Voting in small networks with cross-pressure*

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Abstract

We present a model of participation in elections in small networks, in which citizens suffer from cross-pressures if voting against the alternative preferred by some of their social contacts. We analyze how the existence of cross-pressures may shape voting decisions, and so, political outcomes; and how parties may exploit this effect to their interest. We characterize the strong perfect equilibria of the game and show that, in equilibrium, the social network determines which party wins the election. We also show that to dispose of the citizens better connected in the network with the other faction is not a guarantee to win the election.

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"To go against the dominant thinking of your friends, of most of the people you see every day, is perhaps the most difficult act of heroism you can perform".

Theodore H. White

1 Introduction

Empirical research on the US (Lazarsfeld et al. (1944), Simmel (1955), Campbell et al. (1960), and, more recently, Mutz (2002, 2006)) shows that people experiencing conflicts and inconsistencies between their opinions and those of their relatives, friends or co-workers, are less likely to participate in politics. Mutz (2002), in an empirical study for the US presidential and congressional elections of the 1992 and 1996, finds that the probability of voting in an election is positively related to some of the usual predictors such as political interest, partisanship, education or age. But she also finds that the decision to cast a vote is strongly negatively related to the exposition to dissonant political opinions within one's personal network. Mutz (2002) documents two theories to explain this effect. First, that cross-cutting exposure is likely to engender attitudinal ambivalence to an individual, inducing political inaction. Second, that cross-pressures arising from one's personal network can create the need to be socially accountable, and then, it may bring about uncomfortable feelings when facing a decision that does not please everybody in one's network. As a result, individuals embedded in networks that supply them with political information that challenges their views can be discouraged from involving and participating in politics.

Despite this evidence, and to the best of our knowledge, there is no model of voter turnout in which this phenomenon has been formally introduced. This paper is intended to be a first step in this direction. More precisely, our aim is to analyze how the existence of cross-pressure may shape voting decisions, and so, political outcomes; and how political parties may exploit this effect to their interest.

To this aim, we consider a finite population of citizens who are to vote between two alternatives: the one currently being implemented and a new alternative. There is a network representing the structure of friendship relationships among the citizens in the community, *i.e.*, a link between two agents represents the idea that they are close friends and not merely acquaintances. We assume that the network architecture and the preferences of the citizens are common-knowledge. The model thus fits most clearly voting situations where the number of people involved is not too large. Citizens select their preferred alternative in a winner-take-all election. Each citizens' utility depends upon the alternative implemented and the identity of the representatives running for office. The latter captures the possibility that a citizen gets utility from holding office himself or that he encounters costs from confronting ideologically a friend. The political process is modeled as a three-stage game. At stage 1 the challenger chooses a list of citizens to represent its alternative. All the citizens observe the nominations and the nominees decide whether to accept to run for office or not. If at least one nominee does not accept, no election is held and the incumbent keeps in office. Otherwise, we move to the second stage. At stage 2, the incumbent chooses a list of citizens to contest the challenger. As previously, all nominees have to accept to run for office in order to move to the third stage. At stage 3 citizens decide whether to vote or to abstain. This decision is based on the benefits of having their preferred alternative implemented and on the cross-cutting cost that a citizen may incur when he votes for other alternative rather than the one preferred by some of his friends.

A number of situations fit into our analysis of the voting game. For example, in the field of the academics, we can think of elections for the board of directors in a department (chair, vice-chair and secretary) or in a college (dean and associate deans). Similarly, the elections for the board of directors in private firms or soccer clubs work, in some cases, this way. Finally, the elections for mayor in small villages of continental Europe (*e.g.* Austria, Finland, Italy or Spain), where parties make lists of representatives to be elected, resembles, to some extend, the structure of our game.

Our stylized model delivers some stark results. We show that there always exist a (strong perfect) equilibrium and characterize the set of pure strategy (strong perfect) equilibria of the game. We then show that all the equilibria are outcome equivalent in terms of which alternative gets into office. Thus, in equilibrium, the primitives of the model determine which group wins the election. We also provide some results on how parties exploit the cross-cutting effect to their interest. In particular, we show that if parties choose one representative each, it is optimal for a party to nominate the citizen who has more friends in the other faction, therefore, who can generate the highest amount of cross-pressure. However, if they have to choose more than one representative each, more complex strategic considerations enter into the game and the previous rule is not always optimal. The model thus predicts that, in general, to dispose of the citizens better connected in the network with the other faction is not a guarantee to win the election. In this complex environment, we are able to bound the set of lists that parties may use in equilibrium, which simplifies the process of identification of the unique equilibrium outcome in terms of which alternative gets into office. Finally, we devote some space to analyze the last mover advantage and to study the equilibria of the game for low and high values of the cross-cutting cost.

Our model fits into the literature on voter turnout, pioneered by Riker and Ordeshook (1968), Ledyard (1981, 1984) and Palfrey and Rosenthal (1983, 1985), who model participation in elections as an individual activity. These models predict low voter turnout in costly elections, a prediction that contradicts mass participation. The consensus that seem to hold today is, however, that in order to generate voter turnout in costly elections, participation has to be understood as a group activity.[1] These "group-based" models of turnout, such as Uhlaner (1989), Morton (1991), Shachar and Nalebuff (1999), or Herrera and Martinelli (2006), consider that voters may be mobilized by leaders, who happen to affect the voting decisions of affine citizens by means of some kind of consumption benefit (they suggest reasons such a social pressure or side-payments). We have in common with this literature the idea that leaders (representatives in our

[1] See Feddersen (2004) for an excellent survey of this literature.

context) influence voter turnout. One important problem of mobilization models is, however, that they do not explicitly model how leaders affect voting decisions. We contribute to this literature by explaining how leaders may generate social pressure and by analyzing how they can strategically use their influence to serve their own interests. Additionally, we endogenize the choice of leaders and are the first to introduce a network architecture into a model of turnout. The model is thus pioneer in highlighting the importance that variables such as location and connectivity have in voting games.

In a similar vein, the literature on coalition formation and vote buying considers that leaders (vote buyers) may affect the voting decisions of citizens (vote sellers) by means of side-payments. Groseclose and Snyder (1996) present a model of sequential vote buying in a legislature with a last mover advantage. This last mover advantage also appears in our game.[2] In their paper, however, vote sellers only care about how they vote, not about which alternative wins. This is in sharp contrast to our analysis, where citizens care about their voting behavior (which determines whether they incur the cross-cutting cost), as well as about the winner of the election. Their focus is also different to ours: they study whether buying a supermajority coalition may be cheaper than buying a minimal winning coalition, and show that, indeed, it may well be the case.

Finally, Fowler (2005) is our closest reference within the literature on social networks. He considers a *small world* network in which citizens imitate each other's voting decisions, and studies how turnout cascades arise, providing an explanation to the striking turnout numbers in large elections.

