^n}$, and both are finite, then $\lim_{n\to\infty} \mathbb{P}(A^n \mid \boldsymbol{\pi}^n)/\mathbb{P}(B^n \mid \boldsymbol{\pi}^n) = \lim_{n\to\infty} e^{n\mu_{A^n}}/e^{n\mu_{B^n}} = \lim_{n\to\infty} e^{n(\mu_{A^n}-\mu_{B^n})} = 0$. ☐

Lemma 3.2. [Myerson (2000)] *Let* M *be an event in* $\mathbb{Z}_+^3$*. Then*

$$\begin{aligned}\mu_M &= \lim_{n\to\infty} \max_{\mathbf{m}^n\in M} \ln(\mathbb{P}(\mathbf{m}^n \mid n\boldsymbol{\pi}^n))/n \\ &= \lim_{n\to\infty} \max_{\mathbf{m}^n\in M} \sum_{k\in K} \pi_k^n \psi\left(\frac{m^n(k)}{n\pi_k^n}\right)\end{aligned}$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}$ *is defined as follows.*

$$\psi(w) = w(1-\ln w) - 1,\ \forall w > 0,\ \text{and } \psi(0) = -1.$$

Lemma 3.3. *Consider the following event generated by a sequence of voting equilibria* $\boldsymbol{\pi}^n$ *that converges to* $\boldsymbol{\pi}$*:* $M = \{\mathbf{m}^n : m^n(i) = m^n(j) + z_1,\ m^n(i) \geq m^n(h) + z_2\}$*, where* z_1, z_2 *are integers. Then the magnitude of* M *is*

$$\mu_M = \begin{cases} \pi_h + 2\sqrt{\pi_i\pi_j} - 1 & \text{if } \pi_i\pi_j > \pi_h^2 \\ 3\sqrt[3]{\pi_h\pi_i\pi_j} - 1 & \text{if } \pi_i\pi_j \leq \pi_h^2 \end{cases}$$

Proof. See appendix. ☐

Proposition 3.4. *Let* $\boldsymbol{\pi}$ *be a limit voting equilibrium. If* $\pi_i < \min\{\pi_j, \pi_h\}$*, then among all* ij*-sensible events, the close race between* j *and* h

$$\Lambda_{jh} = \{\mathbf{m}^n : m^n(h) = m^n(j) + 1 > m^n(i) + 1\} \cup \{\mathbf{m}^n : m^n(j) = m^n(h) > m^n(i) + 1\}$$

has the largest magnitude.

Proof. See appendix. □

Proposition 3.4 says that in a Poisson voting game, if a candidate is perceived to be the weakest in the sense that his expected equilibrium vote share is the smallest, then a close race, if occurring, is almost certain to be between the two strong candidates as the number of voters becomes very large. Note that Proposition 3.4 incorporates more cases as compared with Lemma 2 in Palfrey (1989), which only computes the probability ratio of the pivot events $\{\mathbf{m}^n : m^n(h) = m^n(i) + 1 > m^n(j) + 1\}$ and $\{\mathbf{m}^n : m^n(h) = m^n(j) + 1 > m^n(i) + 1\}$, that is, events ix. and x. in Table 1. In other words, assuming a uniform tie-breaking rule is more general than assuming an alphabetic tie-breaking rule as in Palfrey (1989). Palfrey's proof would have been considerably complicated if a uniform tie-breaking rule is used. This indeed illustrates one advantage of using Poisson games and computing the magnitudes of pivot events.

3.2 Equilibria in the large Poisson game

In this subsection, we study equilibria in voting subgames. If there is only one participating candidate, he gets all votes since we assume no abstention. When there are two competing candidates, the only voting equilibrium is that each voter votes for the one whom she prefers since voting for one's least preferred among all participating candidates is a weakly dominated action, which is not used by any voter in equilibrium due to our assumption. We call voting according to one's true preference *sincere voting.*

Our main result here is a complete characterization of the limit equilibria for a sequence of Poisson voting games with three candidates (Proposition 3.5). It is well known that plurality rules are susceptible to strategic voting, i.e. Palfrey's generic case. But surprisingly, to our best knowledge, no theorem fully describes equilibrium voting behavior in plurality voting games. We show that in one of Palfrey's non-generic cases where all candidates are expected to have the same equilibrium vote share, sincere voting is the dominant action for each voter. In the other non-generic case where one candidate has the largest expected equilibrium vote share, and the other two have equal share of the remaining votes, supporters of a weak candidate either vote for her favorite or for the strongest, whereas supporters of the strong candidate always vote sincerely.

Denote by t_k the type of voters whose most preferred candidate is k.

Proposition 3.5. *Suppose that $(\boldsymbol{\pi}^n)$ is a sequence of voting equilibria converging to some limit $\boldsymbol{\pi}$ as $n \to \infty$. Then $\boldsymbol{\pi}$ satisfies the following properties.*

1. *If $\pi_i < \min\{\pi_j, \pi_h\}$, then $\pi_i = 0$.*

2. *If $\pi_i = \pi_j = \pi_h$, then $\sigma_{t_k}(a_k) = 1,\ k = i, j, h$.*

3. *If $\pi_i = \pi_j < \pi_h$, then $\sigma_{t_i}(a_j) = 0$, $\sigma_{t_h}(a_h) = 1$.*

