On refutability of the Nash-Walras equilibrium hypothesis

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Abstract

This paper studies whether the hypothesis of competitive equilibrium can be rejected upon observation of the response of commodity prices to changes in individual real incomes, for an economy where there are externalities. Some of the results are negative: even under refined classes of preferences, requiring, for instance, strategic complementarity in the demand for the externality, the hypothesis of Nash-Walras equilibrium is not refutable. Some separability, at least between two commodities and the externality, suffices for testable restrictions to exist. Information on individual demands for the commodity that causes the externality yields only extremely mild restrictions. Importantly, for the particular case of a public good some restrictions do exist.

Key words: Nash-Walras equilibrium, externalities, revealed preferences, testable restrictions.

JEL classification numbers: D12, D50, D62.

The results of [3] notwithstanding, the Sonnenschein-Mantel-Debreu theorem, which implies that the response of aggregate excess demand to changes in prices can be arbitrary, was understood to imply that general equilibrium theory imposed no, or hardly any, testable restrictions at the aggregate level.[1]

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It is now clear that this view of the problem of falsifiability of general equilibrium theory is overly pessimistic: in an insightful contribution, [6] have shown that some information at the individual level may generate nontrivial testable restrictions, even if it does not reveal actual individual choices, and only describes individual constraints. Specifically, [6] observed that the object under consideration in the Sonnenschein-Mantel-Debreu analysis, the aggregate excess demand function, is not the appropriate object on which restrictions are to be derived, since (i) it is not observable, under the 'null' hypothesis of general equilibrium, except at points at which it is, by definition, zero; and (ii) it measures the response of some endogenous variable (demand) to another endogenous variable (prices). Instead, [6] study restrictions on how prices respond to changes in individual endowments: a finite subset of the equilibrium manifold overcomes the observability problem, and provides information on how an endogenous variable responds to perturbations on exogenous variables. The key result is that this response is not arbitrary: there exists a (finite) set of non-tautological conditions that is necessary (and sufficient) for the existence of preferences that rationalize a given, finite set of data on prices and individual endowments.

The results of [6] constituted the basis for further developments on the empirical implications of general equilibrium. [10] obtained restrictions in a differential setting, while [14] extended [6] to intertemporal problems under uncertainty, and [7] to a setting where preferences of individuals are allowed to change randomly. [1] derives a test of the hypothesis that a sequence of allocations can be supported by competitive prices, and [2] derives testable implications of Pareto-efficiency and individual rationality.[2]

Importantly, by an application of the methodology of [6] to the analysis of public goods via Lindahl prices, [23] shows that the hypothesis is of Pareto-efficiency in the provision of public goods is falsifiable, whenever information on market prices, production levels and individual incomes is available.

Here, I address the question of testability for the hypothesis of competitive equilibrium for economies with externalities. The equilibrium concept under consideration is Nash-Walras equilibrium, which combines the competitive elements of price-taking behavior with the strategic principle of Nash equilibrium. By an immediate extension, the introduction of externalities will not affect the restrictions derived by [6] if one imposes, for instance, additive separability of individual preferences in the externalities. Here, I study the extent to which one can deviate from this extreme assumption of separability, without completely losing the empirical implications of the model without externalities. Some of the results are negative: even under refined classes of preferences, the hypothesis of Nash-Walras equilibrium is not refutable on the basis of data on prices and individual endowments. Some separability, at least between two commodities and the externality, gives testable restrictions. Information on individuals choices of the externality yields only some extremely mild restrictions. Importantly, for the particular case of a public good some restrictions do exist.

1 General setting and a basic result

There is a finite number, $L+1$, of commodities; these commodities are to be consumed in nonnegative amounts, so the consumption set of each individual is $\mathbb{R}^L_+ \times \mathbb{R}_+$, and a consumption bundle for individual i is denoted by (x^i, y^i). A consumption profile (allocation) is $(x, y) = (x^i, y^i)_{i=1}^I$.

The society is a finite set, $\mathcal{I}$, of consumers denoted by $i = 1, \ldots, I$, with $I \geq 2$. Externalities exist because the utility of each individual is affected by the consumption of the last commodity by all the other consumers: for individual i, preferences are represented by a utility function $u^i : \mathbb{R}^L_+ \times \mathbb{R}_+ \times \mathbb{R}^{I-1}_+ \to \mathbb{R}$, and $u^i(x^i, y^i, y^{\neg i})$ is person i's utility level if her consumption bundle is (x^i, y^i) and the profile of consumption of the last commodity by the other individuals in the society is $y^{\neg i}$. The profile of preferences is denoted by $u = (u^i)_{i=1}^I$.

The endowment of agent i is a bundle, (e^i, k^i), and the profile of endowments is an allocation (e, k). Prices are denoted by (p, q).[3]

An **economy**, $\mathcal{E}$, is a profile of individual preferences and endowments: $\mathcal{E} = (u, e, k)$. For an economy, **a Nash-Walras equilibrium** consists of prices and an allocation of commodities, (p, q, x, y), such that:

1. each consumer's demand is rational, given the prices and the choices of other

individuals: each (x^i, y^i) solves the problem

$$\max_{\tilde{x},\tilde{y}} u^i(\tilde{x}, \tilde{y}, y^{\neg i}) : p(\tilde{x} - e^i) + q(\tilde{y} - k^i) \leq 0;$$

2. all markets clear: $\sum_i (x^i - e^i, y^i - k^i) = 0$.

Let $\mathcal{W}(\mathcal{E})$ denote the set of Nash-Walras equilibria of $\mathcal{E}$, and let $W(\mathcal{E})$ be the projection of this set into the space of prices. A **data set** is a finite sequence of pairs consisting of a vector of prices and a profile of individual endowments, $(p_t, q_t, e_t, k_t)_{t=1}^T$. I assume that all observed prices and endowments are strictly positive. A profile of preferences u **rationalizes** a data set if each observation of prices is consistent with the corresponding (observed) profile of endowments, in the sense of being a vector of Nash-Walras equilibrium prices: for every t, $(p_t, q_t) \in W(u, e_t, k_t)$.

The problem of testability with respect to some class of preferences is whether rationalizability by a profile defined in that class imposes (non-tautological) implications on data sets. Observation of data violating these conditions would contradict the hypothesis that all the observed values of the endogenous variables (prices) are determined by the Nash-Walras mechanism, given the observed values of the exogenous variables (endowments) under fixed unobservable fundamentals (invariant preferences that lie in the relevant class).

The following result says that rationalizability by a profile of preferences defined in the basic class of conditions used to guarantee existence of Nash-Walras equilibrium imposes no testable implications.

Proposition. *Any data set is rationalizable by a profile of continuous preferences that are strongly concave and strictly monotone in own consumption* (x^i, y^i).[4]

This result, the most basic one in the paper, follows in fact as a corollary of proposition 1 below. Rationalizations for any data set are constructed on the basis the following algorithm, where $\mathbf{e} = (1, 0, \ldots, 0) \in \mathbb{R}^L$.

Algorithm. *Input: a data set* $(p_t, q_t, e_t, k_t)_{t=1}^T$.

1. $t = 1$.

2. *If* $y^i_{t'} \neq k^i_t$, *for all* i *and all* $t' < t$, *then let* $x^i_t = e^i_t$ *and* $y^i_t = k^i_t$, *for all* i, *and go to step 7.*

3. *Let* $\mathcal{J} = \{i | \exists t' < t : y^i_{t'} \neq k^i_t\}$.

4. *If* $\mathcal{J} = \varnothing$, *let* $\epsilon = 1$ *and go to step 6.*

5. *Let* $\epsilon = \min_{i \in \mathcal{J}}\{\min_{t'<t: y^i_{t'} \neq k^i_t}\{|y^i_{t'} - k^i_t|\}\}$.

6. *Let* $\gamma = \min_{i \neq 1}\{\frac{(I-1)p_{t,1}e^i_{t,1}}{q_t}\}$, $\delta = \frac{1}{2}\min\{\epsilon, k^1_t, \gamma\}$, $y^1_t = k^1_t - \delta$, $x^1_t = e^1_t + \frac{q_t \delta}{p_{t,1}}\mathbf{e}$, *and, for every* $i \neq 1$, $y^i_t = k^i_t + \frac{\delta}{I-1}$ *and* $x^i_t = e^i_t - \frac{q_t \delta}{p_{t,1}(I-1)}\mathbf{e}$.

7. *If* $t = T$, *stop. Else,* $t = t + 1$ *and go to step 2.*

Output: $(x_t, y_t)^T_{t=1}$.

The crucial properties of the output of the algorithm are in the following lemma.

Lemma 1. *Given a data set, let* $(x_t, y_t)^T_{t=1}$ *be the output of the algorithm. Then,*

1. *for every* i *and every* t, $(x^i_t, y^i_t) \gg 0$;

2. *for every* i *and every* t, $p_t x^i_t + q_t y^i_t = p_t e^i_t + q_t k^i_t$;

3. *for every* t, $\sum_i (x^i_t, y^i_t) = \sum_i (e^i_t, k^i_t)$;

4. *for every* i *and every* t, *and for every* $t' \neq t$, $y^i_{t'} \neq y^i_t$.

Proof. The proofs of this and all other lemmata are in Appendix A2. □

2 Further assumptions

2.1 Concavity and monotonicity

The basic proposition obtained above does not imply that rationalizability with respect to proper subclasses of preferences is not testable. For instance, the possibility that concavity or monotonicity may fail with respect to the externality is allowed by that proposition.

Proposition 1. *Any data set is rationalizable by a profile of continuous, strongly concave and strictly monotone preferences.*

Proof. Given the data set, let $(x_t, y_t)_{t=1}^T$ be the output of the algorithm.

Fix i and let $j \neq i$. For simplicity of notation, and using property 4 of lemma 1, suppose that $y_1^j > y_2^j > \ldots > y_T^j$. Suppose that there exists no solution $(u_t^i, \lambda_t^i, \mu_t^i)_{t=1}^T$ to the system:

1. for every t and every $t' \neq t$,

$$u_{t'}^i < u_t^i + \lambda_t^i(p_t(x_{t'}^i - x_t^i) + q_t(y_{t'}^i - y_t^i)) + \mu_t^i(y_{t'}^j - y_t^j);$$

2. for every t, $\lambda_t^i > 0$.

Then, it follows from the Theorem of the Alternative for strict inequalities, [22, §22.2], that there exist $\alpha_{t,t'} \geq 0$, for all t and all $t' \neq t$, and $\beta_t \geq 0$, for all t, with at least one of these numbers strictly positive, such that:

3. for every t, $\sum_{t' \neq t} \alpha_{t,t'} = \sum_{t' \neq t} \alpha_{t',t}$;
4. for every t, $\sum_{t' \neq t} \alpha_{t,t'}(p_t(x_t^i - x_{t'}^i) + q_t(y_t^i - y_{t'}^i)) = \beta_t$;
5. for every t, $\sum_{t' \neq t} \alpha_{t,t'}(y_t^j - y_{t'}^j) = 0$;

Since $y_1^j > y_t^j$ for all $t \geq 2$, it follows from 5 that $\alpha_{1,t'} = 0$ for every $t' \neq 1$. Then, from 3, it must be that also $\alpha_{t',1} = 0$ for every $t' \neq 1$. Then, 4 implies that $\alpha_{2,t'} = 0$ for every $t' \neq 2$, and hence, from 3, that also $\alpha_{t',2} = 0$ for every $t' \neq 2$. Continuing recursively, it follows that all $\alpha_{t,t'} = 0$, which implies, by 4, that every $\beta_t = 0$, which is impossible. It follows that one can find a solution $(u_t^i, \lambda_t^i, \mu_t^i)_{t=1}^T$ to the system defined by 1 and 2.

