Optimal auction with a general distribution: virtual valuation without densities⋆

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Abstract

We characterize the optimal auction in an independent private values framework for a completely general distribution of valuations. To do this we introduce a new concept: the generalized convex hull. The "derivative" of the generalized convex hull with respect to the distribution of types is the generalized virtual valuation. We present two examples showing how to use the generalized virtual valuation to extend the classical models of Mussa and Rosen and Baron and Myerson for arbitrary distributions.
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This paper is dedicated to the late Prof. Isnard.

1 Introduction

In this paper we consider optimal auctions in an independent private values model. Our first objective is to characterize the optimal auction for general distributions. That is, distributions that may have jumps, that are not absolutely continuous. The classical paper, Myerson (1981), proves existence and characterize the optimal auction if the distribution of types/valuations of each bidder i has a density that is

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continuous, strictly positive on its support, the interval $[a_i, b_i]$. Let us call this condition [2] (M). This condition is also used in Riley and Samuelson (1981). They consider a symmetrical model and look for an optimal mechanism amongst those symmetrical ones that gives the object to the bidder with the highest type. They conclude that the usual auctions (like a first-price sealed bid auction) are optimal if a reserve price is properly chosen. Myerson's paper reaches the most general result under condition (M). He shows that the optimal auction is a Vickrey's type auction: what you pay when you win does not depend on your type but only on the other's bidders types. In Myerson's paper a key role is played by the virtual valuation of bidder i given by $J_i(x_i) = x_i - \frac{1-F_i(x_i)}{f_i(x_i)}$. If the virtual valuation is increasing, the object is delivered to bidder i if $J_i(x_i) \geq 0$ and $J_i(x_i) > J_j(x_j)$ for all bidders $j \neq i$. If the virtual valuation is not increasing, Myerson obtains a surrogate, $\overline{J}_i$, which is increasing and replaces J_i in the allocation rule.

A distribution that satisfies condition (M) is continuous and strictly increasing. However it is not difficult – indeed it is quite common – to find non-pathological meaningful examples of valuations that do not satisfy (M). Let us mention a few simple examples:

a) Suppose that the population of an economy is divided in two groups of equal size. In one group the distribution of types is uniform $[0,1]$. In the other group the distribution of types is uniform in $[2,3]$. In this economy the distribution of types has support $[0,1] \cup [2,3]$. The density the distribution in this case is null in $(1,2)$;

b) Suppose again that the population is divided in two groups. The highest preference group has utility $u = 1$ and is of size $1/2$. The other group has preferences $u \in [0,1)$ with an uniform distribution. Thus the distribution of valuation u has a jump of size $1/2$ at $u = 1$. Alternatively we might think that the types $u \in [0,2]$ is uniformly distributed. But the object at sale gives utility at most 1. Thus the valuation $V_i(u) = \min\{u, 1\}$. In this case the distribution has a jump at valuation $V_i = 1$ as well.

c) In Bose and Daripa (2008) they need to find the optimal auction for a distribution that has a density in an interval $[\underline{v}, \overline{v}]$ is constant in $[\overline{v}, v_h)$ and then has a jump at v_h.

d) On a more conceptual streak, the set of types may be specified by the realization of a family of random variables (i.e. the bidder information set) and therefore the set of types will have many gaps. Whenever there is a gap the density must be null at those gaps (as in (a) above).

It is a growing concern about what we can say for a general set of types. For example Che and Gale (2006) consider revenue equivalence for arbitrary set of types. Suppose the set of types is not an interval. Is there an optimal auction? If there

[2] That is a distribution G satisfies the condition (M) if G has a continuous strictly positive density $g : [a,b] \to (0,\infty)$, $\int_a^b g(u)\,du = 1$.

is one, can we characterize the optimal auction? Since bidders and the seller only care about expected utilities, it is quite intuitive that what matters is the distribution. Thus two sets of types that generate the same distribution are equivalent. That is, for optimality[3] the concern for arbitrary types translates into a concern about arbitrary distributions.

It goes without saying that the more general a result the harder is to characterize the solution. Thus Myerson (1981) was fortunate in proving existence through characterizing the optimal auction. On the other hand the more general optimal auction existence result is Page (1998). However due to its generality it is quite difficult to be more explicit about the properties of the optimal mechanism. In this paper we want to characterize the optimal auction for a general distribution. It would be specially nice if this characterization is similar to Myerson's virtual valuation characterization. This is what we do and we will need a new concept: the generalized convex hull. To motivate our solution recall how the virtual valuation $\overline{J}_i$ of a given bidder i is constructed. First an auxiliary function is defined: $h(q) = J_i\left(F_i^{-1}(q)\right)$, $0 \le q \le 1$. Then the integral $H(q) = \int_0^q h(r)\,dr$. If J_i is increasing this function is convex. In general we consider $G : [0,1] \to \mathbb{R}$ the convex hull of H. Define $g(q) = G'_+(q)$ the right-hand derivative of G. This always exists. Finally $\overline{J}_i(x_i) = g(F_i(x_i))$ is the surrogate virtual valuation. For a general distribution it will not do however to mimic Myerson's steps. The first difficulty begins with $F_i^{-1}(q)$ which is not well defined if F_i is not strictly increasing. And if F_i has jumps the inverse image $F_i^{-1}(q)$ may be empty. Another difficulty is that when we take the derivative of the convex hull it is with respect to Lebesgue's measure and this may not be well tailored to the discontinuities of F_i. It is necessary to be judicious. How to do this is the second objective of our paper and our main technical contribution. The convex hull of H being a convex function, is the supremum of the family of linear function $q \to \alpha + \beta q$ that are below H. The generalized convex hull will not, in general, be a convex function. But it will be defined as the supremum of the family of functions[4] $q \to \alpha + \beta F(q)$ such that $\alpha + \beta F(\cdot) \le H(\cdot)$. Intuitively this is the proper analogue of the convex hull for a general distribution. We will show that the generalized convex hull, G_{gen} so obtained can be written in the form $G_{\text{gen}}(x) = \int_a^x l(u)\,dF(u)$ and $l(\cdot)$ being an increasing function. It is in this sense that we say that the derivative of G_{gen} is $l(\cdot)$. This derivative will be the generalized virtual valuation. This function $l(\cdot)$ is a general ironing for the one-dimensional case. In Mussa and Rosen (1978) there is perhaps the first example of ironing in the literature. They have a function $G(\theta)$ that is at first increasing, then decreases and finally is increasing again. They call it a "wiggle." This $G(\theta)$ correspond to the virtual valuation. The key restriction is to obtain an increasing $\tilde{G}$ in a way that preserves the average marginal revenue. This is accomplished by making

[3] Naturally this is not true for revenue equivalence. For a general analysis of revenue equivalence in arbitrary type spaces see K. Chung and W. Olszewski (2006).
[4] We might say the family of F linear functions.