Proof. We first show that $\pi_i < \min\{\pi_j, \pi_h\}$ implies $\pi_i = 0$. Note that voters who least prefer i never vote for i, so it suffices to show that any voter whose least preferred candidate is j or h never votes for i either. Without loss of generality, consider a type-t voter whose least preferred candidate is h. We show that this voter's expected payoff from voting for i is strictly less than that from voting for j in equilibrium as the electorate becomes very large. To see this, recall that Proposition 3.4 shows the largest-magnitude event among all ij-sensible events is $\Lambda_{jh} = \{\mathbf{m}^n : m^n(h) = m^n(j)+1 > m^n(i)+1\} \cup \{\mathbf{m}^n : m^n(j) = m^n(h) > m^n(i)+1\}$, that is, events v. and x. in Table 1. Notice Λ_{jh}, v., x., and Ω_{ij} have the same magnitude, hence the limit ratio of probabilities between any pair of these events is positive. For any ij-sensible event $M^n \in \Omega_{ij} \backslash \Lambda_{jh}$ with smaller magnitude, we have $\lim_{n\to\infty} \mathbb{P}(M^n/\mathbb{P}(\Omega_{ij}) = 0$ due to Lemma 3.1. So conditional on some ij-sensible event occurring,

$$\begin{aligned}
& \lim_{n\to\infty} EV_t(a_i, \boldsymbol{\pi}^n) - EV_t(a_j, \boldsymbol{\pi}^n) \\
= & -\frac{1}{2} v_t(x_j) \times \lim_{n\to\infty} \mathbb{P}(\text{v.} \mid \Omega_{ij}) - \frac{1}{2} v_t(x_j) \times \lim_{n\to\infty} \mathbb{P}(\text{x.} \mid \Omega_{ij}) \\
= & -\frac{1}{2} v_t(x_j) \times \lim_{n\to\infty} \frac{\mathbb{P}(\text{v.}) + \mathbb{P}(\text{x.})}{\mathbb{P}(\Omega_{ij})} = -\frac{1}{2} v_t(x_j) < 0,
\end{aligned}$$

where $\mathbb{P}(\text{v.} \mid \Omega_{ij})$ is the conditional probability of event v. given Ω_{ij}, and so on. Therefore, in the limit this voter is strictly better off voting for j if all other voters follow $\boldsymbol{\pi}^n$ in each Γ^n. By a similar argument, voters who least prefer j always vote for h. This implies $\pi_i = 0$. For the rest of the proof, the limit expected payoff calculation is always conditional on some ij-sensible events occurring, so we will omit further mentioning this restriction.

Next, we assume that $\pi_i = \pi_j = \pi_h$. Due to symmetry in candidate strength, it suffices to consider a type-t voter whose preference is $i \succ j \succ h$, and show that voting for i is the dominant action for this voter. Note that in the current case, all ij-sensible events have the same magnitude. In fact, the following pairs of events in Table 1 have the same probabilities: ii. and iv., iii. and v., vi. and vii., ix., and x. A direct computation yields

$$\begin{aligned}
&\lim_{n\to\infty} EV_t(a_i, \boldsymbol{\pi}^n) - EV_t(a_j, \boldsymbol{\pi}^n) \\
= \quad &(v_t(x_i) - v_t(x_j)) \times \lim_{n\to\infty} \frac{\mathbb{P}(\text{i.}) + \mathbb{P}(\text{ii.}) + .5\mathbb{P}(\text{iii.}) + \mathbb{P}(\text{vi.}) + .5\mathbb{P}(\text{viii.}) + .5\mathbb{P}(\text{ix.})}{\mathbb{P}(\Omega_{ij})} > 0.
\end{aligned}$$

Therefore the voter prefers to vote for i over j and h. This proves the second part of the proposition.

Finally, assume that $\pi_i = \pi_j < \pi_h$. We first show that $\sigma_{t_i}(a_j) = 0$. Notice in this case, the largest-magnitude events in Ω_{ij} are iii., v., ix., x. Since iii. and v., ix. and x. have the same probabilities, we have

$$\lim_{n\to\infty} EV_{t_i}(a_i, \boldsymbol{\pi}^n) - EV_{t_i}(a_j, \boldsymbol{\pi}^n) = \frac{1}{2}(v_{t_i}(x_i) - v_{t_i}(x_j)) \times \lim_{n\to\infty} \frac{\mathbb{P}(\text{iii.}) + \mathbb{P}(\text{ix.})}{\mathbb{P}(\Omega_{ij})} > 0.$$

So a type t_i voter always votes for i over j, that is, $\sigma_{t_i}(a_j) = 0$. We are left to show that $\sigma_{t_h}(a_h) = 1$. This is equivalent to showing that $\pi_j = \pi_h < \pi_i \Rightarrow \sigma_{t_i}(a_i) = 1$ (in doing so we only need to refer to Table 1 and examine a voter's decision between voting for i and j). If $\pi_j = \pi_h < \pi_i$, the largest-magnitude events in Ω_{ij} are i., iii., vi., vii., ix., that is, events where a tie for most votes is not between j and h. So

$$\begin{aligned}
&\lim_{n\to\infty} EV_{t_i}(a_i, \boldsymbol{\pi}^n) - EV_{t_i}(a_j, \boldsymbol{\pi}^n) \\
= \quad &(v_{t_i}(x_i) - v_{t_i}(x_j)) \times \lim_{n\to\infty} \frac{\mathbb{P}(\text{i.}) + .5\mathbb{P}(\text{vi.}) + .5\mathbb{P}(\text{vii.})}{\mathbb{P}(\Omega_{ij})} + \frac{1}{2} v_{t_i}(x_i) \lim_{n\to\infty} \frac{\mathbb{P}(\text{iii.}) + \mathbb{P}(\text{ix.})}{\mathbb{P}(\Omega_{ij})} > 0.
\end{aligned}$$

Therefore a type t_i voter always votes for i over j. By symmetry, she always votes for i over h. This proves $\sigma_{t_i}(a_i) = 1$. □

Notice the last two cases in Proposition 3.5 correspond to Palfrey's non-generic case, that is, when there are two or more weak candidates who have the same expected equilibrium vote share. Palfrey (1989) mentions the possibility of such equilibria, but does not give a formal characterization. Furthermore, such non-Duvergerian outcomes are dismissed as empirical rarities. We feel that a more rigorous treatment is necessary for

Palfrey's theory to be complete. This is our task in the next section, where we show that these exceptions can be theoretically eliminated by considering strategic incentives of candidates.