As in [19, Theorem 2], define function $h : \mathbb{R}^L \times \mathbb{R} \times \mathbb{R}^{I-1} \to \mathbb{R}_+$ by

$$h(x, y, y^\neg) = \sqrt{\|(x, y, y^\neg)\|^2 + 1} - 1.$$

Function h is strongly convex and has the properties that $h(x, y, y^\neg) = 0$ only at $(x, y, y^\neg) = 0$, and that all its partial derivatives are less than 1. For each t, let $\epsilon_t^i \in (0, \lambda_t^i \min\{\min_l\{p_{t,l}\}, q_t\})$ be small enough so that for all $t' \neq t$,

$$u_{t'}^i < u_t^i + \lambda_t^i(p_t(x_{t'}^i - x_t^i) + q_t(y_{t'}^i - y_t^i)) + \mu_t^i(y_{t'}^j - y_t^j) - \epsilon_t^i h((x_t^i, y_t^i, y_t^{\neg i}) - (x_{t'}^i, y_{t'}^i, y_{t'}^{\neg i})),$$

and define function $v_t^i : \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1} \to \mathbb{R}$ by

$$\begin{aligned} v_t^i(x^i, y^i, y^{\neg i}) &= u_t^i + \lambda_t^i(p_t(x^i - x_t^i) + q_t(y^i - y_t^i)) \\ &\quad + \mu_t^i(y^j - y_t^j) - \epsilon_t^i h((x_t^i, y_t^i, y_t^{\neg i}) - (x^i, y^i, y^{\neg i})), \end{aligned}$$

which is strictly monotone in (x^i, y^i), and strongly concave.

Define u^i by $u^i(x^i, y^i, y^{\neg i}) = \min_t\{v_t^i(x^i, y^i, y^{\neg i})\}$. It is immediate that u^i is continuous, strongly concave in all arguments and strictly monotone in (x^i, y^i). Fix t, and observe that $u^i(x_t^i, y_t^i, y_t^{\neg i}) = u_t^i$, whereas, whenever $p(\tilde{x} - e_t^i) + q(\tilde{y} - k_t^i) \leq 0$, by condition 2 of lemma 1,

$$\begin{aligned} u^i(\tilde{x}, \tilde{y}, y_t^{\neg i}) &\leq v_t^i(\tilde{x}, \tilde{y}, y_t^{\neg i}) \\ &\leq u_t^i + \lambda_t^i(p_t(\tilde{x} - x_t^i) + q_t(\tilde{y} - y_t^i)) \\ &= u_t^i + \lambda_t^i(p_t(\tilde{x} - e_t^i) + q_t(\tilde{y} - k_t^i)) \\ &\leq u_t^i. \end{aligned}$$

Given condition 3 of lemma 1, the latter implies that profile u rationalizes the data set. To conclude the proof, notice that the result holds if one adds

$$(\max\{1, -\min_t\{\mu_t^i - \epsilon_t^i\}\} + 1) \sum_{j' \neq i} y^{j'}$$

to the definition of each u^i, and that in this case the function is strictly monotone in all arguments. □

The proposition means that the assumption that convexity applies only to the variables that each individual chooses is innocuous from the perspective of testability: the Nash-Walras hypothesis is nontestable with and without that assumption.

2.2 Strategic complementarities

The literature on monotone comparative statics (see, for example, [20] and [25]), and in particular its application to abstract games, [8], have shown that the assumption of strategic complementarity strengthens, significantly, the tests of hypotheses of noncooperative behavior. The following theorem shows that such result does not extend to the present setting.

A utility function u^i will be said to exhibit **strategic complementarity** if, for every x^i,

$$u^i(x^i, y^i, y^{\neg i}) - u^i(x^i, \hat{y}^i, y^{\neg i}) \geq u^i(x^i, y^i, \hat{y}^{\neg i}) - u^i(x^i, \hat{y}^i, \hat{y}^{\neg i})$$

whenever $y^i \geq \hat{y}^i$ and $y^{\neg i} \geq \hat{y}^{\neg i}$.

Proposition 2. *Any data set is rationalizable by a profile of continuous, concave, strictly monotone preferences that satisfy strategic complementarity.*

Proof. From a run of the algorithm, fix the output $(x_t, y_t)_{t=1}^T$.

Fix an individual i. Let $j \neq i$, and assume that $y_1^j < y_2^j < \ldots < y_T^j$. Suppose that there exists no array of real numbers $(u_t^i, \lambda_t^i, \mu_t^i)_{t=1}^T$ that solves the following system of inequalities:

1. for all t, and for all $t' \neq t$,

$$-u_t^i + u_{t'}^i + \lambda_t^i(p_t(x_t^i - x_{t'}^i) + q_t(y_t^i - y_{t'}^i)) + \mu_t^i(y_t^j - y_{t'}^j) < 0;$$

2. for all t, $-\lambda_t^i < 0$;

3. for all $t \leq T-1$, $\lambda_t^i q_t - \lambda_{t+1}^i q_{t+1} < 0$;

4. for all $t \leq T-1$, $-\mu_t^i + \mu_{t+1}^i < 0$.

Then, again by [22, §22.2], there must exist nonnegative real numbers $\alpha_{t,t'}$, for all t and all $t' \neq t$, β_t, for all t, γ_t, for all $t \leq T-1$, and δ_t, for all $t \leq T-1$, not all of which are zero, that solve the following system of equalities

5. for all t, $\sum_{t' \neq t} \alpha_{t,t'} = \sum_{t' \neq t} \alpha_{t,t'}$;

6. $\sum_{t' \neq 1} \alpha_{1,t'}(p_1(x_1^i - x_{t'}^i) + q_1(y_1^i - y_{t'}^i)) + \gamma_1 q_1 = \beta_1$;

7. for all t, $2 \leq t \leq T-1$,

$$\sum_{t' \neq t} \alpha_{t,t'}(p_t(x_t^i - x_{t'}^i) + q_t(y_t^i - y_{t'}^i)) + (\gamma_t - \gamma_{t-1})q_t = \beta_t;$$

8. $\sum_{t' \neq T} \alpha_{T,t'}(p_T(x_T^i - x_{t'}^i) + q_T(y_T^i - y_{t'}^i)) - \gamma_{T-1} q_T = \beta_T$;

9. $\sum_{t'\neq 1} \alpha_{1,t'}(y^j_1 - y^j_{t'}) = \delta_1$;

10. for all t, $2 \leq t \leq T-1$, $\sum_{t'\neq t} \alpha_{t,t'}(y^j_t - y^j_{t'}) + \delta_{t-1} = \delta_t$;

11. $\sum_{t'\neq T} \alpha_{T,t'}(y^j_T - y^j_{t'}) + \delta_{T-1} = 0$.

It follows from 11 that $\delta_{T-1} = 0$, and that $\alpha_{T,t'} = 0$ for all $t \neq T$. Then, 5 implies that $\alpha_{t',T} = 0$ for all $t' \neq T$. Since $\delta_{T-1} = 0$ and $\alpha_{T-1,T} = 0$, then, by 10, $\delta_{T-2} = 0$, and $\alpha_{T-1,t'} = 0$ for all $t' \neq T-1$. Again, 5 then implies that $\alpha_{t',T-1} = 0$ for all $t' \neq T-1$. Following recursively, we get that for all $t \geq 2$ it is true that $\delta_{t-1} = 0$, and that $\alpha_{t,t'} = 0$ for all $t' \neq t$. It follows, then, from 9, that $\alpha_{1,t'} = 0$ for all $t' \neq 1$. Since $q_T > 0$, it follows from item 8 that $\gamma_{T-1} = 0$ and $\beta_T = 0$. Immediately, it follows from item 7 that $\gamma_{T-2} = 0$ and $\beta_{T-1} = 0$, since $q_{T-1} > 0$. Recursively, item 7 implies that for all $t \geq 2$, $\gamma_{t-1} = 0$ and $\beta_t = 0$. It follows that $\gamma_1 = 0$ and then, from item 6, that $\beta_1 = 0$. All this contradicts the fact that at least one of these numbers should be nonzero, so it follows that the system defined by items 1 to 4 must have a solution.

Define individual preferences u^i, by

$$u^i(x^i, y^i, y^{\neg i}) = \min_t \{u^i_t + \lambda^i_t(p_t(x^i - x^i_t) + q_t(y^i - y^i_t)) + \mu^i_t(y^j - y^j_t)\}.$$

It is immediate that function u^i is continuous and concave in all arguments, and, by 2, it is strictly monotone on (x^i, y^i). To see that u^i satisfies strategic complementarity, it suffices to observe that its right-hand partial derivative with respect to y^i is nondecreasing in $y^{\neg i}$. To see that this is the case, fix x^i, y^i, $y^{\neg i}$ and $\tilde{y}^{\neg i}$, and let t and $\tilde{t}$ be such that the right-hand partial at $(x^i, y^i, y^{\neg i})$ is $\lambda^i_t q_t$, and the the right-hand partial at $(x^i, y^i, \tilde{y}^{\neg i})$ is $\lambda^i_{\tilde{t}} q_{\tilde{t}}$. By construction,

$$\begin{aligned} & u^i_t + \lambda^i_t(p_t(x^i - x^i_t) + q_t(y^i - y^i_t)) + \mu^i_t(y^j - y^j_t) \leq \\ & \quad u^i_{\tilde{t}} + \lambda^i_{\tilde{t}}(p_{\tilde{t}}(x^i - x^i_{\tilde{t}}) + q_{\tilde{t}}(y^i - y^i_{\tilde{t}})) + \mu^i_{\tilde{t}}(y^j - y^j_{\tilde{t}}), \end{aligned}$$

and

$$\begin{aligned} & u^i_{\tilde{t}} + \lambda^i_{\tilde{t}}(p_{\tilde{t}}(x^i - x^i_{\tilde{t}}) + q_{\tilde{t}}(y^i - y^i_{\tilde{t}})) + \mu^i_{\tilde{t}}(\tilde{y}^j - y^j_{\tilde{t}}) \leq \\ & \quad u^i_t + \lambda^i_t(p_t(x^i - x^i_t) + q_t(y^i - y^i_t)) + \mu^i_t(\tilde{y}^j - y^j_t). \end{aligned}$$

Adding these two inequalities yields $(\mu^i_t - \mu^i_{\tilde{t}})(y^j - \tilde{y}^j) \leq 0$, and then, from 3 and 4, it follows that $(\lambda^i_t q_t - \lambda^i_{\tilde{t}} q_{\tilde{t}})(y^j - \tilde{y}^j) \geq 0$.

Strict monotonicity in $y^{\neg i}$ can be obtained as in the proof of proposition 1, without affecting the remainder of the argument.