$G(\theta)$ constant (say $= q$) in a certain interval (u,v) and to find u,v,q we have the equations

$$G(u) = q = G(v), \tag{1}$$

$$\int_u^v G(\theta) f(\theta) d\theta = q \int_u^v f(\theta) d\theta. \tag{2}$$

In Myerson (1981) the function $\overline{J}_i$ is the ironing of J_i. It has properties similar to (1,2). Mussa and Rosen simple approach does not handle efficiently many wrinkles (or wiggles). Myerson's approach handles any number of wrinkles (even infinitely many) under condition (M). However if there are jumps it is not clear what the ironing should be. Even existence is a problem. For example the existence of points u and v is not guaranteed since continuity is not assured due to the possible jump points of the distribution. Under this terminology we might say that the generalized virtual valuation $l(\cdot)$ irons not only infinitely many wrinkles but irons also a cloth that is torn at infinitely many points (i.e. the jumps of the distribution).

The generalized virtual valuation $l(\cdot)$ has a wide applicability. We show in the last section, that we may, without difficulty, generalize the classical nonlinear pricing model (Mussa and Rosen (1978)) and the regulation of a monopolist (Baron and Myerson (1982)) for arbitrary distributions of consumer types or cost respectively.

2 Generalized virtual valuation

In this section a distribution $F : \mathbb{R} \to [0,1]$ is fixed. Define $a := a_F$ and $b := b_F$ where

$$a_F = \inf\{x; F(x) > 0\} \text{ and } b_F = \sup\{x; F(x) < 1\}. \tag{3}$$

We suppose $-\infty < a < b < \infty$. Our main purpose is to define the (generalized) virtual valuation. To do this we need to define the generalized convex hull (g.c.h. for short.) Although we might formally define the g.c.h. of an arbitrary function it will not have the properties we need. Thus we restrict the definition to a case that is sufficient for our needs. Let ν be a bounded signed measure such that $\nu([a,b]^c) = 0$. Thus the support of ν is contained in $[a,b]$. Let $H(x) = H_\nu(x) := \nu((-\infty,x])$ for $x \in (-\infty,\infty)$. Thus $H(x) = 0$ if $x < a$. And $H(x) = H(b)$ if $x \geq b$. We now define the auxiliary set

$$\Gamma = \left\{(\alpha,\beta) \in \mathbb{R}^2; \alpha + \beta F(\cdot) \leq H(\cdot)\right\}.$$

The set Γ is non-empty, closed and convex. If $(\alpha,\beta) \in \Gamma$ then $\alpha \leq 0$ and $\alpha + \beta \leq H(b)$.

Definition 1 (generalized convex hull) *The function* $\phi : \mathbb{R} \to \mathbb{R}$,

$$\phi(x) = \sup\{\alpha + \beta F(x); (\alpha,\beta) \in \Gamma\}$$

is the generalized convex hull of H.

Remark 1 *It is well know (see Rockafellar (1970) for example) that a convex function is the supremum of the set of linear functions that are below it. Moreover the supremum of a family of linear functions is a convex function. Thus the convex hull of a function H is the supremum of the set of linear functions that are below H. If we interpret $\alpha+\beta F$ as a F linear function then it is quite natural to say that ϕ being the supremum of the family of F linear functions that are below H is the generalized convex hull of H.*

It is clear that $\phi \leq H$ and it is constant in $(-\infty, a)$ and in $[b, \infty)$. Define $\Gamma(x) = \{\beta; \exists (\alpha,\beta) \in \Gamma, \alpha+\beta F(x) = \phi(x)\}$.

Definition 2 (subgradient) *The number β is a subgradient of ϕ at x if for every z,*

$$\phi(z) \geq \phi(x) + \beta(F(z) - F(x)).$$

We denote by $\partial\phi(x)$ the set of subgradients of ϕ at x. The set $\partial\phi(x)$ is the subdifferential of ϕ at x.

Thus if $\beta \in \partial\phi(x)$ then $(\phi(x) - \beta F(x), \beta) \in \Gamma$ and the supremum in the definition of ϕ is attained. Reciprocally if $(\alpha,\beta) \in \Gamma$ and $\alpha+\beta F(x) = \phi(x)$ then $\beta \in \partial\phi(x)$. Thus $\Gamma(x) = \partial\phi(x)$. It follows easily from the definition that the subdifferential of ϕ at x is a closed convex set. Let[5] $l(x) = \inf \partial\phi(x)$ and $s(x) = \sup \partial\phi(x)$. The following proposition collects elementary properties of ϕ. Define[6] $\beta^* := \sup_{x<b} \frac{H(b)-H(x)}{F(b)-F(x)}$ and $\beta_* := \inf_{z>a} \frac{H(z)-t_a}{F(z)-F(a)}$ where $t_a := \min\{H(a), 0\}$.

Proposition 1 (basic properties)

a) *If $0 < F(x) < 1$ then $\partial\phi(x) \neq \emptyset$;*
b) *$\Gamma(b) = [\beta^*, \infty)$ and $\phi(b) = H(b)$;*
c) *If $F(x) = 0$ then $\phi(x) = t_a$. In particular $\phi(a-) = t_a$;*
d) *If $F(a) = 0$ then $\Gamma(a) \neq \emptyset$ if and only if $\beta_* > -\infty$;*
e) *If $\Gamma(a) = \emptyset$ then $\lim_{x\downarrow a} s(x) = -\infty$*
f) *$l(\cdot)$ and $s(\cdot)$ are increasing.*

Proof: In the appendix.