4 Equilibria in the Full Game

Our first main result is the non-existence of a three-candidate equilibrium under some fairly general conditions, namely no candidate expects an absolute majority, and the centrist is not expected to be the strongest in the limit voting equilibrium (Theorem 4.3). Our second main result provides a sufficient and necessary condition for the persistence of three parties in India (Theorem 4.4). The following two lemmas identify the largest magnitude events in a two-candidate election and a three-candidate election with two equally weak candidates, respectively.

Lemma 4.1. *Suppose that only i and j participate. Let $M = \{\mathbf{m}^n : m^n(j) \geq m^n(i)\}$ and $M' = \{\mathbf{m}^n : m^n(i) > m^n(j)\}$ be two events in the induced subgame. If the limit voting equilibrium $\boldsymbol{\pi}$ satisfies $\pi_i \leq \pi_j$, then $\mu_M > \mu_{M'}$. In particular, $\lim_{n\to\infty} \mathbb{P}(M \mid n\boldsymbol{\pi}^n) = 1$, $\lim_{n\to\infty} \mathbb{P}(M' \mid n\boldsymbol{\pi}^n) = 0$.*

Proof. See appendix. □

Lemma 4.2. *Suppose that $\mathbf{s} = (I, I, I)$. If the limit voting equilibrium $\boldsymbol{\pi}$ in $\Gamma_{\mathbf{s}}$ satisfies $\pi_i = \pi_j < \pi_h$, then the event $M = \{\mathbf{m}^n : m^n(h) > \max\{m^n(i), m^n(j)\}\}$ has the largest magnitude. In particular, $\lim_{n\to\infty} \mathbb{P}(M \mid n\boldsymbol{\pi}^n) = 1$, and $\lim_{n\to\infty} \mathbb{P}(M' \mid n\boldsymbol{\pi}^n) = 0$ for any smaller magnitude event M'.*

Proof. See appendix. □

Theorem 4.3. *There does not exist a limit equilibrium $(\mathbf{s}, \boldsymbol{\pi})$, where $\mathbf{s} = (I, I, I)$, and $\boldsymbol{\pi}$ satisfies the following two properties: P1. $\pi_i = \pi_j < \pi_h \Rightarrow \pi_h \leq \frac{1}{2}$; P2. $\pi_1 = \pi_3 \Rightarrow \pi_2 \leq \frac{1}{3}$.*

Proof. Suppose that $\boldsymbol{\pi}$ is the limit of some sequence of voting equilibria $\boldsymbol{\pi}^n$ and satisfies P1 and P2. We show that $\mathbf{s} = (I, I, I)$ cannot be an entry equilibrium given $\boldsymbol{\pi}$. Without loss of generality, suppose that candidate 1 is (one of) the weakest under $\boldsymbol{\pi}$. There are three cases to consider: (i) $\pi_1 < \min\{\pi_2, \pi_3\}$; (ii) $\pi_1 = \pi_2 < \pi_3$; (iii) $\pi_1 = \pi_2 = \pi_3$. Notice the case $\pi_1 = \pi_3 < \pi_2$ is ruled out due to the assumption of P2. We show that in each of the three cases, some candidate prefers to withdraw, thus $\mathbf{s}$ is not an entry equilibrium.

First, if $\pi_1 < \min\{\pi_2, \pi_3\}$, then Proposition 3.5 shows that $\pi_1 = 0$. So candidate 1 does not enter by assumption. Second, if $\pi_1 = \pi_2 < \pi_3$, then $\sigma_{t_1}(a_1) = \sigma_{t_3}(a_3) = 1$. To see this, note that Proposition 3.5 implies that $\sigma_{t_3}(a_3) = 1$, and a type-t_1 voter chooses between 1 and 3, but 3 is her least preferred candidate, so $\sigma_{t_1}(a_1) = 1$. Furthermore, a type-t_2 voter may vote for 2 or 3, but not for 1. So we must have $\pi_1 = \lambda(t_1),\ \pi_2 \leq \lambda(t_2),\ \pi_3 \geq \lambda(t_3)$. This implies that $\lambda(t_1) + \lambda(t_2) \geq \pi_1 + \pi_2 \geq \pi_3 \geq \lambda(t_3)$, where $\pi_1 + \pi_2 \geq \pi_3$ is due to the assumption of P1. Now consider candidate 1's incentive to participate. Since $\pi_1 = \pi_2 < \pi_3$, Lemma 4.2 implies that the event when 3 is the winner has the largest magnitude and the limit probability of 1. So 1's expected payoff under $(\mathbf{s}, \boldsymbol{\pi})$ equals $u_1(x_3) = 0$. If 1 does not participate, voters vote sincerely between 2 and 3. Denote the limit voting equilibrium by $\boldsymbol{\pi}'$, then $\pi_2' = \lambda(t_1) + \lambda(t_2) \geq \lambda(t_3) = \pi_3'$. Lemma 4.1 implies that the event when 2 wins or ties with 3 has the largest magnitude and the limit probability of 1. So 1's expected payoff from $((O, \mathbf{s}_{-1}), \boldsymbol{\pi}')$ is at least $\frac{1}{2}u_1(x_2) > 0$. Therefore, 1 prefers to stay out. Finally, if $\pi_1 = \pi_2 = \pi_3$, then voting is sincere in equilibrium. So each candidate has a winning probability of $\frac{1}{3}$. If a left or a right candidate does not participate, his supporters will vote for the centrist. Lemma 4.1 then implies that the centrist wins for sure in a two-candidate competition. Recall that $x_2 = \theta x_1 + (1-\theta)x_3,\ \theta \in (0, 1)$. If $\theta \in [\frac{1}{2}, 1)$, then 1's expected payoff increase from staying out is

$$u_1(x_2) - \frac{1}{3}(1 + u_1(x_2)) = \frac{1}{3}(2u_1(x_2) - 1) > \frac{1}{3}(2\theta - 1) \geq 0$$

where the first inequality is due to the assumption that candidates are averse to extreme outcomes. This shows that 1 prefers not to participate for $\theta \in [\frac{1}{2}, 1)$. If $\theta \in (0, \frac{1}{2})$, then 3's expected payoff increase from staying out is

$$u_3(x_2) - \frac{1}{3}(1 + u_3(x_2)) = \frac{1}{3}(2u_3(x_2) - 1) > \frac{1}{3}(2(1-\theta) - 1) \geq 0$$