To complete the proof, it suffices that (p_t, q_t, x_t, y_t) be a Nash-Walras equilibrium of economy (u, e_t, k_t), for every t. By part 3 of lemma 1, it suffices that for each individual, (x_t^i, y_t^i) solve the problem

$$\max_{\tilde{x},\tilde{y}} u^i(\tilde{x}, \tilde{y}, y_t^{\neg i}) : p_t\tilde{x} + q_t\tilde{y} \leq p_t e_t^i + q_t k_t^i.$$

That (x_t^i, y_t^i) is feasible follows from parts 1 and 2 of lemma 1. From item 1 above, it is immediate that $u^i(x_t^i, y_t^i, y_t^{\neg i}) = u_t^i$. Now, suppose that $(\tilde{x}, \tilde{y})$ satisfies that $p_t\tilde{x} + q_t\tilde{y} \leq p_t e_t^i + q_t k_t^i$; then, by construction,

$$\begin{aligned} u^i(\tilde{x}, \tilde{y}, y_t^{\neg i}) &\leq u_t^i + \lambda_t^i(p_t(\tilde{x} - x_t^i) + q_t(\tilde{y} - y_t^i)) + \mu_t^i(y_t^j - y_t^j) \\ &= u_t^i + \lambda_t^i(p_t(\tilde{x} - e_t^i) + q_t(\tilde{y} - k_t^i)) \\ &\leq u_t^i \\ &= u^i(x_t^i, y_t^i, y_t^{\neg i}). \end{aligned}$$

□

It follows that even the combination of overall concavity, strategic complementarity and monotonicity imposes no testable implications. The following result exploits the fact that the results above do not assume that endowments are different across observations.

Corollary. *Any finite set of prices is a subset of Nash-Walras equilibrium prices for some economy with a given profile of endowments and a profile of continuous, concave, strictly monotone preferences that satisfy strategic complementarity. Formally, for any finite set of strictly positive prices,* $\{(p_s, q_s)\}_{s=1}^S$*, and any profile of strictly positive endowments,* (e, k)*, there exists a profile of preferences,* u*, satisfying the conditions above, such that* $\{(p_s, q_s)\}_{s=1}^S \subseteq W(u, e, k)$*. Alternatively, all preferences can be taken to be strongly concave and strictly monotone.*

Proof. The result is immediate from the two previous propositions, by letting the data set be $(p_s, q_s, e, k)_{s=1}^S$. □

This corollary resembles the results of [17], although the presence of externalities allows for further properties on preferences.

3 Separability

Obviously, additive separability of each u^i in $y^{\neg i}$ restores the testable implications of the model without externalities. I now show that weaker separability assumptions do impose some restrictions as well.

Let $L \geq 2$. A utility function u^i will be said to be weakly separable (in x^i) if there exist a strictly monotone function $V^i : \mathbb{R}^L_+ \to \mathbb{R}$ and a function $U^i : V[\mathbb{R}^L_+] \times \mathbb{R} \times \mathbb{R}^{I-1}_+ \to \mathbb{R}$, strictly monotone in its first two arguments, such that $u^i(x^i, y^i, y^{\neg i}) = U^i(V^i(x), y^i, y^{\neg i})$.[5] We will say that u^i is weakly, smoothly separable if it is differentiable and has interior contour sets, and if functions U^i and V^i are differentiable.

3.1 A characterization

Given a data set, $(p_t, q_t, e_t, k_t)_{t=1}^T$, define the following condition (see [27] and [6]).

Condition 1. *There exist vectors x^i_t and ρ^i_t, and real numbers y^i_t, u^i_t, v^i_t, θ^i_t, λ^i_t and μ^i_t, for every individual i and every observation t, such that*

1. *for all i and t, $x^i_t \geq 0$, $y^i_t \geq 0$, $\lambda^i_t > 0$ and $\mu^i_t > 0$;*

2. *for all i and t, $\rho^i_t = \lambda^i_t p_t$ and $\theta^i_t = \mu^i_t q_t$;*

3. *for all i and t, $p_t x^i_t + q_t y^i_t = p_t e^i_t + q_t k^i_t$;*

4. *for all i and t, and for all t' such that $y^{\neg i}_{t'} = y^{\neg i}_t$,*

$$u^i_{t'} \leq u^i_t + \frac{\mu^i_t}{\lambda^i_t}(V^i_{t'} - V^i_t) + \theta^i_t(y^i_{t'} - y^i_t);$$

5. *for all i, t and t',*

$$v^i_{t'} \leq v^i_t + \rho^i_t(x^i_{t'} - x^i_t);$$

6. *for all t, $\sum_i (x^i_t, y^i_t) = \sum_i (e^i_t, k^i_t)$.*

The condition is said to be satisfied **in the interior** if, furthermore, $x^i_t \gg 0$ and $y^i_t > 0$, for all i and t.

The following lemma does not give tests of rationalizability (because of the existential quantifiers involved), but will be instrumental for the proposition that follows.

Lemma 2. *Condition 1 is a (weak) characterization of rationalizability under weak separability:*

1. *if a data set is rationalizable by a profile of weakly, smoothly separable preferences that satisfy concavity and strong monotonicity in own demands, then it satisfies condition 1 in the interior;*

2. *if a data set satisfies condition 1, then it is rationalizable by a profile of weakly separable, concave and strongly monotone preferences.*

3.2 The test

In order to obtain a proper test, all existential quantifiers must be eliminated from the characterization given by lemma 2.

Proposition 3. *Rationalizability by a profile of weakly separable preferences is testable using a finite set of inequalities on data. Formally, for any finite sequence of profiles of strictly positive individual endowments, $d = (e_t, k_t)_{t=1}^T$, there exists a semialgebraic set of sequences of strictly positive prices for all commodities, Δ_d, such that:*

1. *if $(p_t, q_t, e_t, k_t)_{t=1}^T$ is rationalizable by a profile of weakly, smoothly separable preferences that satisfy concavity and strict monotonicity in own demands, then, then $(p_t, q_t)_{t=1}^T \in \Delta_d$;*

2. *if $(p_t, q_t)_{t=1}^T \in \Delta_d$, then $(p_t, q_t, e_t, k_t)_{t=1}^T$ is rationalizable by a profile of weakly separable, concave and strictly monotone preferences.*

Also, there exist d for which Δ_d is a proper subset of $(\mathbb{R}^L_{++} \times \mathbb{R}_{++})^T$.

Proof. Consider the set of $(p_t, q_t, (x_t^i, y_t^i, \rho_t^i, \nu_t^i, \lambda_t^i, \mu_t^i, u_t^i, v_t^i)_{i=1}^I)_{t=1}^T$ such that for all i and all t,

1. $x_t^i \geq 0$, $y_t^i \geq 0$, $\lambda_t^i > 0$ and $\mu_t^i > 0$;

2. $\rho_t^i = \lambda_t^i p_t$ and $\theta_t^i = \mu_t^i q_t$;

3. $p_t x_t^i + q_t y_t^i = p_t e_t^i + q_t k_t^i$;

4. for all t' such that $y^{\neg i}_{t'} = y^{\neg i}_t$,

$$\lambda^i_t u^i_{t'} \leq \lambda^i_t u^i_t + \mu^i_t(v^i_{t'} - v^i_t) + \lambda^i_t \theta^i_t (y^i_{t'} - y^i_t);$$

5. for all t', $v^i_{t'} \leq v^i_t + \rho^i_t(x^i_{t'} - x^i_t)$;

6. $\sum_i (x^i_t, y^i_t) = \sum_i (e^i_t, k^i_t)$.

This is a finite set of polynomial inequalities, so it follows from the Tarski-Seidenberg theorem (see [21, Theorem 8.6.6]) that its projection into the space of prices only, Δ_d, is semialgebraic.

Parts 1 and 2 follow, respectively, from parts 1 and 2 of lemma 2. Example 1 below illustrates d for which $\Delta_d \subsetneq (\mathbb{R}^L_{++} \times \mathbb{R}_{++})^T$. □

3.3 Non-rationalizable data

In order to complete the argument of proposition 3, we must show that there exist data sets which cannot be rationalized under the separability hypothesis. The following example is one such data set.

Example 1. *Suppose that $I = L = 2$. Suppose that a data set includes the following two observations:*

$$\begin{array}{llll} e^1_1 = (1,4), & e^1_2 = (4,1), & e^2_1 = (2,1), & e^2_2 = (1,2), \\ k^1_1 = 0.01, & k^1_2 = 0.005, & k^2_1 = 0.01, & k^2_2 = 0.005, \\ p_1 = (1,10), & p_2 = (10,1), & q_1 = 0.1, & q_2 = 0.2. \end{array}$$

Suppose that the data set is rationalized by the profile of weakly separable preferences (u^1, u^2). Let (x^1_t, y^1_t) solve

$$\max_{x^1, y^1} u^1(x^1, y^1, y^2_t) : p_t \cdot (x^1 - e^1_t) + q_t(y^1 - k^1_t) \leq 0,$$

and suppose that $u^1(x^1, y^1, y^2) = U^1(V^1(x^1), y^1, y^2)$. Then, it must be that each x^1_t solves the problem $\max_x V^1(x) : p_t x \leq T^1_t$, with $T^1_t = p_t e^1_t - q_t(y^1_t - k^1_t)$. Since, by aggregate feasibility, $T^1_t \in [p_t e^1_t - q_t k^2_t, p_t e^1_t + q_t k^1_t]$, it follows that $T^1_t \in [40.999, 41.001]$. Also, since $e^1_1 + e^2_1 = (3,5)$ and $e^1_2 + e^2_2 = (5,3)$, feasible values of x^1_1 and x^1_2 can only be, respectively, in

$$X_1 = \{(x_1, x_2) | x_1 \in [0,3], x_2 = \frac{T^1_1}{10} - 0.1 x_1\}$$

and

$$X_2 = \{(x_1,x_2)|x_2\in[0,3], x_1=\frac{T_2^1}{10}-0.1x_2\}$$
$$\subset \{(x_1,x_2)|x_1\in[3.7999,4.2], x_2=T_2^1-10x_1\}.$$

Since $X_1\cap X_2=\varnothing$, necessarily $x_1^1\neq x_2^1$. Since $x_1^1\in X_1$, then

$$p_2x_1^1=10x_{1,1}^1+\frac{T_1^1}{10}-0.1x_{1,1}^1\leq 9.9(3)+4.1001<T_2^1=p_2x_2^1,$$

whereas, since $x_2^1\in X_2$, then

$$p_1x_2^1=x_{2,1}^1+10(T_2^1-10x_{2,1}^1)\leq 410.01-99(3.7999)<T_1^1=p_1e_1^1,$$

which is impossible, as it violates the weak axiom of revealed preferences.

It follows that under weak separability the hypothesis of competitive equilibrium does impose testable restrictions that take the form of a finite set of inequalities on data. The results will still hold as long as for each individual there are two of the $L+1$ commodities for which the person's preferences are weakly separable.