In the analysis that follows, we present the model and the structure of the game in Section 2. In Section 3 we define the equilibrium notion and characterize the set of equilibria of the game. In Section 4 we analyze how the parties strategically create cross-pressure to serve their own interest and study the last mover advantage. In Section 5 we relax some of the assumptions and analyze the equilibria of the game for low and high values of the cross-cutting cost. Finally, in Section 6 we discuss how our results depend on certain key features of the model.

2 The model

A finite and small population of citizens N is to vote between two alternatives, the one currently being implemented (B) and a new alternative (A). The set of citizens N is divided into two disjoint groups N_A and N_B. We index by $a \in N_A = \{1, ..., |N_A|\}$ the citizens favoring alternative A and we index by $b \in N_B = \{|N_A| + 1, ..., |N|\}$ the citizens favoring alternative B. There is an undirected network $g : N \times N \to \{0, 1\}$ representing the structure of friendship relationships among the elements of N, where $g(i, j) = 1$ if i and j are friends, and $g(i, j) = 0$ otherwise.[3] We assume that the structure

[2] Recently, Dekel et al. (2008) analyze a sequential vote buying game where the number of "bribing" rounds is determined endogenously and so, no advantage is given, *a priori*, to any of the parties.

[3] The fact that g is undirected means that, for each $i, j \in N$, $g(i, j) = g(j, i)$. See Section 6 for a discussion on this issue.

of friendship relationships (g) and each other's preferences (N_A and N_B) are common knowledge. These assumptions are to be interpreted in a context of small networks, where they are easily satisfied (see Section 6 for a discussion on this issue).

The citizens select their preferred alternative in a winner-take-all election. There are k seats in office for the winning alternative.[4] An incumbent, party B, and a challenger, party A, have to choose k representatives, with $k \in \{1, ..., \min\{|N_A|, |N_B|\}\}$, to run for office and implement, if elected, their preferred alternative. We assume that party A, the challenger, moves before party B, the defender of the alternative currently being implemented. Although we use a sequential structure primarily for analytic convenience,[5] this sequentiality can be justified in our setup, where one of the parties supports the *status quo*, which stays in place if no removal process is opened. Groseclose and Snyder (1996) justify a similar sequence of events based on the fact that it can represent the equilibrium outcome of a more general game in which, *a priori*, no player moves first.[6]

The political process is modeled as the following three-stage game:

Stage 1 (challenger stage). The challenger, party A, chooses k citizens, from the set N_A, to represent alternative A. All players become informed of the identity of the k proposed representatives. Then each nominee decides, simultaneously, whether to accept or not to be part of the list. If at least one of the nominees does not accept, the game finishes: no challenging list forms and the incumbent keeps in office.[7] In contrast, if all the nominees accept, the list forms, it is publicly observed, and we move to the second stage.

Stage 2 (incumbent stage). The incumbent, party B, chooses k citizens, from the set N_B, to represent alternative B and contest the challenger. All players become informed of the identity of the k proposed representatives. Then each nominee decides, simultaneously, whether to accept or not to be part of the list. If at least one of the nominees does not accept, the game finishes: the incumbent does not form a list and the challenger gets into office. In contrast, if all the nominees accept, the list forms, it is publicly observed, and we move to the third stage.

Stage 3 (voting stage). Each citizen $i \in N$ decides whether to vote for his preferred alternative or to abstain. The alternative with more votes gets into office. In case of a tie, a coin flip determines the winner.

We next define the strategies for parties and citizens. Let L_A and L_B denote the set

[4]For example, elections for board of directors or governance committees in university departments or firms, where there is usually more than one seat in office (chair and vice-chair, dean and associate deans, etc.).

[5]In the simultaneous version of the game, pure strategy equilibria may fail to exist.

[6]The game would be one like ours with an initial stage in which both parties decide, simultaneously, whether to open the game or not. If just one party chooses to open, it moves first. If both parties choose to open, a coin flip determines who is the first mover. Finally, if none of them chooses to open, the game is not played and the *status quo* stays in place. As pointed out by Groseclose and Snyder (1996), the defender of the *status quo* (party B in our game) has no incentive to initiate such a game, and so, if it happens to play, it will necessarily be the last mover.

[7]This specific formulation of the model might be interpreted as a reduced form of a more general game in which parties are allowed to propose a new list if at least one nominee does not accept to run for office (see Section 6).

of lists available to parties A and B respectively, *i.e.*,

$$L_A = \{l_A \subseteq N_A : |l_A| = k\} \quad \text{and} \quad L_B = \{l_B \subseteq N_B : |l_B| = k\}.$$

A strategy for party A, s_A, is an element from L_A, and a strategy for party B is a function $s_B : L_A \to L_B$. Let S_A and S_B be the sets of strategies for parties A and B, respectively.

Regarding citizens, for each $a \in N_A$, a strategy s_a is a pair of functions (r_a, v_a),[8] where $r_a : L_A \to \{0,1\}$ and $v_a : L_A \times L_B \to \{0,1\}$ are such that

$$r_a(l_A) = \begin{cases} 1 & \text{if citizen } a \text{ accepts to run for office in } l_A \\ 0 & \text{otherwise.} \end{cases}$$

$$v_a(l_A, l_B) = \begin{cases} 1 & \text{if citizen } a \text{ votes for } A \text{ in an election between } l_A \text{ and } l_B \\ 0 & \text{otherwise.} \end{cases}$$

Let S_a be the set of strategies of citizen a. Analogously, for each $b \in N_B$, a strategy s_b is a pair (r_b, v_b), where $r_b : L_A \times L_B \to \{0,1\}$ and $v_b : L_A \times L_B \to \{0,1\}$.[9] Let S_b the set of strategies of citizen b. Let $S = S_A \times S_B \times \prod_{a \in N_A} S_a \times \prod_{b \in N_B} S_b$.

Now, we define the payoffs. For each $s \in S$, let

$$r_A(s) = \min_{a \in s_A}\{r_a(s_A)\} \quad \text{and} \quad r_B(s) = \min_{b \in s_B(s_A)}\{r_b(s_A, s_B(s_A))\}.$$

In words, for each party $j \in \{A, B\}$, $r_j(s) = 1$ ($r_j(s) = 0$) indicates that j runs (does not run) for office. For each $s \in S$, let $P_A(s)$ be the probability that alternative A gets into office. We consider that the objective of parties is to win office, which provides a payoff that we normalize to one. Hence, given $s \in S$, the payoff to party A is

$$P_A(s) = \begin{cases} 0 & \text{if} \begin{cases} r_A(s) = 0 \text{ or} \\ r_A(s) = r_B(s) = 1 \text{ and } \sum_{a \in N_A} v_a(s_A, s_B(s_A)) < \sum_{b \in N_B} v_b(s_A, s_B(s_A)) \end{cases} \\ \frac{1}{2} & \text{if } r_A(s) = r_B(s) = 1 \text{ and } \sum_{a \in N_A} v_a(s_A, s_B(s_A)) = \sum_{b \in N_B} v_b(s_A, s_B(s_A)) \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, the payoff to party B is $P_B(s) = 1 - P_A(s)$.