Remark 2 *In particular note that item (a) implies that the subdifferential $\partial\phi(x)$ is non-empty for $x \in (a,b)$. Item (b) shows that $\partial\phi(b)$ is non-empty if and only if $\beta^* < \infty$. And $\partial\phi(a)$ is empty if and only if $F(a) = 0$ and $\beta_* = -\infty$.*

Remark 3 *Note that $l(x) \neq s(x)$ only on a countable set since the family of intervals $\{(l(x), s(x)); a \leq x \leq b\}$ is pairwise disjoint.*

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[5] It is convenient to define $l(x) = -\infty$ if $\partial\phi(x) = \emptyset$

[6] We allow $\beta_* = -\infty$ and $\beta^* = \infty$.

The following propositions shows that ϕ has the same continuity properties as F. The proofs are in the appendix.

Proposition 2 *ϕ is right-continuous.*

Proposition 3 *The left-limit $\phi(x-) = \lim_{y \uparrow x} \phi(y)$ exists. Moreover,*

$$s(x-)(F(x) - F(x-)) \leq \phi(x) - \phi(x-) \leq l(x)(F(x) - F(x-)).$$

Having defined the g.c.h. we now define the virtual valuation.

Definition 3 *The function $l : [a,b] \to [-\infty, \beta^*]$ is the virtual valuation of H.*

It is well know [7] that a convex function is the integral of its subgradients. The next theorem below shows that the generalized convex hull is the integral of the virtual valuation. But before we need

Proposition 4 $l(x) = \frac{\phi(x)-\phi(x-)}{F(x)-F(x-)}$ *if $F(x) - F(x-) > 0$.*

Proof: The corollary 3 shows that $s(x-) \leq \frac{\phi(x)-\phi(x-)}{F(x)-F(x-)} \leq l(x)$. Suppose $s(x-) < \gamma < l(x)$. Since $\gamma \notin \Gamma(x)$ there exists a z' such that $\phi(x) + \gamma(F(z') - F(x)) > H(z')$. Necessarily $z' < x$. Since
$H(z') \geq \phi(x-) + s(x-)(F(z') - F(x-))$ we get

$$\phi(x) - \phi(x-) + \gamma\left(F(z') - F(x)\right) > s(x-)\left(F\left(z'\right) - F(x-)\right) \geq \gamma\left(F\left(z'\right) - F(x-)\right).$$

implying that $\phi(x) - \phi(x-) > \gamma(F(x) - F(x-))$ and that $\frac{\phi(x)-\phi(x-)}{F(x)-F(x-)} > \gamma$. Since γ is arbitrary, $\frac{\phi(x)-\phi(x-)}{F(x)-F(x-)} = l(x)$. QED

Theorem 1 *It is true that $\phi(x) = \int_a^x l(u)\, dF(u), x \geq a$.*

Proof: Let $b > x > a' > a$. If $y \in (a', x)$ we have $l(a') \leq l(y) \leq l(x)$. For each integer N we divide $[l(a'), l(x)]$ in N equal length intervals, $L_1 = [c_0, c_1), \ldots, L_N = [c_{N-1}, c_N]$ where $c_N = l(x)$. Let $I_j = \left\{y \in [a', x]; l(y) \in L_j\right\}$, $1 \leq j \leq N$. Let $\varepsilon > 0$ and choose N large enough such that $\frac{l(x)-l(a')}{N} < \varepsilon$. Let $x_0 = a' \leq x_1 \leq \ldots \leq x_m = x$ be the extremes of the intervals [8] I_i. We may write

$$\phi(x) - \phi\left(a'\right) = \sum_j \left(\phi\left(x_{j+1}\right) - \phi\left(x_j\right)\right). \tag{4}$$

[7] See Corollary 24.2.1 of Rockafellar (1970).

[8] Some I_i may be empty. And some may contain only one element.

Consider the interval $\left(x_j, x_{j+1}\right), x_j < x_{j+1}$. It is contained in a unique interval $I' = I_k$ for some k. If $x_j < z < w < x_{j+1}$ then it is true that

$$\phi(w) - \phi(z) \geq l(z)(F(w) - F(z)) = \int_{(z,w]} l(z) dF(u) \geq \int_{(z,w]} (l(u) - \varepsilon) dF(u).$$

Now making $w \uparrow x_{j+1}$ and $z \downarrow x_j$ we have that

$$\phi(x_{j+1}-) - \phi(x_j) \geq \int_{(x_j, x_{j+1})} (l(u) - \varepsilon) dF(u).$$

Now from Proposition 4 we get,

$$\phi(x_{j+1}) - \phi(x_j) \geq \int_{(x_j, x_{j+1}]} (l(u) - \varepsilon) dF(u).$$

Therefore adding termwise in j we get from (4),

$$\phi(x) - \phi(a') \geq \int_{(a',x]} (l(u)) dF(u) - \varepsilon.$$

We now make a' decreases to a and get using right-continuity (Proposition 2):

$$\phi(x) - \phi(a) \geq \int_{(a,x]} l(u) dF(u) - \varepsilon.$$

Since $\phi(a) = \phi(a) - \phi(a-) = \int_{\{a\}} l(u) dF(u)$ we have $\phi(x) \geq \int_{[a,x]} l(u) dF(u) - \varepsilon$. Now make ε go to zero to get $\phi(x) \geq \int_{[a,x]} l(u) dF(u)$. Analogously we prove the other inequality. QED

Remark 4 *Since $\#\partial\phi(x) = 1$ except on a countable set it is true from the above theorem that the g.c.h. is the integral of the subgradient except that at the distribution discontinuities we do not have a choice in $\partial\phi(x)$ but to use $l(x)$.*

The next two theorems gives the key properties of the virtual valuation.

Theorem 2 *For every increasing function $Q : [a,b] \to \mathbb{R}$,*

$$\int Q(s) d\nu(s) \leq \int Q(s) l(s) dF(s).$$

Proof: Let $Q : [a,b] \to \mathbb{R}$ be increasing. Since $\int_{\mathbb{R}} d\nu = \int_{\mathbb{R}} l dF$ we may suppose without loss of generality [9] that $Q(a) = 0$. For a given integer N divide $[0, Q(b)]$ in N intervals of the same length: $H_1 = [0, \frac{Q(b)}{N})$ and so on. Let $H_N = [\frac{N-1}{N} Q(b), Q(b)]$.

[9] Just consider $Q' = Q - Q(a)$.