In the analysis of [6] the data identify individual budget constraints at each observation, and the tests exploit an existing tension between the principles of market-clearing (in the form of the nonnegativity constraints) and individual rationality (in the form of the weak axiom of revealed preferences). Here, the overall budget constraint of each individual is observed, but not the 'reduced' budget that constrains the individual's choice of the commodities in which her preferences are separable; nonnegativity constraints on all other commodities, however, do impose bounds on the position of these reduced budgets, and, then, within these bounds, the tension between market clearing and individual rationality yields the testable implications.

4 Public goods

Another canonical case is when the externality is actually a public commodity: its consumption is nonrival and nonexclusive. In this case, preferences are $u^i:\mathbb{R}_+^L\times\mathbb{R}_+\to\mathbb{R}$, and the utility level of individual i given a consumption allocation (x,y) is $u^i(x^i,\sum_j y^j)$.

For this situation to be of interest, I introduce production in the economy: I assume that there exists an aggregate technology $\mathcal{F} \subseteq \mathbb{R}^L \times \mathbb{R}$, so that a production plan of netputs (X, Y) is technologically feasible if, and only if, $(X, Y) \in \mathcal{F}$. The ownership structure of the economy is $\theta = (\theta^i)_{i=1}^I$, a vector of nonnegative numbers such that $\sum_i \theta^i = 1$.

An **economy** is now a profile of individual preferences and endowments, a technology and an ownership structure: $\mathcal{E} = (u, e, k, \theta, \mathcal{F})$. For a given economy, **a Nash-Walras equilibrium** is a vector of prices, a profile of demands and a production plan, (p, q, x, y, X, Y), such that

1. the firm maximizes profits: plan (X, Y) solves the problem
$$\max_{\tilde{X}, \tilde{Y}} p\tilde{X} + q\tilde{Y} : (\tilde{X}, \tilde{Y}) \in \mathcal{F};$$
2. every consumer is rational, given prices and the demands of other consumers: each (x^i, y^i) solves the problem
$$\max_{\tilde{x}, \tilde{y}} u^i(\tilde{x}, \tilde{y} + \sum_{j \neq i} y^j) : p\tilde{x} + q\tilde{y} \leq pe^i + qk^i + \theta^i(pX + qY);$$
3. all markets clear: $\sum_i (x^i - e^i, y^i - k^i) = (X, Y)$.

Under nonnegativity of consumption, it is immediate that the second condition in this definition is equivalent to, and can be replaced by,

2. for each i, $(x^i, \sum_j y^j)$ solves the problem
$$\max_{\tilde{x}, \tilde{y}} u^i(\tilde{x}, \tilde{y}) : \begin{cases} p\tilde{x} + q\tilde{y} \leq pe^i + q(k^i + \sum_{j \neq i} y^j) + \theta^i(pX + qY), \\ \tilde{y} \geq \sum_{j \neq i} y^j. \end{cases}$$

As before, $\mathcal{W}(\mathcal{E})$ denotes the set of Nash-Walras equilibria of $\mathcal{E}$, and $W(\mathcal{E})$ is the projection of this set into the space of prices.

A **data set** is now $(p_t, q_t, e_t, k_t, \theta_t)_{t=1}^T$, a finite sequences of prices, endowments and ownership structures. As before, all prices and endowments are taken to be strictly positive. A data set is **rationalizable** if there exist a profile of preferences u and a technology $\mathcal{F}$ such that
$$(p_t, q_t) \in W(u, e_t, k_t, \theta_t, \mathcal{F}),$$
for every observation t.

4.1 A characterization

I follow the same strategy as before, in order to show that the hypothesis of Nash-Walras equilibrium in this setting does impose testable implications: first, I obtain a (partial) characterization of rationalizability mediated by existential quantifiers; then, I argue that quantified variables can be eliminated to obtain an equivalent set of conditions on observable variables only, and use an example to show that these conditions are not tautological.

The following two lemmas are helpful for the first characterization, but are also of interest on their own. First, I obtain revealed-preference conditions, in the form of Afriat inequalities, for a rational consumer facing a choice problem with a public good.

Lemma 3. *Fix a sequence of prices, nominal incomes, m_t, demands for all commodities, (x_t, y_t), and aggregate demand for the public good by the rest of the consumers, $\bar{y}_t$, denoted by $(p_t, q_t, m_t, x_t, y_t, \bar{y}_t)_{t=1}^T$.*

1. *If there exists a continuous, strictly monotone utility function u such that each (x_t, y_t) solves the problem*

$$\max_{\tilde{x},\tilde{y}} u(\tilde{x}, \tilde{y}) : p_t\tilde{x} + q_t\tilde{y} \leq m_t + q_t\bar{y}_t \text{ and } \tilde{y} \geq \bar{y}_t,$$

   then, $p_t x_t + q_t y_t = m_t + q_t\bar{y}_t$ and there exist numbers u_t and $\lambda_t > 0$, for all t, such that

$$u_{t'} \leq u_t + \lambda_t(p_t(x_{t'} - x_t) + q_t(\max\{y_{t'}, \bar{y}_t\} - y_t)),$$

   for all t and t'.

2. *If, for all t, $p_t x_t + q_t y_t = m_t + q_t\bar{y}_t$ and $y_t \geq \bar{y}_t$, and there exist numbers u_t and $\lambda_t > 0$ such that*

$$u_{t'} \leq u_t + \lambda_t(p_t(x_{t'} - x_t) + q_t(\max\{y_{t'}, \bar{y}_t\} - y_t)),$$

   for all t and t', then there exists a continuous, monotone utility function u such that each (x_t, y_t) solves the problem

$$\max_{\tilde{x},\tilde{y}} u(\tilde{x}, \tilde{y}) : p_t\tilde{x} + q_t\tilde{y} \leq m_t + q_t\bar{y}_t \text{ and } \tilde{y} \geq \bar{y}_t.$$

If, furthermore,

$$u_{t'} \leq u_t + \lambda_t(p_t(x_{t'} - x_t) + q_t(y_{t'} - y_t)),$$

for all t and t', then all utility functions can be taken to be concave and strictly monotone.

The following lemma extends the analysis of [28] to derive testable implications of profit maximization to no-free-lunch technologies.

Lemma 4. *Fix a sequence of prices and production plans, $(p_t, q_t, X_t, Y_t)_{t=1}^T$. There exists a nonempty, closed, convex, negative monotonic no-free-lunch technology $\mathcal{F}$ such that each production plan (X_t, Y_t) solves the program*

$$\max_{\tilde{X},\tilde{Y}} p_t\tilde{X} + q_t\tilde{Y} : (\tilde{X}, \tilde{Y}) \in \mathcal{F},$$

if, and only if,

1. *for all t and t', $p_tX_{t'} + q_tY_{t'} \leq p_tX_t + q_tY_t$;*

2. *for some $\rho \in \mathbb{R}^L_{++}$ and some $\varphi \in \mathbb{R}_{++}$, $\rho X_t + \varphi Y_t \leq 0$ for all t.*

Now, given a data set, $(p_t, q_t, e_t, k_t)_{t=1}^T$, define the following condition (see [6] and [12]).

Condition 2. *There exist vectors x_t^i and X_t, and real numbers y_t^i, u_t^i, Y_t and λ_t^i, for every individual i and every observation t, such that*

1. *for all i and t, $x_t^i \geq 0$, $y_t^i \geq 0$ and $\lambda_t^i > 0$;*

2. *for all i and t, $p_tx_t^i + q_ty_t^i = p_te_t^i + q_tk_t^i + \theta_t^i(p_tX_t + q_tY_t)$;*

3. *for all i, t and t', either*

$$u_{t'}^i \leq u_t^i + \lambda_t^i(p_t(x_{t'}^i - x_t^i) + q_t(\sum_j y_{t'}^j - \sum_j y_t^j)),$$

or

$$u_{t'}^i \leq u_t^i + \lambda_t^i(p_t(x_{t'}^i - x_t^i) - q_ty_t^i);$$

4. *for all* t *and* t', $p_t X_{t'} + q_t Y_{t'} \leq p_t X_t + q_t Y_t$;

5. *for all* t, $\sum_i (x_t^i, y_t^i) = \sum_i (e_t^i, k_t^i) + (X_t, Y_t)$.

Define also the following, stronger, condition.

Condition 3. *There exist vectors* x_t^i, X_t *and* ρ, *and real numbers* y_t^i, u_t^i, Y_t, φ *and* λ_t^i, *for every individual* i *and every observation* t, *that satisfy items 1, 2, 4 and 5 in condition 2 and, also,*

3'. *for all* i, t *and* t',

$$u_{t'}^i \leq u_t^i + \lambda_t^i (p_t (x_{t'}^i - x_t^i) + q_t (\sum_j y_{t'}^j - \sum_j y_t^j));$$

6. $\rho \gg 0$, $\varphi > 0$ *and for all* t, $\rho X_t + \varphi Y_t \leq 0$.

These conditions provide a (weak) characterization of rationalizable data sets, which is mediated by existential quantifiers; the characterization is similar in spirit to lemma 2.

Lemma 5. *Conditions 2 and 3 offer a (weak) characterization of rationalizability with public goods:*

1. *if a data set is rationalizable by a nonempty technology and a profile of continuous, strictly monotone preferences, then it satisfies condition 2;*

2. *if a data set satisfies condition 2, then it is rationalizable by a nonempty, closed, convex and negative monotonic technology, and a profile of continuous monotone preferences;*

3. *if a data set satisfies condition 3, then it is rationalizable by a nonempty, closed, convex, negative monotonic, no-free-lunch technology, and a profile of continuous, strictly monotone and concave preferences.*

4.2 The test

The derivation of the test now follows, again, by eliminations of the quantifiers of the characterization.

Proposition 4. *In the case of public goods, rationalizability is testable using a finite set of inequalities on data. Formally, for any finite sequence consisting of profiles of strictly positive individual endowments and an ownership structure, $d = (e_t, k_t, \theta_t)_{t=1}^T$, there exist semialgebraic sets of sequences of strictly positive prices for all commodities, Δ_d^C and Δ_d, with $\Delta_d^C \subseteq \Delta_d$, such that:*

1. *if $(p_t, q_t, e_t, k_t, \theta_t)_{t=1}^T$ is rationalizable by a nonempty technology and a profile of continuous, strictly monotone preferences, then $(p_t, q_t)_{t=1}^T \in \Delta_d$;*
2. *if $(p_t, q_t)_{t=1}^T \in \Delta_d$, then $(p_t, q_t, e_t, k_t, \theta_t)_{t=1}^T$ is rationalizable by a nonempty, closed, convex and negative monotonic technology, and a profile of continuous, monotone preferences;*
3. *if $(p_t, q_t)_{t=1}^T \in \Delta_d^C$, then $(p_t, q_t, e_t, k_t, \theta_t)_{t=1}^T$ is rationalizable by a nonempty, closed, convex, negative monotonic no-free-lunch technology, and a profile of strictly monotone and concave preferences.*

Also, there exist d for which Δ_d is a proper subset of $(\mathbb{R}^L_{++} \times \mathbb{R}_{++})^T$.