Regarding citizens, payoffs depend upon the alternative implemented and the identity of the representatives running for office.[10] We consider two sources of positive payoffs: the benefit of holding office, $h > 0$, and the benefit of having their preferred alternative implemented, $d > 0$. We assume $h > d$, which implies that, in the model, a citizen's most important source of benefit is to hold office (which can take the form of supplementary payments, lower teaching duties, more power to allocate resources to own's interests, etc.). These two payoffs are consequentialist, therefore a necessary

[8]Note that s_a is a strategy of the citizen with index $a \in \{1, ..., |N_A|\}$. Different citizens in N_A may therefore have different strategies.

[9]In order to define the strategy profiles in a compact way, we consider that each $a \in N_A$ chooses $r_a(l_A) \in \{0,1\}$ for each $l_A \in L_A$, regardless of whether $a \in l_A$ or not. Analogous considerations hold for each $b \in N_B$. Clearly, the choices of all agents in $N_A \backslash l_A$ and $N_B \backslash l_B$ are irrelevant.

[10]The latter captures the possibility that a citizen gets utility from holding office himself or that he encounters costs from confronting ideologically a friend.

condition for a citizen enjoying these benefits is that his preferred alternative wins, independently of whether the citizen votes or abstains. In contrast, we consider that the act of voting may imply a cross-cutting cost, $c > 0$, that is expressive, *i.e.*, independent of the election outcome. In particular, we assume that $i \in N$ incurs this cost if at least one of i's friends runs for office for the alternative i does not favor and i either runs for office or votes. This assumption is easily justified in our model, where the relative high value of holding office helps to explain why the citizens running for the election may find it profitable to exert pressure on the friends that do not support their "political" career (see Section 6 for a discussion on our formulation of the cross-cutting cost).

In order to introduce a compact expression for the payoff to each $a \in N_A$, let us define $f(a)$ as the set of a's friends that favor his cross-preferred ideology, *i.e.*,

$$f(a) = \{b \in N_B : g(a, b) = 1\}.$$

Analogously, for each $b \in N_B$, $f(b) = \{a \in N_A : g(a, b) = 1\}$. Formally, given $s \in S$, the payoff to each $a \in N_A$ is

$$\pi_a(s) = P_A(s)(h_a(s) + d) - c_a(s),$$

where

$$\begin{aligned} h_a(s) &= \begin{cases} h & \text{if citizen } a \in s_A \\ 0 & \text{otherwise.} \end{cases} \\ c_a(s) &= \begin{cases} c & \text{if} \begin{cases} r_A(s) = r_B(s) = 1, \\ f(a) \cap s_B(s_A) \neq \emptyset \text{ and} \\ \text{either } a \in s_A \text{ or } v_a(s_A, s_B(s_A)) = 1 \end{cases} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The payoff to each $b \in N_B$ is $\pi_b(s) = P_B(s)(h_b(s) + d) - c_b(s)$, where $h_b(s)$ and $c_b(s)$ are defined analogously.

Note that the structure of links among citizens within the same group (those with the same preferences) does not affect players' payoffs. Therefore, in the remainder of the paper, we restrict our attention to the bipartite graph $\tilde{g} : N_A \times N_B \to \{0, 1\}$ induced by g, where $\tilde{g}(a, b) = g(a, b)$ for each $a \in N_A$ and $b \in N_B$.

Finally, let us introduce two assumptions that hold in the main body of the paper: (A1) $c > d$ and (A2) $\frac{h+d}{2} > c$. By (A1), any citizen that is not in a list and has a friend running for office for the alternative he does not favor, gets a higher payoff by abstaining than by voting. By (A2), any nominee who anticipates that his list will win or tie in an election, gets a higher payoff by accepting than by rejecting to run for office. Hence, our model represents a situation where the ideological benefit (itself) does not compensate citizens for incurring the cross-cutting cost, but where the (additional) benefit of holding office does. (A1)-(A2) imply $h > \frac{h+d}{2} > c > d$, *i.e.*, an intermediate level of the cross-cutting cost. These assumptions are relaxed in Section 5, where we analyze the cases $c < d$ and $c > \frac{h+d}{2}$.

3 Equilibrium analysis

3.1 The equilibrium notion

In the multistage setup proposed in our game it is reasonable to require subgame perfection to the Nash concept. However, the subgame perfect Nash equilibrium is a very weak concept for our model: in general, multiple outcomes can be sustained as subgame perfect Nash equilibria due to coordination problems.[11] To address these problems we need a refinement that requires equilibria to be immune to coalitional deviations. The two most well-known refinements that capture this idea are, possibly, the *strong Nash equilibrium* (Aumann (1959)) and the *coalition-proof Nash equilibrium* (Bernheim et al. (1987)).[12] The precise equilibrium concept that we shall use is the *strong perfect equilibrium*, which adapts the strong Nash equilibrium notion to the class of sequential-move games. The notion of strong perfect equilibrium was first introduced by Rubinstein (1979) for repeated games with infinite horizon and posteriorly generalized by Brusco (1997). Formally, for each $C \subseteq N$, let $S_C = \prod_{i \in C} S_i$ and $S_{-C} = \prod_{i \notin C} S_i$.

Definition 1. *(Aumann (1959)) A strong Nash equilibrium is a strategy profile $s \in S$ such that, for each $C \subseteq N$ and $\hat{s}_C \in S_C$, there exists at least one $i \in C$ such that $\pi_i(\hat{s}_C, s_{-C}) \leq \pi_i(s)$.*

Definition 2. *(Brusco (1997)) A strong perfect equilibrium is a strategy profile $s \in S$ which is a strong Nash equilibrium for each proper subgame of the game.*

3.2 Equilibrium characterization

Let $S^* \subset S$ be the set of strong perfect equilibria of our model. We focus on pure strategy equilibria and solve the three period game by backward induction.

The voting stage

Given $l_A \in L_A$ and $l_B \in L_B$, we define the sets $V_A(l_A, l_B)$ and $V_B(l_A, l_B)$ as those encompassing the citizens in favor of alternatives A and B, respectively, that either incur or do not incur the cross-cutting cost c regardless of their behavior in the voting stage. Formally, for each $(l_A, l_B) \in L_A \times L_B$

$$\begin{aligned} V_A(l_A, l_B) &= l_A \cup \{a \in N_A : l_B \cap f(a) = \emptyset\} \\ V_B(l_A, l_B) &= l_B \cup \{b \in N_B : l_A \cap f(b) = \emptyset\}. \end{aligned}$$

We illustrate this concept in Example 1.