Define $I_n = Q^{-1}(H_n)$, $1 \leq n \leq N$. Since Q is increasing I_n is an interval. The function $Q^N = \sum_{n=1}^N \alpha_n \chi_{I_n}$, where $\alpha_n = \frac{n}{N} Q(b)$, is increasing and $|Q - Q^N|_\infty \leq \frac{Q(b)}{N}$. Defining $L_n = I_n \cup I_{n+1} \cup \ldots \cup I_N$ we may rewrite Q^N as

$$Q^N = \sum_{n=1}^{N} \alpha_n \left(\chi_{L_n} - \chi_{L_{n+1}} \right) = \sum_n (\alpha_n - \alpha_{n-1}) \chi_{L_n}.$$

Where $L_{N+1} := \emptyset$. Thus

$$\int Q^N(s) l(s) \, dF(s) = \sum_n (\alpha_n - \alpha_{n-1}) \int_{L_n} l(s) dF(s) \geq$$
$$\sum_n (\alpha_n - \alpha_{n-1}) \int_{L_n} d\nu(s) = \int Q^N(s) \, d\nu(s).$$

In the limit $N \to \infty$ we get $\int Q(s) l(s) \, dF(s) \geq \int Q(s) \, d\nu(s)$.

For the next theorem let $\sigma(l)$ be the smallest sigma-algebra that makes l measurable. That is

$$\sigma(l) = \left\{ B \in \mathcal{B}; l^{-1}(B) \in \mathcal{B} \right\}.$$

It is well known that a function is $\sigma(l)$ measurable if and only if it is a Borelean function composed with l.

Theorem 3 *If $\psi : [a,b] \to \mathbb{R}$ is measurable with respect to $\sigma(l)$ then:*

$$\int \psi(u) \, d\nu(u) = \int \psi(u) l(u) \, dF(u).$$

To prove this theorem we need some preliminary work.

Proposition 5 *(i) Suppose $\phi(x-) < H(x-)$. Then there exists a $z^* > x$ such that $s(z) = l(z) = s(x-)$ whenever $x < z < z^*$.*

(ii) If $\phi(x) < H(x)$ then there is an interval $[x, z^*)$ such that $l(z) = l(x)$ for every $z \in (x, z^*)$.

Proof: In the appendix

Proof of Theorem 3: Since $\sigma(l)$ is generated by sets of the form $\{u; l(u) > \lambda\}$, $\lambda \in \mathbb{R}$ it suffices to prove that $\int_{\lambda < l(u)} d\nu(u) = \int_{\lambda < l(u)} l(u) dF(u)$. The set $\{u; l(u) > \lambda\}$ is an interval. First case: $(x^*, b]$. If $\phi(x^*) < H(x^*)$ there is a $z^* > x^*$ such that $l(z) = l(x^*)$ if $x < z < z^*$ and this implies that $l(x^*) > \lambda$. Thus $\phi(x^*) = H(x^*)$. Now since $\phi(b) = H(b)$ the taking the difference $\phi(b) - \phi(x^*) = H(b) - H(x^*)$ implies that $\int \chi_{(x^*,b]} l(s) dF(s) = \int \chi_{(x^*,b]} d\nu(s)$. Second case: $[x^*, b]$. In this case $l(x^*) > \lambda$ and $s(x^*-) \leq \lambda$. Therefore $\phi(x^*-) = H(x^*-)$. QED

We now collect the results above in our main technical result.

Theorem 4 *There is an increasing function $l : [a,b] \to [-\infty,\infty]$ such that*

(1) For every increasing function $Q : [a,b] \to \mathbb{R}$,

$$\int Q(u)\, d\nu(u) \leq \int Q(u)\, l(u)\, dF(u);$$

(2) If $Q : [a,b] \to \mathbb{R}$ is $\sigma(l)$ measurable then

$$\int Q(u)\, d\nu(u) = \int Q(u)\, l(u)\, dF(u);$$

(3) $l(a) > -\infty$ if and only if $\inf_{z>a} \frac{\nu([a,z]) - \min\{\nu\{a\},0\}}{F(z)-F(a)} > -\infty$;
(4) $l(b) = \sup_{x<b} \frac{H(b)-H(x)}{F(b)-F(x)}$.

3 The independent private values model

An object is to be sold at an auction with I bidders. The seller is risk neutral and wants to maximize expected revenue. Bidder i has a valuation s_i with distribution $F_i : \mathbb{R} \to [0,1]$. Let $a_i = a_{F_i}$ and $b_i = b_{F_i}$ as in (3) above. We suppose [10] $-\infty < a_i < b_i < \infty$. It is clear from the definition that $F_i(a_i-) = 0$ and that $F_i(a_i + \varepsilon) > 0$ for every $\varepsilon > 0$. Also $F_i(b_i) = 1$ and $F_i(b_i - \varepsilon) < 1$ for every $\varepsilon > 0$.

Remark 5 *Note that we are not supposing that the support of F_i is $[a_i, b_i]$.*

Let $S := \Pi_{i=1}^{I} [a_i, b_i]$. The distributions $F_1, \ldots, F_n$ are independent. Each bidder knows his valuation s_i. The revelation principle simplifies the search for the optimal auction. The seller has to choose individually rational, incentive compatible direct mechanisms $(q,p) := \left(q_i, p^i\right)_{i=1}^{I}$. We have that:

(1) $q = (q_1, \ldots, q_n) : S \to [0,1]^I$;
(2) $\sum_{i=1}^{I} q_i(s) \leq 1$;
(3) $p = \left(p^1, \ldots, p^n\right)$, $p^i(s) \in \mathbb{R}$.

The auction proceeds as follows:

(i) Each bidder $i, 1 \leq i \leq I$ announces privately and confidentially $s_i \in [a_i, b_i]$ to the seller. Let $s = (s_1, \ldots, s_I)$;
(ii) Bidder $i, 1 \leq i \leq I$ pays $p^i(s)$;
(iii) Bidder i receives the object with probability $q_i(s)$.

Remark 6 *The restriction in (**i**) above to $s_i \in [a_i, b_i]$ is without loss of generality.*

[10] We may easily consider $b_i = \infty$ or $a_i = -\infty$ at the price of slightly longer proofs.