Proof. As before, notice that given $(e_t, k_t, \theta_t)_{t=1}^T$, the systems defined by conditions 2 and 3 are finite sets of polynomial inequalities, so the sets of their solutions are semialgebraic. Let Δ_d be the projection, into the space of sequences of prices, of the set of solutions to the system defined by condition 2, and let Δ_d^C be the projection of the set defined by condition 3. By the Tarski-Seidenberg theorem, these sets are semialgebraic. The proposition then follows from lemma 5 and example 2 below. □

4.3 Non-rationalizable data

The argument that the test is nontautological is given, once again, by an example of data that cannot be rationalized under the hypothesis of public goods. Consider the following case.

Example 2. *Suppose that $I = 2$ and $L = 1$. Suppose that a data set includes the following two observations:*

$$\begin{array}{llll} e_1^1 = 9, & e_2^1 = 1, & e_1^2 = 1, & e_2^2 = 1, \\ k_1^1 = 1, & k_2^1 = 9, & k_1^2 = 1, & k_2^2 = 1, \\ \theta_1^1 = 1, & \theta_2^1 = 1, & \theta_1^2 = 0, & \theta_2^2 = 0, \\ p_1 = 100, & p_2 = 1, & q_1 = 1, & q_2 = 100. \end{array}$$

Suppose that a data set is rationalized by a nonempty technology and a profile of continuous, strictly monotone preferences. Let $x_t^i \geq 0$ and $y_t^i \geq 0$, for $i, t = 1, 2$, be the associated consumption levels, and let (X_t, Y_t), for $t = 1, 2$, be the associated production plans. Profit maximization, monotonicity of preferences and market clearing immediately imply the following conditions:

$$\begin{aligned} p_1 X_1 + q_1 Y_1 &\geq p_1 X_2 + q_1 Y_2, && (1) \\ p_2 X_2 + q_2 Y_2 &\geq p_2 X_1 + q_2 Y_1, && (2) \\ p_1 x_1^1 + q_1 y_1^1 &= p_1(e_1^1 + X_1) + q_1(k_1^1 + Y_1), && (3) \\ p_2 x_2^1 + q_2 y_2^1 &= p_2(e_2^1 + X_2) + q_2(k_2^1 + Y_2), && (4) \\ p_1 x_1^2 + q_1 y_1^2 &= p_1 e_1^2 + q_1 k_1^2, && (5) \\ p_2 x_2^2 + q_2 y_2^2 &= p_2 e_2^2 + q_2 k_2^2, && (6) \\ x_1^1 + x_1^2 &= e_1^1 + e_1^2 + X_1, && (7) \\ x_2^1 + x_2^2 &= e_2^1 + e_2^2 + X_2, && (8) \\ y_1^1 + y_1^2 &= k_1^1 + k_1^2 + Y_1, && (9) \\ y_2^1 + y_2^2 &= k_2^1 + k_2^2 + Y_2. && (10) \end{aligned}$$

By direct computation (see appendix A1), these conditions imply, furthermore, that

$$\begin{aligned} p_1 x_2^1 + q_1(y_2^1 + y_2^2) &< p_1(e_1^1 + X_1) + q_1(k_1^1 + Y_1) + q_1 y_1^2, && (11) \\ p_2 x_1^1 + q_2(y_1^1 + y_1^2) &< p_2(e_2^1 + X_2) + q_2(k_2^1 + Y_2) + q_2 y_2^2, && (12) \\ p_1 x_2^1 &< p_1 e_1^1 + q_1 k_1^1 + p_1 X_1 + q_1 Y_1, && (13) \\ p_2 x_1^1 &< p_2 e_2^1 + q_2 k_2^1 + p_2 X_2 + q_2 Y_2. && (14) \end{aligned}$$

But this is impossible, as it contradicts individual rationality for individual 1.

Indeed, since each $(x_t^1, y_t^1 + y_t^2)$ solves the problem

$$\max_{\tilde{x},\tilde{y}} u^1(\tilde{x},\tilde{y}) : \begin{cases} p_t\tilde{x} + q_t\tilde{y} \leq p_t(e_t^1 + X_t) + q_t(k_t^i + Y_t) + q_t y_t^2 \\ \tilde{y} \geq y_t^2, \end{cases}$$

it must also solve the problem

$$\max_{\tilde{x},\tilde{y}} u^1(\tilde{x},\tilde{y}) : \begin{cases} p_t\tilde{x} + q_t\tilde{y} \leq p_t(e_t^1 + X_t) + q_t(k_t^i + Y_t) + q_t y_t^2 \\ p_t\tilde{x} \leq p_t(e_t^1 + X_t) + q_t(k_t^i + Y_t). \end{cases}$$

But then, it follows from [12, proposition 1] that the following acyclicity condition must be observed: for $t \neq t'$, if

$$p_t x_{t'}^1 + q_t(y_{t'}^1 + y_{t'}^2) \leq p_t(e_t^1 + X_t) + q_t(k_t^1 + Y_t) + q_t y_t^2$$

and

$$p_t x_{t'}^1 \leq p_t(e_t^1 + X_t) + q_t(k_t^1 + Y_t)$$

then, either

$$p_{t'} x_t^1 + q_{t'}(y_t^1 + y_t^2) \geq p_{t'}(e_{t'}^1 + X_{t'}) + q_{t'}(k_{t'}^1 + Y_{t'}) + q_{t'} y_{t'}^2,$$

or

$$p_{t'} x_t^1 \geq p_{t'}(e_{t'}^1 + X_{t'}) + q_{t'}(k_{t'}^1 + Y_{t'}).$$

It follows, once again, that the assumption that the externality is in the form of a produced public good gives testable restrictions on prices, given the observation of real endowments and of the ownership structure of the economy.

In a production economy, one might expect refutability to fail because: (i) if profits are not observed, individual incomes are undetermined and hence the restrictions imposed by individual rationality may be weakened; and (ii) in the presence of production, nonnegativity constraints on consumption are less informative: production may transform endowments so as to allow consumption allocations *outside* the original Edgeworth boxes. Introducing production, however, adds profit maximization as an element of the model and this additional structure may as well be a source of refutability for the hypothesis.

On the other hand, the assumption of a public good yields sufficient structure to impose empirical implications on individually-rational behavior. While it is true that the decisions of other agents in the economy induce changes in the utility function of each individual in terms of the variables that she chooses, the fact that these effects have the public-good structure implies that one can see the problem as the one of an individual who always maximizes the same, given utility function, only subject to not-necessarily-linear budget constraints.

5 Further information

The last results show that further restrictions may (but need not) yield testability for the competitive hypothesis under externalities. Alternatively, I now show that if one observes individual demands for the externality some restrictions arise even under the more general classes of preferences. I consider again the simpler case of an exchange economy with general types of externalities.

In this case, some concepts need to be redefined, although I keep the existing notation, for simplicity. Now, a data set is $(p_t, q_t, e_t, k_t, y_t)_{t=1}^T$, a finite sequence that includes information on y_t^i for every individual and observation. I assume, again, that all observed prices, endowments and consumptions are strictly positive, and that $\sum_i (y_t^i - k_t^i) = 0$, for all t, and $p_t e_t^i + q_t(k_t^i - y_t^i) \geq 0$, for all i and all t.

Consistently, $W(\mathcal{E})$ now denotes the projection of the set of Nash-Walras equilibria of economy $\mathcal{E}$ into the space of prices and individual consumptions of the externality. Also, a data set is rationalized by a profile of preferences u, if $(p_t, q_t, y_t) \in W(u, e_t, k_t)$, for every observation t.

5.1 A characterization and the test

As before, given a data set, define the following condition.

Condition 4. *There exist vectors x_t^i and ρ_t^i, and numbers u_t^i, θ_t^i and λ_t^i, for each individual i and each observation t, such that:*

1. *for every i and t, $x_t^i \geq 0$, $\rho_t^i \gg 0$, $\theta_t^i > 0$ and $\lambda_t^i > 0$*

2. *for every i and t, $\rho_t^i = \lambda_t^i p_t$ and $\theta_t^i = \lambda_t^i q_t$;*

3. *for every i and t, $p_t x_t^i = p_t e_t^i + q_t(k_t^i - y_t^i)$;*

4. *for every i and t, and for all t' such that $y_{t'}^{\neg i} = y_t^{\neg i}$,*

$$u_{t'}^i \leq u_t^i + \rho_t^i(x_{t'}^i - x_t^i) + \theta_t^i(y_{t'}^i - y_t^i);$$

5. *for every t, $\sum_i x_t^i = \sum_i e_t^i$.*

The following lemma does not give tests of rationalizability, but will yield such test upon quantifier elimination.

Lemma 6. *Condition 4 characterizes rationalizability if demands for the externality are observed:*

1. *if a data set is rationalizable by a profile of continuous preferences that are locally nonsatiated in own consumption, it satisfies condition 4.*

2. *if a data set satisfies condition 4, then it is rationalizable by a profile of continuous, concave and strongly monotone preferences.*

Now, the testability result is the following proposition.

Proposition 5. *Rationalizability under observation of demands for the externality is testable using a finite set of inequalities on data. Formally, for any finite sequence of profiles of strictly positive individual endowments and demands for the externality, $d = (e_t, k_t, y_t)_{t=1}^T$, there exists a semialgebraic set of sequences of strictly positive prices for all commodities, Δ_d, such that:*

1. *if $(p_t, q_t, e_t, k_t, y_t)_{t=1}^T$ is rationalizable by a profile of continuous preferences that satisfy concavity and strong monotonicity on own demands, then $(p_t, q_t)_{t=1}^T \in \Delta_d$;*

2. *if $(p_t, q_t)_{t=1}^T \in \Delta_d$, then $(p_t, q_t, e_t, k_t, y_t)_{t=1}^T$ is rationalizable by a profile of continuous, concave and strongly monotone preferences.*

Also, there exist d for which Δ_d is a proper subset of $(\mathbb{R}_{++}^L \times \mathbb{R}_{++})^T$.

Proof. (The argument is similar to the proof of propositions 3 and 4.) Condition 4 defines a semialgebraic set, so its projection into the space of prices is semialgebraic as well. Lemma 6 then yields statements 1 and 2, while the fact that there are d for which $\Delta_d \subsetneq (\mathbb{R}_{++}^L \times \mathbb{R}_{++})^T$ follows from example 3 below. □

5.2 Non-rationalizable data

Example 3. *Suppose that $I = L = 2$. The information of the data set with partial observation includes the following two observations:*

$$\begin{array}{cccc} e_1^1 = (1,4) & e_2^1 = (4,1) & e_1^2 = (2,1) & e_2^2 = (1,2) \\ k_1^1 = 1 & k_2^1 = 0.5 & k_1^2 = 1 & k_2^2 = 1.5 \\ p_1 = (1,10) & p_2 = (10,1) & q_1 = 1 & q_2 = 2 \\ y_1^1 = 0 & y_2^1 = 0 & y_1^2 = 2 & y_2^2 = 2 \end{array}$$

Suppose that the data set is rationalized by preference profile (u^1, u^2). Since $y_1^2 = y_2^2$, it follows that consumer 1 maximizes the same utility function, $u^1(\cdot,\cdot,y_1^2)$, at both observations. Let x_t^1 be such that (x_t^1, y_t^1) solves the problem

$$\max_{x^1,y^1} u^1(x^1, y^1, y_t^2) : p_t(x^1 - e_t^1) + q_t(y^1 - k_t^1) \leq 0.$$

Define $X_1 = \{x | x_1 \in [0,3], x_2 = 4.2 - 0.1x_1\}$ and $X_2 = \{x | x_1 \in [3.9, 4.2], x_2 = 42 - 10x_1\}$. Since $p_1 e_1^1 + q_1(k_1^1 - y_1^1) = 42 = p_2 e_2^1 + q_2(k_2^1 - y_2^1)$, $e_1^1 + e_1^2 = (3,5)$ and $e_2^1 + e_2^2 = (5,3)$, it follows that $x_1^1 \in X_1$ and $x_2^1 \in X_2$. Since $X_1 \cap X_2 = \varnothing$, then $x_1^1 \neq x_2^1$. Since $x_1^1 \in X_1$, then

$$p_2 x_1^1 + q_2 y_1^1 = 9.9 x_{1,1}^1 + 4.2 \leq 9.9(3) + 4.2 < 42 = p_2 e_2^1 + q_2 k_2^1,$$

whereas, since $x_2^1 \in X_2$, by a similar argument, $p_1 x_2^1 + q_1 y_2^1 < p_1 e_1^1 + q_1 k_1^1$. But this is impossible, since it violates the weak axiom of revealed preferences.