[11] For example, if $k > 2$ and there are $a, a' \in s_A$ with $r_a(s) = r_{a'}(s) = 0$, then, for each $a'' \in s_A \setminus \{a, a'\}$, all $r_{a''} \in \{0, 1\}$ can be sustained in a subgame perfect Nash equilibrium (in all the cases, $r_A(s) = 0$ and B keeps in office). Analogously, if $r_a(s_A) = 1$ for all $a \in s_A$, the nominees in $s_B(s_A)$ face the same coordination problem (so A can get into office). Note that similar coordination problems appear in the third stage of the game (*e.g.*, each $a \in N_A$ abstains and at least two players in N_B vote, or *vice versa*).

[12] The strong Nash equilibrium is more restrictive than the coalition-proof Nash equilibrium. While the strong Nash equilibrium considers all possible coalitional deviations, the coalition-proof Nash equilibrium only considers deviations that are self-enforcing.

Example 1. *Let $N_A = \{1, ..., 5\}$, $N_B = \{6, ..., 10\}$ and $\widetilde{g}$ be represented in Figure 1, where for each $a \in N_A$ and $b \in N_B$, $\widetilde{g}(a, b) = 1$ if and only if there is an arc between a and b. Suppose $k = 2$ and consider a pair of lists, for instance, $\{1, 5\}$ and $\{9, 10\}$. Then, $V_A(\{1, 5\}, \{9, 10\}) = \{1, 5\} \cup \{1, 3\} = \{1, 3, 5\}$ and $V_B(\{1, 5\}, \{9, 10\}) = \{9, 10\} \cup \{8, 9\} = \{8, 9, 10\}$.*

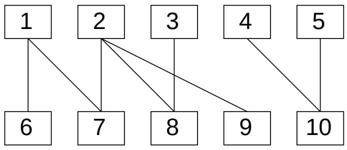

Figure 1.

The next lemma characterizes the (strong perfect) equilibria of the voting stage.

Lemma 1. *Assume (A1)-(A2). Given $(l_A, l_B) \in L_A \times L_B$, s is a strong perfect equilibrium of the voting stage if and only if, for each $i \in N \backslash (V_A(l_A, l_B) \cup V_B(l_A, l_B))$, $v_i(l_A, l_B) = 0$ and,*

i) if $|V_A(l_A, l_B)| > |V_B(l_A, l_B)|$, $\sum_{a \in N_A} v_a(l_A, l_B) > |V_B(l_A, l_B)|$,

ii) if $|V_A(l_A, l_B)| < |V_B(l_A, l_B)|$, $\sum_{b \in N_B} v_b(l_A, l_B) > |V_A(l_A, l_B)|$ and,

iii) if $|V_A(l_A, l_B)| = |V_B(l_A, l_B)|$, for each $j \in V_A(l_A, l_B) \cup V_B(l_A, l_B)$, $v_j(l_A, l_B) = 1$.

Proof. In the Appendix. □

Lemma 1 says that, in equilibrium, any citizen that is not in a list and incurs the cross-cutting cost if voting, abstains. Hence, given two lists running for office, the outcome of an election depends on the cardinalities of $V_A(l_A, l_B)$ and $V_B(l_A, l_B)$. More precisely, if $V_A(l_A, l_B) > (<) \ V_B(l_A, l_B)$, party A (B) wins the election. In case of equality, there is a tie.

The incumbent stage

Given $l_A \in L_A$, by Lemma 1 we define the set of lists of party B that win the election, $W_B(l_A)$, and the set of lists of party B that tie in the election, $T_B(l_A)$, as follows

$$\begin{aligned} W_B(l_A) &= \{l_B \in L_B : |V_B(l_A, l_B)| > |V_A(l_A, l_B)|\} \text{ and} \\ T_B(l_A) &= \{l_B \in L_B : |V_B(l_A, l_B)| = |V_A(l_A, l_B)|\}. \end{aligned}$$

As previously, we can illustrate these concepts by means of Example 1. Consider again the list $\{1, 5\}$ of party A. The reader can check that $W_B(\{1, 5\}) = \{\{7, 10\}, \{8, 10\}\}$ and $T_B(\{1, 5\}) = \{\{6, 7\}, \{6, 8\}, \{6, 10\}, \{7, 8\}, \{9, 10\}\}$.

The next lemma characterizes the (strong perfect) equilibria of the incumbent stage.

Lemma 2. *Assume (A1)-(A2). Given $l_A \in L_A$, s is a strong perfect equilibrium of the incumbent stage if and only if*

i) for each $l_B \in L_B$,

- if $|V_B(l_A, l_B)| \geq |V_A(l_A, l_B)|$, $r_B(l_A, l_B) = 1$ and,

- if $|V_B(l_A, l_B)| < |V_A(l_A, l_B)|$ and there is $b' \in l_B$ s.t. $l_A \cap f(b') \neq \emptyset$, $r_B(l_A, l_B) = 0$.

ii) and party B chooses,
- if $W_B(l_A) \neq \emptyset$, $s_B(l_A) \in W_B(l_A)$ and,
- if $W_B(l_A) = \emptyset$ and $T_B(l_A) \neq \emptyset$, $s_B(l_A) \in T_B(l_A)$.

Proof. In the Appendix. □

Lemma 2 characterizes, in turn, the optimal behavior of citizens in l_B and party B. Part i) requires that all the nominees accept to run for office if, given Lemma 1, l_B wins or ties in an election against l_A. If, on the contrary, l_B looses, part i) requires that l_B does not run for office if at least one member of l_B incurs the cross-cutting cost. Part ii) requires that party B chooses a winning list, if available, and chooses a list that procures a tie, if there are no winning lists.

The challenger stage
We finally define the set of lists of party A that, given Lemmas 1 and 2, allow party A to get into office, W_A, and the set of lists of party A that procure a tie, T_A, as follows[13]

$$\begin{aligned} W_A &= \{l_A \in L_A : W_B(l_A) \cup T_B(l_A) = \emptyset\} \text{ and} \\ T_A &= \{l_A \in L_A : W_B(l_A) = \emptyset \wedge T_B(l_A) \neq \emptyset\}. \end{aligned}$$

The next lemma characterizes the (strong perfect) equilibria of the challenger stage. The interpretation of the results is completely analogous to that of Lemma 2.

Lemma 3. *Assume (A1)-(A2). s is a strong perfect equilibrium of the challenger stage if and only if*
i) for each $l_A \in L_A$,
- if $|V_A(l_A, s_B(l_A))| \geq |V_B(l_A, s_B(l_A))|$, $r_A(l_A) = 1$ and,
- if $|V_A(l_A, s_B(l_A))| < |V_B(l_A, s_B(l_A))|$ and there is $a \in l_A$ s.t. $s_B(l_A) \cap f(a) \neq \emptyset$, $r_A(l_A) = 0$.
ii) and party A chooses,
- if $W_A \neq \emptyset$, $s_A \in W_A$ and,
- if $W_A = \emptyset$ and $T_A \neq \emptyset$, $s_A \in T_A$.

The proof of Lemma 3 is omitted as it follows analogous arguments to those used to prove Lemma 2.