Remark 7 *The payment $p^i(s)$ can be understood as the expected payment of a random payment $\tilde{P}^i(s)$. This allows the inclusion of mixed strategy equilibria as in a first-price auction with discrete distribution of types.*

Define

$$p_i(s_i) := \int p^i(s)\, dF_{-i}(s_{-i}) \text{ and } Q_i(s_i) := \int q_i(s)\, dF_{-i}(s_{-i}).$$

The direct mechanism (q,p) has to satisfy incentive compatibility (IC) and voluntary participations (VP) constraints. That is, for every $s_i, s_i' \in [a_i, b_i]$,

$$s_i Q_i(s_i) - p_i(s_i) = \int \left(s_i q_i(s) - p^i(s)\right) dF_{-i}(s_{-i}) \geq 0; \tag{VP}$$

$$s_i Q_i(s_i) - p_i(s_i) \geq s_i Q_i\left(s_i'\right) - p_i\left(s_i'\right). \tag{IC}$$

The seller's expected revenue is $R = \sum_{i=1}^n R_i$ where $R_i = E[p^i(s)]$. The proof of the next two lemmas are identical to Myerson's (1981) proof and is omitted.

Lemma 1 *If (q,p) satisfies (IC) then:*

i) *$Q_i(\cdot)$ is increasing;*
ii) *$T_i(x) := xQ_i(x) - p_i(x)$, $x \in [a_i, b_i]$ is such that $T_i(x) = \int_{a_i}^x Q_i(u)\, du + \alpha_i$.*

Thus the payment of bidder i is a function of Q_i:

$$p^i(s) = s_i q(s) - \int_{a_i}^{s_i} q(y, s_{-i})\, dy - \alpha_i \text{ and} \tag{5}$$

$$p_i(s_i) = s_i Q_i(s_i) - \int_{a_i}^{s_i} Q_i(y)\, dy - \alpha_i. \tag{6}$$

Lemma 2 *Let $R_i = \int p_i(x)\, dF_i(x)$. Then*

$$R_i = \int s_i Q_i(s_i)\, dF_i(s_i) - \int (1 - F_i(s_i))\, Q_i(s_i)\, ds_i - \alpha_i. \tag{7}$$

The voluntary participation constraint is true if and only if $\alpha_i \geq 0$. To maximize revenue we set from now on $\alpha_i = 0$. Let ν_i be the (signed) measure defined by

$$\nu_i(A) = \int_A s\, dF_i(s) - \int_A (1 - F_i(s))\, \chi_{[a_i,b_i]}(s)\, ds. \tag{8}$$

Thus we may rewrite the revenue as $R_i = \int Q_i(x)\, d\nu_i(x)$. Theorem 4 guarantees the existence of an increasing function $l_i : [a_i, b_i] \to [-\infty, b_i]$ such that for every increasing function $Q : [a_i, b_i] \to \mathbb{R}$,

$$\int Q(u) d\nu_i(u) \leq \int Q(u) l_i(u) dF_i(u). \tag{9}$$

Moreover we have equality if Q is $\sigma(l_i)$ measurable.

Let $l_0(s_0) \equiv 0$ and $q_0(s) \equiv 1 - \sum_{i=1}^{I} q_i(s)$. We are now able to characterize the optimal auction. We define the allocation rule and the payment rule.

Definition 4 *For each $s \in S$ let $H(s) = \{i \geq 1; l_i(s_i) = \max_{j \geq 0} l_j(s_j)\}$. Define*

$$\overline{q}_i(s) = \begin{cases} \frac{1}{\#H(s)} & \text{if } i \in H(s); \\ 0 & \text{if } i \notin H(s). \end{cases} \tag{10}$$

Definition 5 *The payment of bidder i is for $s \in S$ given by*

$$\overline{p}^i(s) = s_i \overline{q_i}(s) - \int_{a_i}^{s_i} \overline{q}_i(y, s_{-i})\, dy. \tag{11}$$

Theorem 5 *The optimal mechanism is $(\overline{q}, \overline{p})$ where $\overline{p}^i$ is defined by (11) and $\overline{q}(s) = (\overline{q}_1(s), \ldots, \overline{q}_I(s))$ by (10).*

Proof: Let (q, p) be an incentive compatible, voluntary participation mechanism. From (9) we have that

$$R_i = \int Q_i(s_i)\, d\nu_i(s_i) \leq \int Q_i(s_i)\, l_i(s_i)\, dF_i(s_i).$$

The expected revenue is majored by:

$$R = \sum_i \int Q_i(s_i)\, d\nu_i(s_i) \leq \sum_i \int l_i(s_i)\, Q_i(s_i)\, dF_i(s_i) =$$
$$\sum_{i=1}^{I} \int l_i(s_i)\, q_i(s)\, dF(s) \leq \int \max_{j \geq 0} l_j(s_j) dF(s).$$

Thus

$$R \leq \int \sum_{j=0}^{I} \bar{q}_j(s)\, l_j(s_j)\, dF(s) = \sum_{i=1}^{I} \int l_i(s_i)\, \overline{Q}_i(s_i)\, dF_i(s_i).$$

Thus it suffices to prove that $\overline{Q}_i(s_i)$ is increasing and that

$$\int l_i(s_i)\, \overline{Q}_i(s_i)\, dF_i(s_i) = \int \overline{Q}_i(s_i)\, d\nu_i(s_i) \tag{*}$$

Let $s_i' > s_i$. For a given s_{-i} define $\lambda := \max_{0 \leq j \neq i} l_j(s_j)$ and $h = \#\{j \geq 1; j \neq i, l_j(s_j) = \lambda\}$. Then $H(s_i', s_{-i}) \subset H(s)$ and therefore

$$\overline{q}_i(s_i', s_{-i}) \geq \overline{q}_i(s).$$

Integrating in s_{-i} we get $\overline{Q}_i(s_i') \geq \overline{Q}_i(s_i)$. Thus $\overline{Q}_i(\cdot)$ is increasing. And the equality (*) follows from the l_i measurability of $\overline{Q}_i$. QED

We finish this section with an example.