This shows that 3 prefers not to participate for $\theta \in (0, \frac{1}{2})$. Therefore, in all the cases, $\mathbf{s}$ is not an entry equilibrium. This proves the non-existence of a three-candidate equilibrium in the full game. □

Theorem 4.4. *There exists a limit equilibrium* $(\mathbf{s}, \boldsymbol{\pi})$*, where* $\mathbf{s} = (I, I, I)$*, and* $\pi_1 = \pi_3 < \pi_2$*, if and only if* $\lambda(t_1) = \lambda(t_3) < \lambda(t_2)$*.*

Proof. We first prove the necessary part. Suppose that $\boldsymbol{\pi}$ is a limit voting equilibrium satisfying $\pi_1 = \pi_3 < \pi_2$. Then Proposition 3.5 implies that a type-t_1 voter votes for either 1 or 3 as $n \to \infty$, but voting for 3 is a weakly dominated action, so she votes for 1. The same argument applies to a type-t_3 voter. A type-t_2 voter always votes for 2. This shows that $\pi_k = \lambda(t_k), \forall k$, so $\lambda(t_1) = \lambda(t_3) < \lambda(t_2)$.

Next, we prove the sufficient part. Suppose that $\lambda(t_1) = \lambda(t_3) < \lambda(t_2)$. Let $(\boldsymbol{\pi}^n)$ be a sequence generated by the voting behavior $\sigma_{t_k}(a_k) = 1, \forall k, \forall n$. That is, in each Γ^n, voting is sincere. It is clear that $\pi_k^n = \lambda(t_k),\ \forall n$, so the limit $\pi_k = \lambda(t_k)$, which implies that $\pi_1 = \pi_3 < \pi_2$. We claim that $\boldsymbol{\pi}$ is a limit equilibrium. To see this, simply note that if all other voters follow $\boldsymbol{\pi}^n$, then a type-t_k voter maximizes her expected payoff by voting for k as $n \to \infty$. This shows that all the probabilities of voting for k come from types that are maximizing, so $\boldsymbol{\pi}$ is a limit equilibrium. To see that $\mathbf{s} = (I, I, I)$ is an entry equilibrium given $\boldsymbol{\pi}$, note that when all three candidates participate, the winner is 2. If either 1 or 3 does not participate, voters vote sincerely in a two-candidate election, in which case 2 gets the extra votes from supporters of 1 or 3, and remains to be the winner. Therefore, 1 and 3 are indifferent between participating and not participating. 2 obviously has no incentive to withdraw. This shows that $\mathbf{s}$ is an entry equilibrium, so $(\mathbf{s}, \boldsymbol{\pi})$ is an equilibrium in the full game. □

5 Conclusion

In this paper, we study the implication of strategic candidacy on Duverger's law. Combining this with the waste vote argument in a framework with uncertain voter population, we have explained both generic and non-generic cases in Palfrey (1989), as well as the persistence of three parties in India.

For a more complete study, we would allow voters to have weak preferences, as well as the possibility of abstention. Since indifferent voters can vote in any way they want or abstain, the model will naturally have multiple equilibria, which begs the question how compelling the Duvergerian equilibrium is, and which equilibrium refinement is proper, if necessary, to obtain Duverger's law.

In our model, we assume that there are only three candidates. Real life elections, however, often involve more than three candidates competing for one winning position. A typical problem then involves multiple equilibria in the voting subgame. For example, suppose that $K = \{1, 2, 3, 4\}$, and $\pi_1 = \pi_2 = \pi_4 < \pi_3$, $\pi_1 + \pi_2 > \pi_3$. It is tempting to think that 1 prefers to withdraw in order to let 2 win over 3. However, this is true only if 1 expects his supporters to vote sincerely when he does not participate. If instead all votes are concentrated on 3 and 4, which is nevertheless a voting equilibrium, then 1 is indifferent between in and out of the competition. To address this problem, we may resort to stronger solution concepts such as coalition-proof equilibrium in the entry stage. Thus, in the above example, we would expect 1 and 2, 3 and 4 form coalitions, instead of standing as individual competitors, and the result is again a two-party outcome. An interesting example to look at is three Australian federal elections in 1901, 1903 and 1906 under plurality voting, the winner of which was Protectionist party and Labor coalition.

Another natural extension is to let policies be a strategic variable for candidates. Palfrey (1984), Callander and Wilson (2007) study models where two incumbent candidates first choose a policy platform, followed by plurality voting. They obtain equilibria in which a third party entry is deterred. However, both papers assume sincere voting. It would be interesting to examine Duverger's law under such circumstances while voting is strategic.

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TABLES

	Pivot events $\{\mathbf{m}^n\}$	Outcomes for $\mathbf{m}^n + a_i$	Outcomes for $\mathbf{m}^n + a_j$
i.	$i = j \geq h$	i	j
ii.	$i = h = j + 1$	i	$\frac{1}{3}i + \frac{1}{3}j + \frac{1}{3}h$
iii.	$i = h > j + 1$	i	$\frac{1}{2}i + \frac{1}{2}h$
iv.	$j = h = i + 1$	$\frac{1}{3}i + \frac{1}{3}j + \frac{1}{3}h$	j
v.	$j = h > i + 1$	$\frac{1}{2}j + \frac{1}{2}h$	j
vi.	$i = j + 1 \geq h + 1$	i	$\frac{1}{2}i + \frac{1}{2}j$
vii.	$j = i + 1 \geq h + 1$	$\frac{1}{2}i + \frac{1}{2}j$	j
viii.	$h = i + 1 = j + 1$	$\frac{1}{2}i + \frac{1}{2}h$	$\frac{1}{2}j + \frac{1}{2}h$
ix.	$h = i + 1 > j + 1$	$\frac{1}{2}i + \frac{1}{2}h$	h
x.	$h = j + 1 > i + 1$	h	$\frac{1}{2}j + \frac{1}{2}h$

Note: The following examples explain the notations in the table.