At this point, we have the complete characterization of the strong perfect equilibria of the game. Note, however, that in view of Lemmas 1-3, the set of strong perfect equilibria may be large. Despite this multiplicity, the next proposition shows that all the equilibria are outcome equivalent in terms of which alternative gets into office.

Proposition 1. *Assume (A1)-(A2). Then $S^* \neq \emptyset$ and, for each $(s, s') \in S^* \times S^*$, $P_A(s) = P_A(s')$.*

Proof. The fact that $S^* \neq \emptyset$ follows, by construction, from Lemmas 1-3. By Lemmas 1-3, for each $s \in S^*$: If $W_A \neq \emptyset$, then $P_A(s) = 1$; if $W_A = \emptyset$ and $T_A \neq \emptyset$, then $P_A(s) = 1/2$ and; if $W_A = T_A = \emptyset$, then $P_A(s) = 0$. □

[13] In the case of Example 1, the reader can check that $W_A = \emptyset$ and $T_A = \{\{1, 2\}, \{2, 4\}, \{2, 5\}\}$.

Proposition 1 thus shows that, given (A1)-(A2), there always exist (at least) one strong perfect equilibrium, and that even though there may be (and generally will be) multiplicity of equilibria, all of them are outcome equivalent in terms of which alternative gets into office. This result allows us to focus on a (reasonable) subset of strong perfect equilibria, without restricting the set of equilibrium outcomes. Formally, let $S^{**} \subset S^*$ be the subset of strong perfect equilibria such that $s \in S^{**}$ if and only if it satisfies the following conditions (1)-(3):

(1) For each $i \in N$, $v_i(l_A, l_B) = 1$ if and only if $i \in V_A(l_A, l_B) \cup V_B(l_A, l_B)$.

(2) For each $l_A \in L_A$, $l_B \in L_B$, $a \in N_A$ and $b \in N_B$,
$r_a(l_A) = 1$ if and only if $|V_A(l_A, s_B(l_A))| \geq |V_B(l_A, s_B(l_A))|$ and
$r_b(l_A, l_B) = 1$ if and only if $|V_B(l_A, l_B)| \geq |V_A(l_A, l_B)|$.

(3) $s_A \in \arg\max_{l_A \in L_A} |V_A(s_B(l_A), l_A)| - |V_B(s_B(l_A), l_A)|$ and, for each $l_A \in L_A$,
$s_B(l_A) \in \arg\max_{l_B \in L_B} |V_B(l_A, l_B)| - |V_A(l_A, l_B)|$.

We obtain conditions (1)-(3) by imposing some restrictions on the conditions obtained in Lemmas 1-3. Basically, we require citizens to use weakly dominant strategies and parties to maximize the net plurality defined as the difference between the number of votes obtained by the parties. Note that, by construction, $S^{**} \neq \emptyset$.

4 The parties' game

In the previous section we have analyzed how the existence of cross-pressure affects voting decisions and have characterized the equilibrium behavior of citizens and parties. In the following, we study the strategic behavior of parties when choosing their representatives and the implications that the sequential structure assumed in the model has for the parties' game.

4.1 The strategic use of cross-pressure

Here, we analyze how the parties exploit the cross-pressure effect to their interest. To this aim, we assume that citizens' behavior is determined by (1)-(2), and analyze the game where parties maximize net plurality, *i.e.*, we focus on the set S^{**}.

Since, by (1), citizens with cross-pressure do not vote (unless they run for office), it seems natural to think that the optimal list of a party should generate the highest amount of cross-pressure to the people on the other faction. This quite intuitive idea, however, falls short in some cases. The reason being that it says nothing about the special status of the representatives. The parties thus have to take into account these two aspects if they aim to win the election.

In order to formalize this idea and thoroughly analyze the behavior of the parties, we need to define some concepts. For expositional reasons, we define them exclusively for party A. The *covering of a list* $l_A \in L_A$, $F(l_A)$, consists of those citizens of N_B linked to at least one citizen of l_A.

$$F(l_A) = \bigcup_{a \in l_A} f(a).$$

The *maximal covering* for party A, m_A, is the maximal number of citizens in N_B than can be covered by a list in L_A, *i.e.*,

$$m_A = \max_{l_A \in L_A} |F(l_A)|.$$

Hence, the set of *lists of maximal covering* for party A is

$$L_A^m = \{l_A \in L_A \mid |F(l_A)| = m_A\}.$$

The definitions of $F(l_B)$, m_B and L_B^m for party B are completely analogous. Last, for each $(l_A, l_B) \in L_A \times L_B$, we define the set of *covered representatives* of a party as those citizens in its list who are linked to at least one citizen in the opposite list.

$$C_A(l_A, l_B) = l_A \cap F(l_B) \quad \text{and} \quad C_B(l_A, l_B) = l_B \cap F(l_A).$$

With these definitions at hand, we say that $s \in S^{**}$ is an *equilibrium in maximal covering* if and only if $s_A \in L_A^m$ and $s_B(s_A) \in L_B^m$.

To illustrate these concepts, consider the situation presented in Example 1 above. Let us focus on the pair of lists $\{1, 5\} \in L_A$ and $\{9, 10\} \in L_B$. Then $F(\{1, 5\}) = \{6, 7\} \cup \{10\} = \{6, 7, 10\}$, $F(\{9, 10\}) = \{2\} \cup \{4, 5\} = \{2, 4, 5\}$; and $C_A(\{1, 5\}, \{9, 10\}) = \{1, 5\} \cap \{2, 4, 5\} = \{5\}$, $C_B(\{1, 5\}, \{9, 10\}) = \{9, 10\} \cap \{6, 7, 10\} = \{10\}$. Straightforward calculations show that $m_A = m_B = 4$, $L_A^m = \{\{1, 2\}, \{2, 4\}, \{2, 5\}\}$ and $L_B^m = \{\{7, 10\}, \{8, 10\}\}$.

We are now in position to characterize the equilibrium behavior of parties.

Proposition 2. *Assume (1)-(2). Then $s \in S^{**}$ if and only if*
i) $s_A \in \arg\max_{l_A \in L_A} |F(l_A)| - |F(s_B(l_A))| + |C_A(l_A, s_B(l_A))| - |C_B(l_A, s_B(l_A))|$ and,
ii) for all $l_A \in L_A$, $s_B(l_A) \in \arg\max_{l_B \in L_B} |F(l_B)| + |C_B(l_A, l_B)| - |C_A(l_A, l_B)|$.

Proof. In the Appendix. □

By Proposition 2, the two main forces driving the optimal behavior of parties are clear. On the one hand, a party may benefit from increasing the covering of its list, as it can allow it to inhibit (more) voters of the other faction. On the other hand, a party must take into account that it does not pay to cover representatives of the other faction, since these citizens always vote. Ceteris paribus, a party thus prefers to have the highest number of covered representatives in its list.

Note that in the case $k = 1$, the number of covered representatives in the two lists is necessarily the same. Thus, if $k = 1$, the characterization of S^{**} turns out to be much simpler, as the following corollary shows.