Example 1 *Suppose we have two bidders with valuations distributed as*

$$F(u) = \begin{cases} \frac{u}{3} & \text{if } 0 \le u < 1; \\ \frac{u+1}{3} & \text{if } 1 \le u \le 2. \end{cases}$$

This distribution has a jump of size $1/3$ *at* $u = 1$. *If* ν *is defined as in (8) the function* $H(x) = H_\nu(x) = -x(1 - F(x))$ *is given by*

$$H(x) = \begin{cases} \frac{x^2-3x}{3} & \text{if } 0 \le x < 1 \\ \frac{x^2-2x}{3} & \text{if } 1 \le x \le 2. \end{cases}$$

If $\beta \in \mathbb{R}$ *define* $\alpha(\beta) := \inf\{H(x) - \beta F(x); 0 \le x \le 2\}$. *Thus* $(\alpha, \beta) \in \Gamma$ *if and only if* $\alpha \le \alpha(\beta)$. *We omit the calculations that find*

$$\alpha(\beta) = \begin{cases} 0 & \text{if } \beta \le -3; \\ -\frac{(3+\beta)^2}{12} & \text{if } -3 \le \beta \le -1; \\ -\frac{2+\beta}{3} & \text{if } -1 \le \beta \le 2\left(\sqrt{2}-1\right); \\ -\frac{\beta}{3} - \frac{(2+\beta)^2}{12} & \text{if } 2\left(\sqrt{2}-1\right) \le \beta \le 2. \end{cases}$$

Thus $\phi(x) = \sup\{\alpha(\beta) + \beta F(x); -3 \le \beta \le 2\}$. *We may easily show that* $\phi(x) = \frac{2(\sqrt{2}-1)x-2}{3}$ *if* $1 \le x \le \sqrt{2}$ *and* $\phi = H$ *otherwise. The virtual valuation is given by:*

$$l(x) = \begin{cases} 2x - 3 & \text{if } 0 \le x < 1; \\ 2\left(\sqrt{2}-1\right) & \text{if } 1 \le x \le \sqrt{2}; \\ 2(x-1) & \text{if } \sqrt{2} \le x \le 2. \end{cases}$$

Let us compare the optimal revenue with the second-price revenue with reserve $r = 1$.

$$2\int_{x_1 \ge r > x_2} r\,dF(x_1)\,dF(x_2) + \int_{\min\{x_1,x_2\} \ge r} \min\{x_1, x_2\}\,dF(x_1)\,dF(x_2) =$$
$$2F(1-)(1 - F(1-)) + \int_{x_1 \ge x_2 \ge 1} x_2\,dF(x_1)\,dF(x_2) + \int_{x_2 > x_1 \ge 1} x_1\,dF(x_1)\,dF(x_2) =$$
$$\frac{4}{9} + \int_{x_2 \ge 1} (1 - F(x_2-))\,x_2\,dF(x_2) + \int_{x_1 \ge 1} (1 - F(x_1))\,x_1\,dF(x_1) =$$
$$\frac{7}{9} + \int_{x_2 > 1} \left(1 - \frac{x_2+1}{3}\right)\frac{x_2}{3}dx_2 + \int_{x_1 > 1} \left(1 - \frac{x_1+1}{3}\right)\frac{x_1}{3}dx_1 =$$
$$\frac{7}{9} + \frac{2}{3}\int_1^2 \left(z - \frac{z^2+z}{3}\right)dz = \frac{7}{9} + \frac{2}{3}\int_1^2 \frac{2z - z^2}{3}dz = \frac{7}{9} + \frac{2}{9}\left(z^2 - \frac{z^3}{3}\right)\Big|_1^2 = \frac{25}{27}.$$

In contrast, the revenue in the optimal auction can be calculated as follows:

$$\int \max\{l(x), l(y), 0\}\, dF(x)\, dF(y) =$$
$$\int_{\max\{x,y\}\geq 1} l(\max\{x,y\})\, dF(x)\, dF(y) = \int_1^2 l(z)\, dF^2(z) =$$
$$l(1)\frac{1}{3} + \int_1^{\sqrt{2}} 2\left(\sqrt{2}-1\right)\frac{2(z+1)}{9} dz + \int_{\sqrt{2}}^2 2(z-1)\frac{2(z+1)}{9} dz =$$
$$2\left(\sqrt{2}-1\right)\frac{1}{3} + 2\left(\sqrt{2}-1\right)\left(\frac{z^2}{9}+z\right)\Big|_1^{\sqrt{2}} + \frac{4}{9}\left(\frac{z^3}{3}-z\right)\Big|_{\sqrt{2}}^2 =$$
$$2\left(\sqrt{2}-1\right)\frac{1}{3} + 2\left(\sqrt{2}-1\right)\left(\frac{1}{9}+\sqrt{2}-1\right) + \frac{4}{9}\left(\frac{8}{3} - \frac{2\sqrt{2}}{3} - 2 + \sqrt{2}\right)$$
$$= 2\left(\sqrt{2}-1\right)\left(\sqrt{2}-\frac{5}{9}\right) + \frac{4}{9}\left(\frac{2}{3}+\sqrt{2}\frac{1}{3}\right) = \frac{3286}{2700} > \frac{25}{27}.$$

3.1 The function $l_i(\cdot)$ generalizes Myerson's virtual valuation.

We now show that the function $l_i(\cdot)$ obtained above coincides with the virtual valuation $\overline{J}_i$ under condition (M). Recall that $\overline{J}_i(x)$ is the derivative of the convex hull of the function $H(x) = \int_0^x h_i(q)\, dq$ and $h_i(q) = J_i\left(F_i^{-1}(q)\right)$. Let G be the convex hull of H. The function G is the supremum of the linear functions $\alpha + \beta x$ such that $\alpha + \beta x \leq H(x)$. Since F_i is continuous $F_i([a_i, b_i]) = [0,1]$ and therefore we may consider $G(F_i(\cdot))$ as the supremum of functions of the form $\alpha + \beta F(\cdot)$. Therefore the generalized convex hull $\phi(x) = G(F_i(x))$. Thus $l_i(x) f_i(x) = \phi'(x) = G'(F_i(x)) f_i(x) = \overline{J}_i(x) f_i(x)$.

4 Application to a few classical results.

To show the wider applicability of the generalized virtual valuation we review how we can generalize a couple of classical results to distributions without density. We begin with nonlinear pricing.