It follows from the proposition that, upon observation of prices, individual endowments and individual demands for the externality, the hypothesis of Nash-Walras equilibrium is testable. However, it is important to notice how mild the (exhaustive) restrictions are: any feasible data set generated randomly with non-atomic measures is rationalizable with full probability.

6 Concluding remarks

The results obtained here show that the assumption that there exist consumption externalities significantly weakens the testability results obtained by [6]. Under basic assumptions on individual preferences, the equilibrium hypothesis imposes no restrictions on finite subsets of the equilibrium manifold. Intuitively, this occurs because the presence of externalities allows for the construction of utility functions whose cross-sections with respect to own consumption are maximized by each agent at only one of the observations; immediately, revealed preference arguments are vacuous, and the tensions existing between individual rationality and market clearing, which yields the results in [6], disappears. Importantly, this is the case even when one imposes the hypothesis that all individual preferences exhibit strategic complementarity in the consumption of the externality commodity. This result contrasts with the literature on monotone comparative statics, and with the application of this literature to abstract games.

Stronger restrictions on unobservables, or the observation of individual consumptions of the externality, are required to restore testability. Separability of preferences in own consumption would yield the results trivially: if consumption of the externality commodity by other agents of the economy is assumed to not affect each individual's behavior, the restrictions obtained for the case when there are no externalities immediately hold. What is interesting, though, is how much less separability needs to be assumed if some restrictions are to be maintained. The result obtained here shows that it suffices for preferences to be weakly separable in two of the commodities traded in competitive markets (including, possibly, the externality commodity itself).

Another important case corresponds to the hypothesis that the externality commodity is a public good. Here, I have considered the case in which production takes place, and have, for simplicity, assumed the existence of only one firm. In terms of data, I have assumed that the ownership structure over this aggregate technology is observed as well. The results show that the structure imposed by the principles of profit maximization and individual rationality, along with the structure imposed by the hypothesis of the public good, suffice to yield testable implications on prices, endowments and stock distribution.

Finally, I have argued that for general classes of preferences, under observation of individual demands for the externality, some nontautological restrictions do exist. Inspection of these conditions, however, shows that the exhaustive set of restrictions imposed is extremely mild.

Some of the results obtained here are easy to generalize. For example, at the cost of some technical complication, one can substitute the assumption that the externalities come from consumption of some commodity, for one in which they stem from some abstract action on continuous individual sets. In this case, of course, there are no prices attached to the externality, but arbitrary bounds may be imposed instead. The results obtained here extend to that setting, under the proviso that preferences be restricted to be locally Lipschitzian. The case considered here, consumption externalities, is then a particular case of that abstract setting, although some of the results obtained here can only be derived from the general case for compact arbitrary subsets of the domain.

Appendix A1: Proofs of the claims

A preliminary fact is the following:

Claim 1. *$X_1 \geq X_2 \geq -2$ and $Y_2 \geq Y_1 \geq -2$. Moreover, $y_2^1 + y_2^2 > k_1^1 + y_1^2$.*

Proof. From equations 1 and 2, by direct substitution,

$$100X_1 \geq 100X_2 + (Y_2 - Y_1) \geq 100X_2 + \frac{1}{100}(X_1 - X_2),$$

so $X_1 \geq X_2$. Also, by equation 8, given nonnegativity, $X_2 = x_2^1 + x_2^2 - e_2^1 - e_2^2 \geq -e_2^1 - e_2^2 = -2$. That $Y_2 \geq Y_1 \geq -2$ is proven similarly. Now, equation 10 and the previous inequality imply that

$$y_2^1 + y_2^2 = 10 + Y_2 \geq 10 + Y_1 > 3 + Y_1 \geq 3 + Y_1 - y_1^1.$$

But then, by direct substitution and equation 9,

$$y_2^1 + y_2^2 > (2k_1^1 + k_1^2) + Y_1 - y_1^1 = k_1^1 + y_1^2.$$

□

Claim 2. *$p_1 x_2^1 + +q_1(y_2^1 + y_2^2) < p_1(e_1^1 + X_1) + q_1(k_1^1 + Y_1) + q_1 y_1^2$.*

Proof. By direct substitution and equation 4,

$$\begin{aligned} p_1 x_2^1 + q_1(y_2^1 + y_2^2) &= 100x_2^1 + y_2^1 + y_2^2 \\ &= 100x_2^1 + \frac{1}{100}(901 + X_2 + 100Y_2 - x_2^1) + y_2^2 \\ &= 9.01 + \frac{1}{100}(X_2 + 100Y_2) + (100 - \frac{1}{100})x_2^1 + y_2^2 \\ &= 9.01 + \frac{1}{100}(X_2 + 100Y_2) \\ &\quad + (100 - \frac{1}{100})(e_2^1 + e_2^2 + X_2 - x_2^2) + y_2^2, \end{aligned}$$

where the last equality comes from equation 8. By direct substitution, nonnegativity and equation 6, then,

$$\begin{aligned} p_1 x_2^1 + q_1(y_2^1 + y_2^2) &= 208.99 + 100X_2 + Y_2 - (100 - \frac{1}{100})x_2^2 + y_2^2 \\ &\geq 208.99 + 100X_2 + Y_2 + y_2^2 \\ &= 208.99 + p_1 X_2 + q_1 Y_2 + \frac{p_2 e_2^2 + q_2 k_2^2 - p_2 x_2^2}{q_2} \\ &\leq 208.99 + p_1 X_2 + q_1 Y_2 + \frac{p_2 e_2^2 + q_2 k_2^2}{q_2} \\ &= 211 + p_1 X_2 + q_1 Y_2. \end{aligned}$$

By equation 1, then,

$$\begin{aligned}
p_1 x_2^1 + q_1(y_2^1 + y_2^2) &\le 211 + p_1 X_1 + q_1 Y_1 \\
&< 901 + p_1 X_1 + q_1 Y_1 \\
&= p_1 e_1^1 + q_1 k_1^1 + p_1 X_1 + q_1 Y_1 \\
&\le p_1 e_1^1 + q_1 k_1^1 + p_1 X_1 + q_1 Y_1 + q_1 y_1^2,
\end{aligned}$$

where the last inequality comes from nonnegativity. □

Claim 3. $p_2 x_1^1 + +q_2(y_1^1 + y_1^2) < p_2(e_2^1 + X_2) + q_2(k_2^1 + Y_2) + q_2 y_2^2$.

Proof. By direct substitution, using equations 3 and 9,

$$\begin{aligned}
p_2 x_1^1 + q_2(y_1^1 + y_1^2) &= x_1^1 + 100(y_1^1 + y_1^2) \\
&= \frac{p_1(e_1^1 + X_1) + q_1(k_1^1 + Y_1 - y_1^1)}{p_1} + 100(y_1^1 + y_1^2) \\
&= 9.01 + \frac{1}{100}(100X_1 + Y_1) + (100 - \frac{1}{100})y_1^1 + 100y_1^2 \\
&= 9.01 + \frac{1}{100}(100X_1 + Y_1) \\
&\quad +(100 - \frac{1}{100})(2 + Y_1 - y_1^2) + 100y_1^2 \\
&= 208.99 + X_1 + 100Y_1 + \frac{1}{100}y_1^2 \\
&= 208.99 + p_2 X_1 + q_2 Y_1 + \frac{1}{100}\frac{p_1(e_1^2 - x_1^2) + q_1 k_1^2}{q_1},
\end{aligned}$$

where the last equality comes from equation 5. By equation 2 and nonnegativity,

$$\begin{aligned}
p_2 x_1^1 + q_2(y_1^1 + y_1^2) &\le 208.99 + p_2 X_2 + q_2 Y_2 + \frac{1}{100}\frac{p_1 e_1^2 + q_1 k_1^2}{q_1} \\
&= 211 + p_2 X_2 + q_2 Y_2 \\
&< 901 + p_2 X_2 + q_2 Y_2 \\
&\le 901 + p_2 X_2 + q_2 Y_2 + q_2 y_2^2 \\
&= p_2 e_2^1 + q_2 k_2^1 + p_2 X_2 + q_2 Y_2 + q_2 y_2^2.
\end{aligned}$$

□

Claim 4. $p_1 x_2^1 < p_1 e_1^1 + q_1 k_1^1 + p_1 X_1 + q_1 Y_1$.

Proof. By claim 2,

$$p_1 x_2^1 + q_1(y_2^1 + y_2^2) < p_1(e_1^1 + X_1) + q_1(k_1^1 + Y_1) + q_1 y_1^2.$$

By claim 1, $y_2^1 + y_2^2 > k_1^1 + y_1^2$, so it must be that

$$p_1 x_2^1 < p_1(e_1^1 + X_1) + q_1 Y_1 < p_1(e_1^1 + X_1) + q_1(k_1^1 + Y_1).$$

□

Claim 5. $p_2 x_1^1 < p_2 e_2^1 + q_2 k_2^1 + p_2 X_2 + q_2 Y_2$.