Corollary 1. *Assume (1)-(2) and let $k = 1$. Then $s \in S^{**}$ if and only if $s_A \in L_A^m$ and, for each $l_A \in L_A$, $s_B(l_A) \in L_B^m$.*

Proof. If $k = 1$, for each $l_A \in L_A$ and $l_B \in L_B$, $|C_A(l_A, l_B)| = |C_B(l_A, l_B)|$. Then, $s_B(l_A) \in L_B^m$. This implies that, in equilibrium, $|F(s_B(l_A))| = m_B$. Then, $s_A \in L_A^m$. □

By Corollary 1, if $k = 1$, each $s \in S^{**}$ is an equilibrium in maximal covering. The model thus predicts that if parties have to choose one representative each, they optimally nominate the citizen that is better connected in the network with the other faction. Moreover, given N_A, N_B and g, by Corollary 1, if $k = 1$, it is immediate to obtain the (unique) equilibrium outcome in terms of which alternative gets into office. In particular, if $k = 1$, for each $s \in S^{**}$ we have: (i) if $|N_A| - m_B > |N_B| - m_A$, then $P_A(s) = 1$; (ii) if $|N_A| - m_B = |N_B| - m_A$, then $P_A(s) = 1/2$; and (iii) otherwise, $P_A(s) = 0$.

However, if $k \geq 2$, the number of covered representatives in the list of party A may differ from that of party B. Here, as Proposition 2 shows, both, the covering of the lists *and* the difference in the number of covered representatives, determine the optimal strategy of parties. There is therefore no guarantee that an equilibrium in maximal covering exists, as Example 2 bellow illustrates. The reason is that a list of no-maximal covering may procure a favorable difference of covered representatives that outweighs its smaller covering.

Example 2. *Let* $N_A = \{1, ..., 8\}$, $N_B = \{9, ..., 16\}$ *and* $\widetilde{g}$ *be represented in Figure 2. Suppose* $k = 2$. *Then,* $L_A^m = \{\{3, 6\}\}$ *and* $L_B^m = \{\{13, 16\}\}$. *Let* $l_A^m = \{3, 6\}$ *and* $l_B^m = \{13, 16\}$. *Then,* $F(l_A^m) = \{10, 11, 12, 13, 14, 15\}$, $F(l_B^m) = \{3, 4, 5, 6, 7, 8\}$ *and* $m_A = m_B = 6$. *We claim that* $s_A = l_A^m$ *and* $s_B(l_A^m) = l_B^m$ *is not part of any strong perfect equilibrium. First, note that* $C_A(l_A^m, l_B^m) = \{3, 6\}$ *and* $C_B(l_A^m, l_B^m) = \{13\}$. *Hence,* $V_A(l_A^m, l_B^m) = (N_A \backslash F(l_B^m)) \cup C_A(l_A^m, l_B^m) = \{1, 2, 3, 6\}$ *and* $V_B(l_A^m, l_B^m) = (N_B \backslash F(l_A^m)) \cup C_B(l_A^m, l_B^m) = \{9, 13, 16\}$. *Therefore* $|V_A(l_A^m, l_B^m)| - |V_B(l_A^m, l_B^m)| = 1$ *and, given (7)-(8),* $P_A(s) = 1$. *Consider now that party B deviates to* $l_B = \{10, 11\}$. *Then,* $F(l_B) = \{1, 2, 3, 4, 5\}$, $C_A(l_A^m, l_B) = \{3\}$ *and* $C_B(l_A^m, l_B) = \{10, 11\}$. *Hence,* $V_A(l_A^m, l_B) = \{3, 6, 7, 8\}$ *and* $V_B(l_A^m, l_B) = \{9, 10, 11, 16\}$, *then* $|V_A(l_A^m, l_B^m)| = |V_B(l_A^m, l_B)|$. *Given (1)-(2), party B finds it profitable to deviate to* $\bar{s}_B(l_A^m) = l_B$, *which implies a tie. There is therefore no equilibrium in maximal covering.*[14]

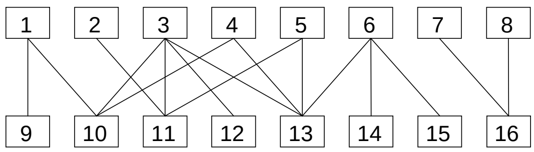

Figure 2.

Given that, when $k \geq 2$, the existence of equilibria in maximal covering is not guaranteed, to identify an equilibrium can be an arduous task (note that $|L_A| = \binom{|N_A|}{k}$ and $|L_B| = \binom{|N_B|}{k}$).[15] In the following proposition, however, we provide lower bounds for the covering of the optimal lists of the parties. It simplifies the characterization of S^{**} and, therefore, the process of identifying the unique equilibrium outcome in terms of which alternative gets into office.

[14] Additionally, it can be shown that if $N_A = \{9, ..., 16\}$ and $N_B = \{1, ..., 8\}$ instead, it is the challenger (A) the party that, in equilibrium, never proposes its list of maximal covering.

[15] For instance, in Example 2, $|L_A| = |L_B| = 28$. This implies that we would have to compare each of the 28 lists of party A to each of the 28 responses of party B.

Proposition 3. *If $s \in S^{**}$, then $|F(s_A)| \geq m_A - 2(k-1)$ and, for all $l_A \in L_A$, $|F(s_B(l_A))| \geq m_B - 2(k-1)$. Moreover, there is $s' \in S^{**}$ such that $|F(s'_A)| > m_A - 2(k-1)$ and, for each $l_A \in L_A$, $|F(s'_B(s'_A))| > m_B - 2(k-1)$.*

Proof. In the Appendix. □

The first part of Proposition 3 says that, in order to characterize S^{**}, we can restrict our attention to the sets $\{l_A \in L_A \mid |F(l_A)| \geq m_A - 2(k-1)\}$ and $\{l_B \in L_B \mid |F(l_B)| \geq m_B - 2(k-1)\}$. Note that, the larger k, the larger the set of possible equilibrium lists; hence, the more complex the characterization is. The second part of the proposition provides a further step in the direction of identifying the unique equilibrium outcome in terms of which alternative gets into office. It says that we do not need to consider those lists in L_A with covering equal to $m_A - 2(k-1)$, nor those lists in L_B with covering equal to $m_B - 2(k-1)$.

For instance, if we aim is to characterize S^{**} in Example 2 above, we can restrict our attention to those lists with covering greater or equal than 4, *i.e.*, those $l_A \in L_A$ and $l_B \in L_B$ such that $|F(l_A)| \in \{4,5,6\}$ and $|F(l_B)| \in \{4,5,6\}$. However, if we just want to identify one equilibrium,[16] we can further restrict our attention to those $l_A \in L_A$ and $l_B \in L_B$ such that $|F(l_A)| \in \{5,6\}$ and $|F(l_B)| \in \{5,6\}$. Making use of these bounds, in the Appendix we obtain that, in Example 2, for each $s \in S^{**}$, $P(s) = 1/2$, *i.e.*, the equilibrium outcome is a tie.