4.1 Nonlinear pricing

In Mussa and Rosen (1978) the set of consumers types is given by an interval $\left[\underline{\theta}, \bar{\theta}\right]$ and types are distributed according to a density $f(\theta)$. There is a monopolist who produces quality q at unit cost $C(q)$. The monopolist's optimization problem is to choose an increasing product quality function $q(\theta)$ to maximize

$$\int [\theta q(\theta) - z(\theta) - C(q(\theta))] dF(\theta) \text{ subject to}$$
$$z(\theta) = z(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(s)\, ds.$$

Here, $q(\theta)$ is the quality level the monopolist intends a type θ to choose. The integral simplifies to

$$\int_{\underline{\theta}}^{\bar{\theta}} [\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) q(\theta) - C(q(\theta))] f(\theta)\, d\theta.$$

If we use the virtual valuation for a general distribution and signed measure

$$d\nu = \theta dF - (1 - F(\theta)) d\theta$$

the analogous problem will be to choose an increasing function $q(\theta)$ to maximize

$$\int_{\underline{\theta}}^{\bar{\theta}} [l(\theta) q(\theta) - C(q(\theta))] dF(\theta).$$

The function $\bar{q}(\theta)$ defined as the [11] solution of $\max_{q \geq 0} l(\theta) q - C(q)$ is increasing. Since it is $l(\theta)$ measurable

$$\int_{\underline{\theta}}^{\bar{\theta}} l(\theta) \bar{q}(\theta) dF(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \theta \bar{q}(\theta) dF(\theta) - \int_{\underline{\theta}}^{\bar{\theta}} \bar{q}(\theta) (1 - F(\theta)) dF(\theta)$$

and solves the original problem as well.

4.2 Regulation of a monopoly

In Baron and Myerson (1982) the virtual valuation has a different aspect. It has the form (see equation 16, page 917)

$$z_\alpha(\theta) = \theta + (1-\alpha) \frac{F(\theta)}{f(\theta)}$$

where the parameter $\alpha \in [0,1]$ is the weight of the monopolist profit in the welfare maximization. For distributions without density we replace $z_\alpha(\theta)$ by the virtual valuation of the measure $d\nu = \theta dF + (1-\alpha) F(\theta) d\theta$.

[11] More precisely the smallest solution if there is more than one.

5 Conclusion

We introduced a new concept: the generalized virtual valuation. The virtual valuation can be interpreted as the "derivative" of the generalized convex hull. It is well adapted to characterize the optimal auction for distributions that are not absolutely continuous. It is an ironing procedure that is possible for any distribution. The continuity of the optimal auction revenue with respect to the distribution is easily proved using the generalized virtual valuation. This result, if restricted to distributions having densities is much less interesting. In particular, by dropping the density requirement, we are able to approximate a distribution by a discrete distribution maintaining proximity of optimal revenue as well.

We expect wider applicability of the generalized convex hull. For example we apply it to classical problems as in Mussa and Rosen (1978) and Baron and Myerson (1982).

A Proofs omitted in the text

We begin with the proof of Proposition 1 (basic properties):

a) If $0 < F(x) < 1$ then $\partial\phi(x) \neq \emptyset$;

Suppose $0 < F(x) < 1$. There is a sequence $(\alpha_n, \beta_n) \in \Gamma$ such that $\phi_n := \alpha_n + \beta_n F(x)$ converges to $\phi(x)$. Since $\alpha_n \leq 0$ it is true that $\frac{\phi_n}{F(x)} \leq \beta_n$. Now from $\phi_n + (1 - F(x))\beta_n = \alpha_n + \beta_n \leq H(b)$ we have that $\beta_n \leq \frac{H(b)-\phi_n}{1-F(x)}$. Thus since $(\phi_n)_n$ is bounded so is $(\beta_n)_n$. Therefore $(\alpha_n)_n$ is bounded as well. Thus we may suppose without loss of generality that $(\alpha_n, \beta_n) \to (\alpha', \beta') \in \Gamma$. From this we conclude that $\phi(x) = \alpha' + \beta' F(x)$ ending the proof that $\partial\phi(x) = \Gamma(x) \neq \emptyset$.

b) $\Gamma(b) = [\beta^*, \infty)$ and $\phi(b) = H(b)$;

If $\beta \geq \beta^*$ we have that $(H(b) - \beta, \beta) \in \Gamma$ since for every $z \leq b$,

$$H(b) + \beta(F(z) - F(b)) \leq H(b) + \beta^*(F(z) - F(b)) \leq H(z).$$

Thus $[\beta^*, \infty) \subset \partial\phi(b)$ and $\phi(b) = H(b)$. Let $\beta \in \partial\phi(b)$. Then there is an α such that $(\alpha, \beta) \in \Gamma$ and $\alpha + \beta = \phi(b) = H(b)$. Thus

$$H(b) = \alpha + \beta F(z) + \beta(1 - F(z)) \leq H(z) + \beta(1 - F(z)).$$

Therefore $\beta \geq \sup_{z<b} \frac{H(b)-H(z)}{1-F(z)} = \beta^*$. Hence $\partial\phi(b) = [\beta^*, \infty)$.

c) If $F(x) = 0$ then $\phi(x) = t_a := \min\{H(a), 0\}$. In particular $\phi(a-) = t_a$;

Let x be such that $F(x) = 0$. Suppose first that $F(a) > 0$. Then if we define $\beta^a := \inf\{\frac{H(z)}{F(z)}; z > a\} > -\infty$ we have that $(0, \beta^a) \in \Gamma$. So $\phi(x) = 0$. Suppose now that $F(a) = 0$. Let $\alpha < \min\{H(a), 0\}$. There exists $z^0 > a$ such that $\alpha < H(z)$

for $z \in [a, z^0]$. Let $\beta^0 := \inf\{\frac{H(z)}{F(z)}; z \geq z^0\}$. Therefore $\alpha + \min\{\beta^0, 0\}F \leq H$ and $\phi(x) \geq \alpha$. Since α is arbitrary, $\phi(x) = \min\{H(a), 0\}$.

d) If $F(a) = 0$ then $\Gamma(a) \neq \emptyset$ if and only if $\beta_* > -\infty$;

Suppose $\Gamma(a) \neq \emptyset$ and $F(a) = 0$. Then $\phi(a) = t_a$. Take $(u, v) \in \Gamma$ such that $\phi(a) = u + vF(a)$. Then

$$t_a + v(F(z) - F(a)) \leq H(z)$$

and this implies $v \leq \frac{H(z)-t_a}{F(z)-F(a)}$. Reciprocally if $\inf_{z>a} \frac{H(z)-t_a}{F(z)-F(a)} > -\infty$ then $\Gamma(a) \neq \emptyset$.