1. $i = j \geq h$ is an abbreviation for $\{\mathbf{m}^n : m^n(i) = m^n(j) \geq m^n(h)\}$.
2. $\frac{1}{2}i + \frac{1}{2}j$ refers to the outcome that i and j each wins with probability $\frac{1}{2}$.

Table 1: The set of all ij-sensible events Ω_{ij}.

6 Appendix

1. Proof of Lemma 3.3.

Proof. Lemma 3.2 implies that the magnitude of M is

$$\mu_M = \lim_{n\to\infty} \max_{\mathbf{m}^n \in M} \sum_{k\in K} \pi_k^n \psi\left(\frac{m^n(k)}{n\pi_k^n}\right).$$

So our first step is to solve

$$\max_{\mathbf{m}^n \in M} \sum_{k\in K} \pi_k^n \psi(\alpha_k^n),$$

where $\alpha_k^n = m^n(k)/(n\pi_k^n)$, and write $\alpha^n = (\alpha_k^n)_{k\in K}$. We form the Lagrange function and rewrite the maximization problem as

$$\begin{aligned}
\max_{\alpha^n,\lambda_1,\lambda_2} \quad & L = \textstyle\sum_{k\in K} \pi_k^n \psi(\alpha_k^n) + \lambda_1 g_1 + \lambda_2 g_2 \\
\text{s.t.} \quad & g_1 = \alpha_i^n \pi_i^n - \alpha_j^n \pi_j^n - \tfrac{z_1}{n} = 0, \\
& g_2 = \alpha_i^n \pi_i^n - \alpha_h^n \pi_h^n - \tfrac{z_2}{n} \geq 0.
\end{aligned}$$

The Kuhn-Tucker conditions are

$$\begin{aligned}
&\frac{\partial L}{\partial \alpha_i^n} = -\pi_i^n \ln(\alpha_i^n) + (\lambda_1 + \lambda_2)\pi_i^n = 0, \\
&\frac{\partial L}{\partial \alpha_j^n} = -\pi_j^n \ln(\alpha_j^n) - \lambda_1 \pi_j^n = 0, \\
&\frac{\partial L}{\partial \alpha_h^n} = -\pi_h^n \ln(\alpha_h^n) - \lambda_2 \pi_h^n = 0, \\
&\alpha_i^n \pi_i^n - \alpha_j^n \pi_j^n - \frac{z_1}{n} = 0, \\
&\alpha_i^n \pi_i^n - \alpha_h^n \pi_h^n - \frac{z_2}{n} \geq 0,\ \lambda_2 \geq 0,\ \lambda_2 \left(\alpha_i^n \pi_i^n - \alpha_h^n \pi_h^n - \frac{z_2}{n}\right) = 0.
\end{aligned}$$

Note that the first three conditions yield

$$\alpha_i^{n*} = e^{\lambda_1+\lambda_2},\ \alpha_j^{n*} = e^{-\lambda_1},\ \alpha_h^{n*} = e^{-\lambda_2}.$$

Substitute these optimal values into the Lagrange function to get

$$L(\alpha^{n*}) = e^{\lambda_1+\lambda_2}\pi_i^n + e^{-\lambda_1}\pi_j^n + e^{-\lambda_2}\pi_h^n - 1 - \lambda_1 \frac{z_1}{n} - \lambda_2 \frac{z_2}{n}.$$

For the last condition, we have two cases to consider.

(i) $g_2 > 0$. Then $\lambda_2 = 0$. So $g_1 = e^{\lambda_1}\pi_i^n + e^{-\lambda_1}\pi_j^n - \frac{z_1}{n} = 0$. This yields

$$e^{\lambda_1} = \frac{\frac{z_1}{n} + \sqrt{\left(\frac{z_1}{n}\right)^2 + 4\pi_i^n\pi_j^n}}{2\pi_i^n},\ e^{\lambda_2} = 1.$$

Substitute these values into $g_2 > 0$, we get

$$\pi_i^n\pi_j^n - (\pi_h^n)^2 > \frac{2z_2 - z_1}{n}\pi_h^n + \frac{z_2(z_2 - z_1)}{n^2}.$$

Since $\boldsymbol{\pi}^n \to \boldsymbol{\pi}$, the above inequality must hold for n sufficiently large if $\pi_i\pi_j > (\pi_h)^2$ is satisfied. Therefore,

$$\mu_M = \lim_{n\to\infty} L(\alpha^{n*}, \lambda^*) = \pi_h + 2\sqrt{\pi_i\pi_j} - 1,\ \text{if}\ \pi_i\pi_j > (\pi_h)^2,$$

and no solution otherwise.

(ii) $g_2 = 0$. We solve

$$g_1 = e^{\lambda_1+\lambda_2}\pi_i^n - e^{-\lambda_1}\pi_j^n - \frac{z_1}{n} = 0,$$
$$g_2 = e^{\lambda_1+\lambda_2}\pi_i^n - e^{-\lambda_2}\pi_h^n - \frac{z_2}{n} = 0,$$

and get

$$\lim_{n\to\infty} e^{\lambda_1} = \sqrt[3]{\frac{(\pi_j)^2}{\pi_i\pi_h}},\ \lim_{n\to\infty} e^{\lambda_2} = \sqrt[3]{\frac{(\pi_h)^2}{\pi_i\pi_j}}.$$

$\lambda_2 \geq 0$ requires that $(\pi_h)^2 \geq \pi_i\pi_j$. Therefore,

$$\mu_M = \lim_{n\to\infty} L(\alpha^{n*}, \lambda^*) = 3\sqrt[3]{\pi_h\pi_i\pi_j} - 1,\ \text{if}\ (\pi_h)^2 \geq \pi_i\pi_j,$$

and no solution otherwise.

Combining the two cases completes the proof. □

2. Proof of Proposition 3.4.

Proof. Table 1 lists all pivot events that are ij-sensible. We first compute the magnitude of each event using Lemma 3.3.