4.2 The advantage of being the last mover

It seems natural to think that the sequential structure of our game may provide an advantage to the last mover (the incumbent) and that it may affect the equilibrium outcome. In the following, we analyze when and why such last mover advantage appears.

First, note that by Corollary 1, there is no last mover advantage in the case $k = 1$, since the outcome of the game is determined by comparing $|N_A| - m_B$ to $|N_B| - m_A$. This result, however, is not generally true when $k \geq 2$. In this case, the ability of the last mover (party B) to strategically place (some of) its representatives within $F(s_A)$ may benefit it. In Proposition 4 we formalize this idea.

To this aim, consider a population $N = X \cup Y$, $X \cap Y = \emptyset$, and a network $g : N \times N \to \{0,1\}$. Consider the following two games. Let Game 1 be such that $X = N_A$ and $Y = N_B$, and let Game 2 be such that $X = N_{B'}$ and $Y = N_{A'}$.[17] Let P^*_A be the probability that, given the unique equilibrium outcome of Game 1, party A (the challenger, supported by group X) gets into office in Game 1; and let $P^*_{B'}$ be the probability that, given the unique equilibrium outcome of Game 2, party B' (the incumbent, supported by group X) gets into office in Game 2.

Proposition 4. *Assume (A1)-(A2). Then, $P^*_{B'} \geq P^*_A$.*

[16] Recall that, by Proposition 1, it suffices to identify one equilibrium to obtain the (unique) equilibrium outcome of the game in terms of which party gets into office.

[17] Note that in Game 2 we denote parties as A' and B'. This is done for notational convenience, in order to facilitate comparisons across the two games.

Proof. We need to consider two cases. i) If $W_A \neq \emptyset$, let $l \in W_A$. Then, for each $l_{A'} \in L_{A'}$, $l \in W_{B'}(l_{A'})$ and, therefore, $W_{A'} = T_{A'} = \emptyset$. ii) If $W_A = \emptyset$ and $T_A \neq \emptyset$, let $\hat{l} \in T_A$. Then, for each $l_{A'} \in L_{A'}$, $\hat{l} \in T_{B'}(l_{A'}) \cup W_{B'}(l_{A'})$ and, therefore, $W_{A'} = \emptyset$. □

In the following example we show that, given (A1)-(A2), if $k \geq 2$, there are situations where being the last mover is strictly better for a party than moving first, *i.e.*, there exist X, Y, k and g such that $P^*_{B'} > P^*_A$.[18]

Example 3. *Let* $X = \{1, ..., 8\}$, $Y = \{9, ..., 16\}$ *and* $\widetilde{g}$ *be represented in Figure 3. Suppose* $k = 2$. *We consider two games: Game 1, with* $X = N_A$ *and* $Y = N_B$ *and Game 2, with* $X = N_{B'}$ *and* $Y = N_{A'}$. *In Game 1, in equilibrium, B gets into office. In other words, the alternative preferred by group X (here the challenger) is never implemented* ($P^*_A = 0$)*. However, in Game 2, in equilibrium,* B' *gets, at least, a tie. In other words, the alternative preferred by group X (now the incumbent) is implemented with positive probability* ($P^*_{B'} > 0$)*. See the Appendix for details.*

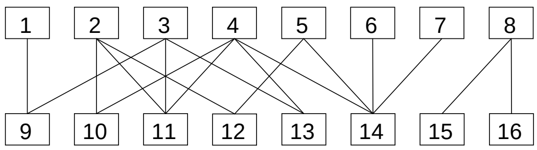

Figure 3.

5 Analysis for extreme values of the cross-cutting cost

In this section we relax assumptions (A1)-(A2) to consider the cases of a low and a high cross-cutting cost. Maintaining $h > d$, we analyze the cases $c < d$ and $c > \frac{h+d}{2}$.

5.1 Low cross-cutting cost

Assume $c < d < \frac{h+d}{2}$. We differentiate two cases, $c > d/2$ and $c < d/2$. We focus on pure strategy equilibria.

Proposition 5. *Let* $c < d < \frac{h+d}{2}$ *and* $\frac{d}{2} < c$. *Then* $S^* \neq \emptyset$ *if and only if, for each* $(l_A, l_B) \in L_A \times L_B$, *one of the following three conditions (I)-(III) holds: (I)* $|V_A(l_A, l_B)| = |V_B(l_A, l_B)|$; *(II)* $|V_A(l_A, l_B)| \geq |N_B|$; *(III)* $|V_B(l_A, l_B)| \geq |N_A|$.

Proof. In the Appendix. □

This result relies on the fact that, if $\frac{d}{2} < c < d$, a citizen may obtain a positive net benefit when he votes, even though he incurs the cross-cutting cost. It raises pivotal considerations that citizens will have to take into account in order to decide whether to vote. This fact will generally create cycles of profitable deviations that may result

[18]Note that there are also cases where $k \geq 2$ and $P^*_{B'} = P^*_A$. For instance, the reader can check that in the case of the network represented in Figure 2, the equilibrium outcome is a tie, regardless of the identity of the party initially in power.

in non-existence of equilibrium. To illustrate this idea, consider a pair of lists (l_A, l_B). Suppose that, within N_A, only those citizens in $V_A(l_A, l_B)$ vote and that, in such a situation, party B wins the election. Then, since $d > c$, if the citizens in $N_A \setminus V_A(l_A, l_B)$ are numerous enough, they will have incentives to (jointly) deviate to vote, so that party A wins the election. But then, since $c > \frac{d}{2}$, a subset of these citizens will have further incentives to deviate (from the coalition) to abstain, so that the result is a tie instead. In such a case, however, the remaining citizens in $N_A \setminus V_A(l_A, l_B)$ would also have incentives to abstain to save the cross-cutting cost. At this point, we are at the initial situation where only those citizens in $V_A(l_A, l_B)$ vote and party B wins.

Proposition 5 thus shows that, if $\frac{d}{2} < c$, a strong perfect equilibrium exists only under very limited conditions. In particular, existence requires that, for each possible pair of lists, either the number of citizens whose voting decision does not imply a cost is the same in the two groups (in which case the equilibrium results in a tie); or that the number of these citizens in one group is greater or equal than the total number of citizens in the other faction (the larger faction wins office). Since these requirements are very strong, in general, equilibrium fails to exist.