e) If $\Gamma(a) = \emptyset$ then $\lim_{x \downarrow a} s(x) = -\infty$;

Let $x^n \downarrow a$. We have that $F(x^n) > 0$ for every n so there exists $(\alpha_n, \beta_n) \in \Gamma$ such that $\phi(x^n) = \alpha_n + \beta_n F(x^n)$. If $(\beta_n)_n$ is bounded without loss of generality $\beta_n \to \beta^*$. Let $(\alpha, \beta) \in \Gamma$, $\alpha + \beta F(x^n) \leq \alpha_n + \beta_n F(x^n)$ which implies that $(\alpha_n)_n$ is bounded. Therefore $\alpha^* + \beta^* 0 \geq \alpha$. Hence $\alpha^* = 0$. And $(\alpha^*, \beta^*) \in \Gamma$ contradiction with $\Gamma(a) = \emptyset$. Since $\beta_n < \beta^*$ necessarily $\beta_n \to -\infty$.

f) Suppose $x < y$. If $F(x) = F(y)$ then $l(x) = l(y)$. Suppose now that $F(x) < F(y)$. Let $\beta \in \partial\phi(x)$ and $\gamma \in \partial\phi(y)$. Then we have

$$\begin{aligned}
&\phi(y) - \phi(x) \geq \beta(F(y) - F(x)) \text{ and} \\
&\phi(x) - \phi(y) \geq \gamma(F(x) - F(y)).
\end{aligned}$$

Adding the inequalities we have that $0 \geq (\beta - \gamma)(F(y) - F(x))$. Thus $\gamma \geq \beta$ and

$$s(y) \geq l(y) = \inf\{\gamma; \gamma \in \partial\phi(y)\} \geq \sup\{\beta; \beta \in \partial\phi(x)\} = s(x) \geq l(x).$$

QED

Proof of Proposition 2: If $F(x) = 1$ there is nothing to prove. Consider now $a \leq x < b$ such that $\partial\phi(x) \neq \emptyset$. Take a sequence $x_n \downarrow x$. Without loss of generality, $F(x^1) < 1$.Then

$$s(x^1)(F(x_n) - F(x)) \geq \phi(x_n) - \phi(x) \geq s(x)(F(x_n) - F(x)).$$

This implies that $\lim_n \phi(x_n) = \phi(x)$. Suppose now that $x < b$ and $\partial\phi(x) = \emptyset$. Then $x = a$ and $F(a) = 0$. The inequality $s(x_1)F(x_n) \geq \phi(x_n) - \phi(a)$ implies that $\phi(a) \geq \limsup_n \phi(x_n)$. From the Proposition 1, item (c) there exists $(\alpha_m, \beta_m) \in \Gamma$ such that $\alpha_m \to t_a$. Thus since $\alpha_m + \beta_m F(x_n) \leq \phi(x_n)$ we have that $\alpha_m \leq \liminf_n \phi(x_n)$ and finally making m goes to ∞ we conclude that

$$\phi(a) = t_a \leq \liminf_n \phi(x_n) \leq \limsup_n \phi(x_n) \leq \phi(a).$$

QED

Proof of Proposition 3: Suppose that $x^n \uparrow x$. Thus if $m = n + p$ and $n \geq n'$,

$$l(x)(F(x_m) - F(x_n)) \geq \phi(x_m) - \phi(x_n) \geq s(x_{n'})(F(x_m) - F(x_n)).$$

Thus $(\phi(x_n))_n$ is Cauchy and therefore converges. Since

$$\phi(x_n) - \phi(x) \geq l(x)(F(x_n) - F(x))$$

we conclude in the limit $n \to \infty$ that $\phi(x-) - \phi(x) \geq l(x)(F(x-) - F(x))$. To prove the other inequality take $y < x$. For n large enough that $x_n > y$,

$$\phi(x) - \phi(x_n) \geq s(x_n)(F(x) - F(x_n)) \geq s(y)(F(x) - F(x_n)).$$

Thus $\phi(x) - \phi(x-) \geq s(y)(F(x) - F(x-))$ and making $y \uparrow x$ we get $\phi(x) - \phi(x-) \geq s(x-)(F(x) - F(x-))$. QED

Proof of Proposition 5: (i) Suppose $\phi(x-) < H(x-)$. Let $\gamma_n = s(x-) + \frac{1}{n}$. Take $y_n \uparrow x$. Then since $\gamma_n \notin \Gamma(y_n)$ there exists a $z_n > y_n$ such that

$$\phi(y_n) + \gamma_n(F(z_n) - F(y_n)) \geq H(z_n).$$

If $z_n \uparrow z^*$:

$$\phi(x-) + s(x-)(F(z^*-) - F(x-)) \geq H(z^*-).$$

Since $\phi(x-) + s(x-)(F(z^*-) - F(x-)) \leq H(z^*-)$ we have the equality

$$\phi(x-) + s(x-)(F(z^*-) - F(x-)) = H(z^*-).$$

Necessarily $z^* > x$. This implies that for every $z \in (x, z^*)$ that $s(x-) \leq s(z) \leq s(x-)$. Suppose now that $z_n \downarrow z^*$. Then $\phi(x-) + s(x-)(F(z^*) - F(x-)) \geq H(z^*)$. Thus $\phi(z^*) = H(z^*)$. Thus $l(x) = s(x-)$. (ii) Suppose now that $\phi(x) < H(x)$. Let $\gamma_n = l(x) + 1/n$ and note that $\phi(x-) + \gamma_n(F(z) - F(x-)) \notin \Gamma$ since at $z = x$ this function is greater than $\phi(x) = \phi(x-) + l(x)(F(x) - F(x-))$. Thus there exists a z_n such that
$\phi(x-) + \gamma_n(F(z_n) - F(x-)) > H(z_n)$. From $\gamma_n > l(x)$ we have that $z_n > x$. If $z_n \downarrow z^*$ in the limit we get $\phi(x-) + l(x)(F(z^*) - F(x-)) \geq H(z^*)$. Or $\phi(x) + l(x)(F(z^*) - F(x)) \geq H(z^*)$. Therefore we have equality and $z^* > x$. Thus for every $z \in [x, z^*)$ it is true that $l(z) = l(x) = s(x)$. Analogously we prove the case $z_n \uparrow z^*$. QED

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