1. i., vi., vii. have the same magnitude

$$\begin{cases} \pi_h + 2\sqrt{\pi_i\pi_j} - 1 & \text{if } \pi_i\pi_j > (\pi_h)^2 \\ 3\sqrt[3]{\pi_i\pi_j\pi_h} - 1 & \text{if } \pi_i\pi_j \leq (\pi_h)^2 \end{cases}$$

2. ii. $\cup$ iii. and viii. $\cup$ ix. have the same magnitude

$$\begin{cases} \pi_j + 2\sqrt{\pi_i \pi_h} - 1 & \text{if } \pi_i \pi_h > (\pi_j)^2 \\ 3\sqrt[3]{\pi_i \pi_j \pi_h} - 1 & \text{if } \pi_i \pi_h \leq (\pi_j)^2 \end{cases}$$

3. iv. has the magnitude $3\sqrt[3]{\pi_i \pi_j \pi_h} - 1$.

4. v. and x. have the same magnitude $\pi_i + 2\sqrt{\pi_j \pi_h} - 1$, since $(\pi_i)^2 < \pi_j \pi_h$.

We are left to show that

$$\pi_i + 2\sqrt{\pi_j \pi_h} - 1 > \max\{\pi_h + 2\sqrt{\pi_i \pi_j} - 1,\ \pi_j + 2\sqrt{\pi_i \pi_h} - 1,\ 3\sqrt[3]{\pi_i \pi_j \pi_h} - 1\}.$$

Note that $\pi_i < \min\{\pi_j, \pi_h\}$ and $\pi_i \pi_j > (\pi_h)^2$ imply that $\pi_j > \pi_h$. So

$$\pi_i + 2\sqrt{\pi_j \pi_h} - 1 - \left(\pi_h + 2\sqrt{\pi_i \pi_j} - 1\right) = \left(\sqrt{\pi_i} - \sqrt{\pi_h}\right)\left(\sqrt{\pi_i} + \sqrt{\pi_h} - 2\sqrt{\pi_j}\right) > 0.$$

This shows that $\pi_i + 2\sqrt{\pi_j \pi_h} - 1 > \pi_h + 2\sqrt{\pi_i \pi_j} - 1$. By symmetry of π_j and π_h, we have $\pi_i + 2\sqrt{\pi_j \pi_h} - 1 > \pi_j + 2\sqrt{\pi_i \pi_h} - 1$. Finally, $\pi_i + 2\sqrt{\pi_j \pi_h} - 1 > 3\sqrt[3]{\pi_i \pi_j \pi_h} - 1$ is proved in Corollary 6.2 after Lemma 6.1. This completes the proof of Proposition 3.4. □

Lemma 6.1. *Suppose that x, y, z satisfy the following conditions:* 1. $0 \leq x < \min\{y, z\} \leq 1$; 2. $x + y + z = 1$; 3. $xz \leq y^2$; 4. $\sqrt[3]{xyz} \leq \frac{1}{3}$. *Then $x + 2\sqrt{yz} > 3\sqrt[3]{xyz}$.*

Proof. First note that if $x = 0$, then $x + 2\sqrt{yz} > 3\sqrt[3]{xyz}$ holds trivially, so it suffices to consider $x > 0$. We define a function $f(x, y, z) = x + 2\sqrt{yz} - 3\sqrt[3]{xyz}$ and show that $f(x, y, z) > 0$, for x, y, z that satisfy conditions 1-4. Note that $x < \min\{y, z\}$ is equivalent to $x < \frac{1}{3}$. Our approach to the proof is to show that f attains a minimum at $x = \frac{1}{3}$ when we replace condition 1 with condition 1′: $0 < x \leq \min\{y, z\} \leq 1$.

The minimization problem is as follows.

$$\begin{aligned} \min_{x,y,z \in \mathbb{R}_{++}} \quad & f(x, y, z) \\ \text{s.t.} \quad & g_1 = 1 - x - y - z = 0, \\ & g_2 = \frac{1}{3} - x \geq 0, \\ & g_3 = y^2 - xz \geq 0, \\ & g_4 = \frac{1}{27} - xyz \geq 0. \end{aligned}$$

We form the Lagrange function, and rewrite the problem as

$$\max_{x,y,z\in\mathbb{R}_{++},\lambda_i} L = -f(x,y,z) + \sum_{i=1}^{4} \lambda_i g_i.$$

The Kuhn-Tucker conditions are

$$\begin{aligned}
\frac{\partial L}{\partial x} &= -1 + \sqrt[3]{\frac{yz}{x^2}} - \lambda_1 - \lambda_2 - z\lambda_3 - yz\lambda_4 = 0,\\
\frac{\partial L}{\partial y} &= -\sqrt{\frac{z}{y}}\left(1 - \sqrt[6]{\frac{x^2}{yz}}\right) - \lambda_1 + 2y\lambda_3 - xz\lambda_4 = 0,\\
\frac{\partial L}{\partial z} &= -\sqrt{\frac{y}{z}}\left(1 - \sqrt[6]{\frac{x^2}{yz}}\right) - \lambda_1 - x\lambda_3 - xy\lambda_4 = 0,
\end{aligned}$$

$$\begin{aligned}
&1 - x - y - z = 0,\\
&\frac{1}{3} - x \geq 0,\ \lambda_2 \geq 0,\ \lambda_2\left(\frac{1}{3} - x\right) = 0,\\
&y^2 - xz \geq 0,\ \lambda_3 \geq 0,\ \lambda_3\left(y^2 - xz\right) = 0,\\
&\frac{1}{27} - xyz \geq 0,\ \lambda_4 \geq 0,\ \lambda_4\left(\frac{1}{27} - xyz\right) = 0.
\end{aligned}$$

We solve the problem by considering whether constraints g_2, g_3, g_4 are binding.

(i) $x = \frac{1}{3}$. Since $x \leq \min\{y, z\}$, we have $x = y = z = \frac{1}{3}$, and $f = 0$.