Proof. By direct substitution and equations 3 and 9,

$$\begin{aligned} p_2 x_1^1 &= x_1^1 \\ &= \frac{901 + 100X_1 + Y_1 - y_1^1}{100} \\ &= 9.01 + X_1 + \frac{1}{100}(y_1^2 - k_1^1 - k_1^2) \\ &= 8.99 + X_1 + \frac{1}{100} y_1^2 \\ &= 8.99 + X_1 + \frac{1}{100}(p_1(e_1^2 - x_1^2) + q_1 k_1^2), \end{aligned}$$

where the last equality comes from equation 5. By nonnegativity and direct substitution,

$$\begin{aligned} p_2 x_1^1 &\leq 8.99 + X_1 + \frac{1}{100}(p_1 e_1^2 + q_1 k_1^2) \\ &= 11 + X_1 \\ &< 701 + X_1. \end{aligned}$$

Since, by claim 1, $Y_1 \geq -2$, using equation 2, it follows that

$$\begin{aligned} p_2 x_1^1 &< 901 + X_1 + 100 Y_1 \\ &= 901 + p_2 X_1 + q_2 Y_1 \\ &\leq 901 + p_2 X_2 + q_2 Y_2 \\ &= p_2 e_2^1 + q_2 k_2^1 + p_2 X_2 + q_2 Y_2. \end{aligned}$$

□

Appendix A2: Proofs the lemmata

Proof of lemma 1: Notice that, since T is finite, if δ is defined at some pass through the algorithm, then $\delta > 0$. The first three properties are immediate. For property 4, it suffices to show that if at the t-th pass through the algorithm, it is true that

$y^i_{t'} \neq y^i_{t''}$ for every distinct $t', t'' < t$, then $y^i_{t'} \neq y^i_t$ for all $t' < t$. This is tautological if $t = 1$, and follows from step 2 if $(x^i_t, y^i_t)^I_{i=1} = (e^i_t, k^i_t)^I_{i=1}$. Now, consider $t \geq 2$ and assume that $(x^i_t, y^i_t)^I_{i=1}$ is given by steps 3-6. Consider three different cases:

1. If $t = 2$ and $\mathcal{J} = \varnothing$. Then, $k^i_2 = y^i_1$ for all i, and since $\delta > 0$, follows that $y^i_2 \neq k^i_2 = y^i_1$.

2. If $t = 2$ and $\mathcal{J} \neq \varnothing$. Then, if $1 \notin \mathcal{J}$, it is true that $y^1_1 = k^1_2$, and, since $\delta > 0$, it follows that $y^1_2 = k^1_2 - \delta = y^1_1 - \delta \neq y^1_1$. Else, $1 \in \mathcal{J}$ and one has that if $y^1_1 = y^1_2$, then, since $y^1_2 = k^1_2 - \delta$, so it would follow that

$$|y^1_1 - k^1_2| = \delta \leq \frac{\epsilon}{2} < \epsilon \leq |y^1_1 - k^1_2|,$$

an obvious contradiction. On the other hand, for each $i \in \mathcal{I} \setminus (\mathcal{J} \cup \{1\})$, it is true that $y^i_1 = k^i_2$, and, since $\delta > 0$, it follows that

$$y^i_2 = k^i_2 + \frac{\delta}{I-1} = y^i_1 + \frac{\delta}{I-1} \neq y^i_1.$$

Finally, for each $i \in \mathcal{J} \setminus \{1\}$, if one had that $y^i_1 = y^i_2$, then, since $y^i_2 = k^i_2 + \frac{\delta}{I-1}$ and $\delta > 0$, it would follow that

$$|y^i_1 - k^i_2| = \frac{\delta}{I-1} \leq \delta \leq \frac{\epsilon}{2} < \epsilon \leq |y^i_1 - k^i_2|,$$

again, a contradiction.

3. If $t \geq 3$. In this case, by the induction assumption, $\mathcal{J} = \mathcal{I}$, from where, if there exists $t' \leq t-1$ with $y^1_{t'} = y^1_t$, then $y^1_t = k^1_t - \delta$ and $\delta > 0$ would imply that $y^1_{t'} \neq k^1_t$ and

$$|y^1_{t'} - k^1_t| = \delta \leq \frac{\epsilon}{2} < \epsilon \leq \min_{t'' \leq t-1 : y^1_{t''} \neq k^1_t} \{|y^1_{t''} - k^1_t|\},$$

an obvious contradiction. Similarly, if for some $i \neq 1$, one had that for some $t' \leq t-1$ it is true that $y^i_{t'} = y^i_t$, then, $y^i_t = k^i_t + \frac{\delta}{I-1}$ and $\delta > 0$ imply that $y^i_{t'} \neq k^i_t$ and

$$|y^i_{t'} - k^i_t| = \frac{\delta}{I-1} \leq \delta \leq \frac{\epsilon}{2} < \epsilon \leq \min_{t'' \leq t-1 : y^i_{t''} \neq k^i_t} \{|y^i_{t''} - k^i_t|\},$$

again a contradiction.

$\square$

Proof of lemma 2. For part 1, let $(x_t^i, y_t^i) \gg 0$ solve the problem

$$\max_{\tilde{x},\tilde{y}} u^i(\tilde{x}, \tilde{y}, y_t^{\neg i}) : p_t\tilde{x} + q_t\tilde{y} \leq p_t e_t^i + q_t k_t^i.$$

and satisfy item 6 of condition 1. Item 3 of the condition follows from monotonicity of u^i.

Since each u^i is smoothly, weakly separable, there exist smooth functions V^i and U^i, such that (i) V^i is concave and strongly increasing; (ii) U^i is concave and strictly increasing in its first two arguments; and (iii) everywhere, $U^i(V^i(x^i), y^i, y^{\neg i}) = u^i(x^i, y^i, y^{\neg i})$.

Individual rationality and strict monotonicity of U^i in its first argument imply that, for all t, x_t^i solves

$$\max_{\tilde{x}} V^i(\tilde{x}) : p_t\tilde{x} \leq p_t e_t^i + q_t(k_t^i - y_t^i).$$

By the Kühn-Tucker theorem, it follows that there exist $\lambda_t^i > 0$ and $\mu_t^i > 0$, for all i and t, such that $\partial_{(x^i,y^i)} u^i(x_t^i, y_t^i, y_t^{\neg i}) = \mu_t^i(p_t, q_t)$ and $\partial V^i(x_t^i) = \lambda_t^i q_t$. Item 1 of condition 1 is satisfied, while item 2 is immediate if we define $\rho_t^i = \lambda_t^i p_t$ and $\theta_t^i = \mu_t^i q_t$.

By concavity of V^i, it follows that $V^i(x^i) \leq V^i(x_t^i) + \theta_t^i(x^i - x_t^i)$, for all y^i. By the chain rule,

$$\partial_{V^i} U^i(V^i(x_t^i), y_t^i, y_t^{\neg i}) = (\partial V^i(x_t^i))^{-1} \partial_{x^i} u^i(x_t^i, y_t^i, y_t^{\neg i}) = \frac{\mu_t^i p_{t,1}}{\lambda_t^i p_{t,1}} = \frac{\mu_t^i}{\lambda_t^i},$$

so it follows, by convexity of U^i, that

$$U^i(V^i, y^i, y_t^{\neg i}) \leq U^i(V^i(x_t^i), y_t^i, y_t^{\neg i}) + \theta_t^i(y^i - y_t^i) + \frac{\mu_t^i}{\lambda_t^i}(V^i - V^i(x_t^i)),$$

for all V^i and y^i. Defining $v_t^i = V^i(x_t^i)$ and $u_t^i = U^i(V_t^i, y_t^i, y_t^{\neg i})$, for all i and t, yields items 4 and 5 of condition 1, and completes the proof of the first part.

For the second part, since the data satisfy condition 1, fix x_t^i, ρ_t^i, u_t^i, v_t^i, θ_t^i, λ_t^i and μ_t^i, for all individual i and observation t, that satisfy the six items of the condition.

Fix an individual i. I first claim that there exist real numbers w_t^i and vectors ν_t^i, for all observation t, such that

$$-w_t^i + w_{t'}^i + \nu_t^i(y_t^{\neg i} - y_{t'}^{\neg i}) \leq \frac{\mu_t^i}{\lambda_t^i}(v_{t'}^i - v_t^i) + \theta_t^i(y_{t'}^i - y_t^i),$$

for all t and t'. If this were not the case, then, by the Theorem of the Alternative for systems of weak inequalities, [22, §22.1], there would exist nonnegative numbers $\alpha_{t,t'}$, for all t and all $t' \neq t$, such that

i. for all t, $\sum_{t' \neq t} \alpha_{t,t'} = \sum_{t' \neq t} \alpha_{t',t}$;

ii. for all t, for all $j \neq i$, $\sum_{t' \neq t} \alpha_{t,t'}(y_t^j - y_{t'}^j) = 0$;

iii. $\sum_t \sum_{t' \neq t} \alpha_{t,t'}(\frac{\mu_t^i}{\lambda_t^i}(v_{t'}^i - v_t^i) + \theta_t^i(y_{t'}^i - y_t^i)) < 0$.

For notational simplicity, define $\alpha_{t,t} = 0$ for all t. Fix $j \neq i$. Let $\mathcal{T}_1^j = \{t | y_{t'}^j \leq y_t^j, \text{ all } t'\}$. Item ii implies that for all $t \in \mathcal{T}_1^j$ and all $t' \notin \mathcal{T}_1^j$, $\alpha_{t,t'} = 0$. Then, from item i,

$$\sum_{t \in \mathcal{T}_1^j} \sum_{t'=1}^{T} \alpha_{t',t} = \sum_{t \in \mathcal{T}_1^j} \sum_{t'=1}^{T} \alpha_{t,t'} = \sum_{t \in \mathcal{T}_1^j} \sum_{t' \in \mathcal{T}_1^j} \alpha_{t,t'} = \sum_{t \in \mathcal{T}_1^j} \sum_{t' \in \mathcal{T}_1^j} \alpha_{t',t},$$

so it follows that for all $t \in \mathcal{T}_1^j$ and all $t' \notin \mathcal{T}_1^j$, $\alpha_{t',t} = 0$. Letting $\mathcal{T}_2^j = \{t \notin \mathcal{T}_1^j | y_{t'}^j \leq y_t^j, \text{ all } t' \notin \mathcal{T}_1^j\}$, a similar argument yields, from ii, that for all $t \in \mathcal{T}_2^j$ and all $t' \notin \mathcal{T}_2^j$, $\alpha_{t,t'} = 0$, and, then, from i, that $\alpha_{t',t} = 0$ for all $t \in \mathcal{T}_2^j$ and all $t' \notin \mathcal{T}_2^j$. We can follow recursively, to obtain that $\alpha_{t,t'} = 0$ whenever $y_{t'}^j \neq y_t^j$. Doing the same for all $j \neq i$, it follows that $\alpha_{t,t'} = 0$ whenever $y_{t'}^{\neg i} \neq y_t^{\neg i}$. But, then, item iii implies that there exists some $y^{\neg i}$ such that

$$\sum_{t: y_t^{\neg i} = y^{\neg i}} \sum_{t': y_t^{\neg i} = y^{\neg i}} \alpha_{t,t'}(\frac{\mu_t^i}{\lambda_t^i}(v_{t'}^i - v_t^i) + \theta_t^i(y_{t'}^i - y_t^i)) < 0,$$

which is impossible since, by item 4 of condition 1 and item i above,

$$\begin{aligned} \sum_{t: y_t^{\neg i} = y^{\neg i}} \sum_{t': y_{t'}^{\neg i} = y^{\neg i}} \alpha_{t,t'}(\frac{\mu_t^i}{\lambda_t^i}(v_{t'}^i - v_t^i) + \theta_t^i(y_{t'}^i - y_t^i)) &\geq \sum_{t: y_t^{\neg i} = y^{\neg i}} \sum_{t': y_{t'}^{\neg i} = y^{\neg i}} \alpha_{t,t'}(u_{t'}^i - u_t^i) \\ &= 0. \end{aligned}$$