(ii) $x < \frac{1}{3}$. This implies that $\lambda_2 = 0$. We have four situations.

a. Only g_3 is binding. Then $y^2 = xz,\ \lambda_3 \geq 0$, and $xyz < \frac{1}{27},\ \lambda_4 = 0$. Solve $y^2 = xz$ and $x + y + z = 1$, we get

$$y = \frac{-z + \sqrt{4z - 3z^2}}{2}.$$

Notice that $f = x + 2\sqrt{yz} - 3\sqrt[3]{xyz} = 1 - 4y - z + 2\sqrt{yz}$, so

$$\frac{df}{dz} = -1 + \sqrt{\frac{y}{z}} + \left(\sqrt{\frac{z}{y}} - 4\right)\frac{dy}{dz},$$

where

$$\frac{dy}{dz} = -\frac{1}{2}\left(1 - \frac{2 - 3z}{\sqrt{4z - 3z^2}}\right).$$

For $z \in [\frac{1}{3}, 1)$, $\frac{df}{dz} = 0$ only if $z = \frac{1}{3}$ and $z \approx .9014$. But $f(.9014) > 0 = f(\frac{1}{3})$, so f attains the minimum at $z = \frac{1}{3}$. But this implies that $x = \frac{1}{3}$, a contradiction to $x < \frac{1}{3}$. Therefore, there is no minimum in this case.

b. Only g_4 is binding. Then $xyz = \frac{1}{27},\ \lambda_4 \geq 0$, and $xz < y^2,\ \lambda_3 = 0$. The Kuhn-Tucker conditions are reduced to

$$-1 + \sqrt[3]{\frac{yz}{x^2}} - \lambda_1 - yz\lambda_4 = 0,$$

$$7 - \sqrt{\frac{z}{y}}\left(1 - \sqrt[6]{\frac{x^2}{yz}}\right) - \lambda_1 - xz\lambda_4 = 0,$$

$$-\sqrt{\frac{y}{z}}\left(1 - \sqrt[6]{\frac{x^2}{yz}}\right) - \lambda_1 - xy\lambda_4 = 0.$$

So we have

$$\frac{z}{y} = \frac{\lambda_1 + xz\lambda_4}{\lambda_1 + xy\lambda_4}.$$

This equation holds only if $y = z$ or $\lambda_1 = 0$. If $y = z$, then

$$xyz = (1 - 2y)y^2 = \frac{1}{27} \Rightarrow y = \frac{1}{3} = z = x,$$

which is a contradiction to $x < \frac{1}{3}$. If $\lambda_1 = 0$, then

$$-27x + 9 = \lambda_4, -\sqrt{27x} + 9x = x\lambda_4 \ \Rightarrow x = \frac{1}{3},$$

which again contradicts $x < \frac{1}{3}$. So there is no minimum in this case.

c. Both g_3 and g_4 are binding. Then $y^2 = xz,\ \lambda_3 \geq 0$, and $xyz = \frac{1}{27},\ \lambda_4 \geq 0$. The two equalities together imply that $x = y = z = \frac{1}{3}$, a contradiction to $x < \frac{1}{3}$. So there is no minimum in this case.

d. Neither of g_3, g_4 is binding. Then $xz < y^2,\ \lambda_3 = 0$, and $7xyz < \frac{1}{27},\ \lambda_4 = 0$. The Kuhn-Tucker conditions are reduced to

$$-1 + \sqrt[3]{\frac{yz}{x^2}} - \lambda_1 = 0,$$

$$-\sqrt{\frac{z}{y}}\left(1 - \sqrt[6]{\frac{x^2}{yz}}\right) - \lambda_1 = 0,$$

$$-\sqrt{\frac{y}{z}}\left(1 - \sqrt[6]{\frac{x^2}{yz}}\right) - \lambda_1 = 0.$$

So we have $y = z$ or $x^2 = yz$. If $y = z$, then $\sqrt[3]{\frac{y^2}{x^2}} = \sqrt[6]{\frac{x^2}{y^2}} \Rightarrow x = y = z = \frac{1}{3}$. If $x^2 = yz$, then again we must have $x = y = z = \frac{1}{3}$. Both cases lead to a contradiction.

So f attains a unique minimum at $x = y = z = \frac{1}{3}$ for x, y, z satisfying conditions $1'$ and 2-4. This implies that $f > 0$, or $x + 2\sqrt{yz} > 3\sqrt[3]{xyz}$, for x, y, z that satisfy conditions 1-4 in Lemma 6.1. □

Corollary 6.2. *Suppose that* $\pi_i < \min\{\pi_j, \pi_h\}$*, and* $\pi_i\pi_h \leq (\pi_j)^2$ *or* $\pi_i\pi_j \leq (\pi_h)^2$*. Then* $\pi_i + 2\sqrt{\pi_j\pi_h} - 1 > 3\sqrt[3]{\pi_i\pi_j\pi_h} - 1$.

Proof. Let $\pi_i = x, \{\pi_j, \pi_h\} = \{y, z\}$. Since π_i, π_j, π_h are probabilities on the set of candidates $K = \{1, 2, 3\}$, they satisfy the first two conditions in Lemma 6.1 (where $\pi_i < \min\{\pi_j, \pi_h\}$ is by assumption). They also satisfy condition 3 in Lemma 6.1 by assumption. Finally, note that by definition, the magnitude of any event is non-positive. So $3\sqrt[3]{\pi_i\pi_j\pi_h} - 1 \leq 0$, which implies that $\sqrt[3]{\pi_i\pi_j\pi_h} - 1 \leq \frac{1}{3}$. Therefore, applying Lemma 6.1, we have $\pi_i + 2\sqrt{\pi_j\pi_h} > 3\sqrt[3]{\pi_i\pi_j\pi_h}$. □

3. Proof of Lemma 4.1.

Proof. We apply the same technique used to prove Lemma 3.3. We first calculate the magnitude of M by solving the following maximization problem.

$$\begin{aligned} \max_{\alpha^n} \quad & \pi_i^n \psi(\alpha_i^n) + \pi_j^n \psi(\alpha_j^n) \\ \text{s.t.} \quad & g = \pi_j^n \alpha_j^n - \pi_i^n \alpha_i^n \geq 0. \end{aligned}$$

We form the Lagrange function

$$\max_{\alpha^n, \lambda} L = \pi_i^n \psi(\alpha_i^n) + \pi_j^n \psi(\alpha_j^n) + \lambda \left(\pi_j^n \alpha_j^n - \pi_i^n \alpha_i^n \right)$$

and solve first order conditions to get $\alpha_i^{n*} = e^{-\lambda}$, $\alpha_j^{n*} = e^{\lambda}$, and $L(\alpha^{n*}) = e^{-\lambda}\pi_i^n + e^{\lambda}\pi_j^n - 1$. The Kuhn-Tucker conditions require that $g \geq 0$, $\lambda \geq 0$, and $\lambda g = 0$.

(i) $g > 0$. Then $\lambda = 0$. This implies that $\alpha_i^{n*} = \alpha_j^{n*} = 1$. So $\mu_M = 0$ for $\pi_j > \pi_i$.

(ii) $g = 0$. Then $\lim_{n \to \infty} e^{\lambda} = \sqrt{\frac{\pi_i}{\pi_j}}$. $\lambda \geq 0$ only when $\pi_i = \pi_j$, in which case $\mu_M = 0$.

To calculate the magnitude of M', we solve the following maximization problem.

$$\begin{aligned} \max_{\alpha_n} \quad & \pi_i^n \psi(\alpha_i^n) + \pi_j^n \psi(\alpha_j^n) \\ \text{s.t.} \quad & g = \pi_i^n \alpha_i^n - \pi_j^n \alpha_j^n - \frac{1}{n} \geq 0. \end{aligned}$$

We have $\alpha_i^{n*} = e^{\lambda}$, $\alpha_j^{n*} = e^{-\lambda}$, and $L(\alpha^{n*}) = e^{\lambda}\pi_i^n + e^{-\lambda}\pi_j^n - 1 - \frac{\lambda}{n}$. If $g > 0$, then $\lambda = 0$, so $\alpha_i^{n*} = \alpha_j^{n*} = 1$. But this implies that $x^n(i) - x^n(j) = n(\pi_i^n - \pi_j^n) \leq 0$, a contradiction to $g > 0$. So we must have $g = 0$. FOC yields $\lim_{n\to\infty} e^{\lambda} = \sqrt{\frac{\pi_j}{\pi_i}}$. So $\mu_{M'} = 2\sqrt{\pi_i\pi_j} - 1 < 0$ since the maximum of the LHS is attained at $\pi_i = \pi_j$. This shows that $\mu_M > \mu_{M'}$.

Finally, $\mu_M > \mu_{M'}$ implies $\lim_{n\to\infty} \mathbb{P}(M')/\mathbb{P}(M \cup M') = 0$. Since $\mathbb{P}(M \cup M') = 1$, and M, M' are mutually exclusive events, we have $\lim_{n\to\infty} \mathbb{P}(M') = 0$, $\lim_{n\to\infty} \mathbb{P}(M) = 1$. □

4. Proof of Lemma 4.2

Proof. We list out all pivot events as follows:

$$\begin{aligned} &i = j = h,\ i = j > h,\ i = h > j,\ j = h > i, \\ &i > \max\{j, h\},\ j > \max\{i, h\},\ h > \max\{i, j\}. \end{aligned}$$

The magnitudes of the first four events can be calculated using Lemma 3.3, which turn out to be all smaller than 0.

In general, to calculate the magnitude of the event $i > \max\{j, h\}$, we need to solve the following problem.

$$\begin{aligned} \max_{\alpha^n} \quad & \sum_{s\in S} \pi_s^n \psi(\alpha_s^n) + \lambda_1 g_1 + \lambda_2 g_2 \\ \text{s.t.} \quad & g_1 = \alpha_i^n \pi_i^n - \alpha_j^n \pi_j^n - \frac{1}{n} \geq 0, \\ & g_2 = \alpha_i^n \pi_i^n - \alpha_h^n \pi_h^n - \frac{1}{n} \geq 0. \end{aligned}$$

(i) $g_1 = 0$, $g_2 = 0$. We have

$$\lim_{n\to\infty} e^{\lambda_1} = \sqrt[3]{\frac{\pi_j^2}{\pi_i\pi_h}},\quad \lim_{n\to\infty} e^{\lambda_2} = \sqrt[3]{\frac{\pi_h^2}{\pi_i\pi_j}}.$$

So $\mu = 3\sqrt[3]{\pi_i\pi_j\pi_h} - 1$ if $\pi_j^2 \geq \pi_i\pi_h$ and $\pi_h^2 \geq \pi_i\pi_j$.

(ii) $g_1 > 0$, $g_2 = 0$. We have

$$\lim_{n\to\infty} e^{\lambda_1} = 1,\quad \lim_{n\to\infty} e^{\lambda_2} = \sqrt{\frac{\pi_h}{\pi_i}}.$$

So $\mu = \pi_j + 2\sqrt{\pi_i\pi_h} - 1$ if $\pi_h \geq \pi_i$ and $\pi_i\pi_h > \pi_j^2$.

(iii) $g_1 = 0$, $g_2 > 0$. We have $\mu = \pi_h + 2\sqrt{\pi_i \pi_j} - 1$ if $\pi_j \geq \pi_i$, and $\pi_i \pi_j > \pi_h^2$.

(iv) $g_1 > 0, g_2 > 0$. We have $\mu = 0$ for $\pi_i > \pi_j$ and $\pi_i > \pi_h$.

If $\pi_i = \pi_j < \pi_h$, only the second case has a solution. So the magnitude of $i > \max\{j, h\}$ and $j > \max\{i, h\}$ is $\pi_j + 2\sqrt{\pi_i \pi_h} - 1 < 0$. Finally, it is straightforward to check that the magnitude of $h > \max\{i, j\}$ is 0, the largest among all events. The limit probability can be verified following the same argument as in the proof of Lemma 4.1. □