It follows that we can find real numbers w_t^i and vectors ν_t^i such that, for all t and t',

$$w_{t'}^i \leq w_t^i + \frac{\mu_t^i}{\lambda_t^i}(v_{t'}^i - v_t^i) + \theta_t^i(y_{t'}^i - y_t^i) + \nu_t^i(y_{t'}^{\neg i} - y_t^{\neg i}).$$

Then, we can define functions $V^i(x^i) = \min_t\{V_t^i + \rho_t^i(x^i - x_t^i)\}$,

$$U^i(V, y^i, y^{\neg i}) = \min_t\{w_t^i + \frac{\mu_t^i}{\lambda_t^i}(V - v_t^i) + \theta_t^i(y^i - y_t^i) + \nu_t^i(y^{\neg i} - y_t^{\neg i})\},$$

and $u^i(x^i, y^i, y^{\neg i}) = U^i(V^i(x^i), y^i, y^{\neg i})$. Each function u^i is continuous, concave and weakly separable, and it is strictly monotone in (x^i, y^i). By construction and item 5 of condition 1, for all t, $u^i(x_t^i, y_t^i, y_t^{\neg i}) = w_t^i$, and if $p_t x^i + q_t y^i \leq p_t e_t^i + q_t k_t^i$, then, by item 3 of condition 1, also $p_t x^i + q_t y^i \leq p_t x_t^i + q_t y_t^i$, which immediately implies, using items 1 and 2 of the condition, that

$$\begin{aligned} u^i(x^i, y^i, y_t^{\neg i}) &\leq w_t^i + \frac{\mu_t^i}{\lambda_t^i}(\min_{t'}\{v_{t'}^i + \rho_{t'}^i(x^i - x_{t'}^i)\} - v_t^i) + \theta_t^i(y^i - y_t^i) \\ &\leq w_t^i + \frac{\mu_t^i}{\lambda_t^i}\rho_t^i(x^i - x_t^i) + \theta_t^i(y^i - y_t^i) \\ &= w_t^i + \mu_t^i p_t \rho_t^i(x^i - x_t^i) + \mu_t^i q_t(y^i - y_t^i) \\ &\leq w_t^i. \end{aligned}$$

It is immediate from item 6 of the condition that profile $(u^i)_{i=1}^I$ rationalizes the data. Strict monotonicity of each u^i in $y^{\neg i}$ can be obtained as in the proof of proposition 2. □

Proof of lemma 3. For part 1, define, for each t, the continuous and increasing function $g_t : \mathbb{R}^L \times \mathbb{R} \to \mathbb{R}$, by

$$g_t(x, y) = \max\{p_t x + q_t y - m_t - q_t \bar{y}_t, p_t x - m_t\}.$$

By construction, since u is strictly monotone, $p_t x + q_t y = m_t + q_t \bar{y}_t$ and it is true that each (x_t, y_t) solves the program

$$\max_{x,y} u(x, y) : p_t x + q_t y \leq m_t + q_t \bar{y}_t \text{ and } p_t x \leq m_t.$$

This means that each (x_t, y_t) solves the program $\max_{x,y} u(x, y) : g_t(x, y) \leq 0$ and $g_t(x_t, y_t) = 0$. It then follows from [12, proposition 1], that there exist numbers u_t and $\lambda_t > 0$, for all t, such that $u_{t'} \leq u_t + \lambda_t g_t(x_{t'}, y_{t'})$, for all t and t'. Since $p_t x_t + q_t y_t = m_t + q_t \bar{y}_t$ and $q_t > 0$, by direct computation,

$$\begin{aligned} g_t(x, y) &= \max\{p_t(x - x_t) + q_t(y - y_t), p_t(x - x_t) + q_t(\bar{y}_t - y_t)\} \\ &= p_t(x - x_t) + q_t(\max\{y, \bar{y}_t\} - y_t), \end{aligned}$$

which proves the result, since $\lambda_t > 0$.

For part 2, define the continuous and monotone function $u : \mathbb{R}_+^L \times \mathbb{R}_+ \to \mathbb{R}$ by

$$u(x, y) = \min_t\{u_t + \lambda_t(p_t(x - x_t) + q_t(\max\{y, \bar{y}_t\} - y_t)).$$

Fix t and suppose that $p_t x + q_t y \leq m_t + q_t \bar{y}_t$ and $y \geq \bar{y}_t$. It is immediate that $u(x, y) \leq u_t$, while, by the conditions on u_t and λ_t, $u(x_t, y_t) = u_t$. Under the extra hypotheses, concavity and monotonicity can be obtained by defining

$$u(x, y) = \min_t\{u_t + \lambda_t(p_t(x - x_t) + q_t(y - y_t))\}.$$

□

Proof of lemma 4. For necessity, the first item follows directly from profit maximization. Now, suppose that for no $(\rho, \varphi) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$ is it true that $\rho X_t + \varphi Y_t \leq 0$ for all t. It follows from [22, Theorem 22.2], again, that there exist $\alpha \in \mathbb{R}^T_+$ and $\beta \in \mathbb{R}^{L+1}_+ \setminus \{0\}$, such that

$$\sum_t \alpha_t (X_t, Y_t) = \beta > 0.$$

It follows that $\alpha > 0$ and, by convexity of $\mathcal{F}$, that

$$(X, Y) = \frac{\sum_t \alpha_t (X_t, Y_t)}{\sum_t \alpha_t} \in \mathcal{F}.$$

But, $(X, Y) = \frac{1}{\sum_t \alpha_t} \beta > 0$, contradicting no-free-lunch.

For sufficiency, as in [28], let $\mathcal{F}$ be the convex hull of the set $\cup_t ((X_t, Y_t) - \mathbb{R}^L_+ \times \mathbb{R}_+)$, which is nonempty, closed and convex. That each (X_t, Y_t) solves $\max_{(X,Y)\in\mathcal{F}} p_t X + q_t Y$ follows from [28, Theorem 2]. To see that Y satisfies no free lunch, suppose that $(X, Y) \in \mathcal{F}$ and $(X, Y) > 0$. By construction, we can find a sequence $(\tilde{X}_t, \tilde{Y}_t, \alpha_t)_{t=1}^T$ such that $(\tilde{X}_t, \tilde{Y}_t) \leq (X_t, Y_t)$ and $0 \leq \alpha_t \leq 1$, for all t, and $\sum_t \alpha_t (\tilde{X}_t, \tilde{Y}_t) = (X, Y)$. Then, by the second item,

$$0 < \rho X + \varphi Y = \rho \sum_t \alpha_t \tilde{X}_t + \varphi^t \tilde{y}^t \leq \rho \cdot \sum_{t=1}^T \alpha^t y^t = \sum_{t=1}^T \alpha^t \left(\rho \cdot y^t\right) \leq 0.$$

□

Proof of lemma 5. For part 1, fix $(u^i)_i$ and $\mathcal{F}$ that rationalize the data. There must exist x^i_t, X_t, y^i_t and Y_t such that

a. (X_t, Y_t) solves $\max_{\tilde{X},\tilde{Y}} p_t \tilde{X} + q_t \tilde{Y} : (\tilde{X}, \tilde{Y}) \in \mathcal{F}$;

b. $(x^i_t, \sum_j y^j_t)$ solves

$$\max_{\tilde{x},\tilde{y}} u^i(\tilde{x}, \tilde{y}) : \begin{cases} p_t \tilde{x} + q_t \tilde{y} \leq p_t e^i_t + q_t k^i_t + \theta^i_t (p_t X_t + q_t Y_t) + q_t \sum_{j\neq i} y^j_t \\ \tilde{y} \geq \sum_{j\neq i} y^j_t; \end{cases}$$

c. $\sum_i (x^i_t - e^i_t, y^i_t - k^i_t) = (X_t, Y_t)$.

From part 1 of lemma 3, condition b implies that

$$p_t x^i_t + q_t \sum_j y^j_t = p_t e^i_t + q_t (k^i_t + \sum_{j\neq i} y^j_t) + \theta^i_t (p_t X_t + q_t Y_t),$$

and that there exist numbers u_t^i and $\lambda_t^i > 0$, such that

$$u_{t'}^i \leq u_t^i + \lambda_t^i(p_t(x_{t'}^i - x_t^i) + q_t(\max\{\sum_j y_{t'}^j, \sum_{j\neq i} y_t^j\} - \sum_j y_t^j)),$$

for all t and t'. Items 1, 2 and 3 of condition 2 follow immediately (since $\lambda_t^i > 0$). Item 4 is follows from profit maximization (condition a), while item 5 follows immediately from market clearing (condition c).

For part 2, fix the vectors x_t^i and X_t, and the numbers y_t^i, u_t^i, Y_t and λ_t^i that yield condition 2. Notice that, by the second part of lemma 2, there exists continuous, monotone utility functions u^i such that each $(x_t^i, \sum_j y_t^j)$ solves the problem

$$\max_{\tilde{x},\tilde{y}} u^i(\tilde{x}, \tilde{y}) : \begin{cases} p_t\tilde{x} + q_t\tilde{y} \leq p_t e_t^i + q_t k_t^i + \theta_t^i(p_t X_t + q_t Y_t) + q_t \sum_{j\neq i} y_t^j \\ \tilde{y} \geq \sum_{j\neq i} y_t^j. \end{cases}$$

Also, by [28, theorem 3], there exists a nonempty, closed, convex and negative monotonic technology, $\mathcal{F}$, such that each (X_t, Y_t) solves the program

$$\max_{\tilde{X},\tilde{Y}} p_t\tilde{X} + q_t\tilde{Y} : (\tilde{X}, \tilde{Y}) \in \mathcal{F}.$$

The conclusion follows, since markets clear by condition 2.

Given condition 3, part 3 is similar, using the extra hypotheses in the second result in lemma 3, and lemma 4. □

Proof of lemma 6. Part 1 it follows from Walras's law and Afriat's theorem (see [26]), by definition of rationalizability.

The proof of part 2 is similar to the proof of the second part of lemma 2, so details can be omitted. Condition 4 implies, again by the Theorem of the Alternative, that there exist numbers w_t^i and vectors ν_t^i, for all individual and observation, such that

$$w_{t'}^i - w_t^i + \nu_t^i(y_t^{\neg i} - y_{t'}^{\neg i}) \leq \lambda_t^i(p_t(x_{t'}^i - x_t^i) + q_t(y_{t'}^i - y_t^i)),$$

for all i, t and t'. Individual preferences can be defined by

$$u^i(x^i, y^i, y^{\neg i}) = \min_t\{w_t^i + \lambda_t^i(p_t(x^i - x_t^i) + q_t(y^i - y_t^i)) + \nu_t^i(y^{\neg i} - y_t^{\neg i})\},$$

and an additive term can be used to guarantee strong monotonicity in all arguments. □

Notes

[1] See [13], for an observation about the empirical implications of the competitive equilibrium models. For the theorem, see [24], [16], [11], and, also, [17].

[2] Moreover, [15], [10], [4], [18] and [9] further showed conditions under which information on individual fundamentals can be uniquely identified from knowledge of the equilibrium manifold.

[3] For simplicity of notation, I'll assume that prices are row vectors and quantities are column vectors.

[4] That is, for every $y^{\neg i}$, function $u^i(\cdot, \cdot, y^{\neg i})$ is strongly concave and strictly monotone.

[5] See [5, §3].

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