Proposition 6. *Let $c < d < \frac{h+d}{2}$ and $c < \frac{d}{2}$. Then $S^* \neq \emptyset$ if and only if either $|N_A| = |N_B|$ or, for each $(l_A, l_B) \in L_A \times L_B$, one of the following two conditions (I')-(II') holds: (I') $|V_A(l_A, l_B)| > |N_B|$; (II') $|V_B(l_A, l_B)| > |N_A|$.*

The proof is omitted as it is very similar to that of Proposition 5. Proposition 6 shows that, if $c < \frac{d}{2}$, the conditions for equilibrium existence, although restrictive, are not so strong as in the previous case.[19] For instance, note that when N_A and N_B are of the same size, there always exists a strong perfect equilibrium (in which parties propose any pair of lists, all nominees accept to run and all citizens vote). In contrast, if $|N_A| \neq |N_B|$, the equilibrium requirements are stronger and, in general, equilibrium fails to exist. Note, however, that if the requirements are satisfied, in equilibrium, the larger faction wins office.

5.2 High cross-cutting cost

Assume $c > \frac{h+d}{2} > d$. We differentiate two cases, $c > h + d$ and $c < h + d$.

Let $c > h + d$. Then, if $s \in S^*$, for each $(l_A, l_B) \in L_A \times L_B$ such that $l_B \cap F(l_A) \neq \emptyset$, $r_B(l_A, l_B) = 0$ and, therefore, the challenger gets into office. Hence, in equilibrium, for each $l_A \in L_A$, the incumbent is "restricted" to choose $s_B(l_A) \subseteq N_B \setminus F(l_A)$. This further implies that, for each $l_A \in L_A$, both $C_A(l_A, s_B(l_A))$ and $C_B(l_A, s_B(l_A))$ are empty. This fact rules out the last mover advantage observed under assumptions (A1)-(A2). In contrast, the current restrictions on parameters may provide an advantage to the first mover, as the choice of l_A restricts the set of lists "available" to party B. On the other hand, if $s \in S^*$, for each $(l_A, l_B) \in L_A \times L_B$ such that $l_B \cap F(l_A) = \emptyset$, party A (B) wins office if $|V_A(l_A, l_B)| > |V_B(l_A, l_B)|$ ($|V_B(l_A, l_B)| > |V_A(l_A, l_B)|$).

[19] Note that when $c < \frac{d}{2}$, no citizen has incentives to deviate to abstain if his preferred alternative is tying. This is in contrast to the previous case.

Now let $c < h+d$. Then, if $s \in S^*$, for each $(l_A, l_B) \in L_A \times L_B$ such that $l_B \cap F(l_A) \neq \emptyset$, there are two possibilities: (i) if $l_B \in W_B(l_A)$, then $r_B(l_A, l_B) = 1$ and party B stays in power and; (ii) if $l_B \notin W_B(l_A)$, then $r_B(l_A, l_B) = 0$ and the challenger gets into office. Hence, the choice of l_A still imposes a restriction on the lists available to party B, since no covered citizen in B finds it profitable to run for office unless $P_B(s) = 1$. Therefore, only if $W_A = T_A = \emptyset$ party B conserves the last mover advantage observed under (A1)-(A2). In contrast, if $T_A \neq \emptyset$, and analogously to the case $c > h+d$, there may be an advantage to the first mover, which, by choosing $s_A \in T_A$, can get into office with a probability of 1.[20] On the other hand, if $s \in S^*$, for each $(l_A, l_B) \in L_A \times L_B$ such that $l_B \cap F(l_A) = \emptyset$, party A (B) wins office if $|V_A(l_A, l_B)| > |V_B(l_A, l_B)|$ ($|V_B(l_A, l_B)| > |V_A(l_A, l_B)|$).

Finally, note that, although we do not prove it formally, if $c > \frac{h+d}{2} > d$, $S^* \neq \emptyset$.

6 Discussion

There is evidence showing that people whose networks involve political disagreement tend to participate less in politics (Lazarsfeld et al. (1944), Simmel (1955) and, more recently, Mutz (2002, 2006)). However, to the best of our knowledge, no model of voter turnout incorporates this fact. This paper is intended to be a first step in this direction. To this aim, we propose a model of voting in small networks and analyze how parties strategically create cross-pressure to serve their own interests as well as its effects on voting outcomes. The results of this paper, however, depend on certain key features of the voting game. This section contains a discussion on the importance and implications of these assumptions for our results.

The assumption of an undirected network

In the paper we assume that citizens are linked through an undirected network. This means that for each $i, j \in N$, $g(i,j) = g(j,i)$. The model thus represents situations where, if two citizens are friends, influence flows in both directions. Consider now a slightly different version of the model in which citizens are linked through a directed network, *i.e.*, pressure only flows in one direction. In this case, the general rule that an equilibrium in maximal covering does not necessarily exist extends to the case $k = 1$. To see it, consider $k = 1$. Here, the number of covered representatives in the two parties is not necessarily the same. Hence, if the parties maximize net plurality, they have to consider both the covering of a list and the number of covered representatives.[21] Thus, the list of maximal covering is not necessarily optimal. This is always the case when the network is directed, independently of k.

The assumption of common knowledge and a small network

In the paper we consider that the preferences of the citizens and the structure of friendship relationships are common knowledge. The model thus fits most clearly voting

[20] Note that, in such a case, for each $l_B \in L_B$, if $l_B \cap F(s_A) \neq \emptyset$, then $r_B(s_A, l_B) = 0$.

[21] If the network is undirected, however, the number of covered representative in the two parties is always the same (provided $k = 1$).

situations where the number of people involved is not too large (a *small network*). The point is whether we can say something about turnout in large elections. We consider that the main force driving our results (the incentives of parties to strategically create cross-pressure to serve their own interest) would still be present in a large election. In this case, however, we conjecture that parties would count on local associations, unions, churches, etc., to exert their influence; rather than directly delegating on representatives. In this sense, we consider that our results on the behavior of parties would apply, at the local level, to a model of large elections. Additionally, we also consider that traditional variables such as money, candidates' endorsement, valence, etc., play a crucial role in the analysis of large elections. As a result, the outcome of a large election would be determined by the interplay between all these variables and the cross-pressures exerted by the parties on the local communities.

The assumption that if a citizen does not accept to be in a list, the game finishes

In the paper we consider that, in the first two stages of the game, the parties propose their lists of representatives. By so doing, the parties can strategically create cross-pressure to serve their own interests. Thus, the model we analyze is game-theoretic in the first two stages and decision-theoretic in the last stage.

In the first two stages it is assumed that if at least one of the nominees does not accept to run for office, the game finishes. The reader can think of this situation as a non contested election in which the only alternative available is the party that is able to form a list. For simplicity we assume that in this case there is no election and that such party gets into office. However, the result would not change if we considered that an election takes place in that case.

Last, consider a slightly modified version of our game in which the parties are allowed to propose a new list if the previous one is not accepted. Assume that this process continues until either a list is formed, or the party exhausts all its possible lists; and that, in the latter case, the game finishes (the other alternative is implemented). In this new version of the game, there are two possibilities. First, at some stage of the process the party is able to form a list to run for office. In such a case, in our original game, the party will also be able to run for office. Second, the party exhausts all its possible lists and is not able to run for office. In such a case, in our original game, the party will neither be able to run for office. Hence, the game we analyze can be interpreted as a *reduced form* of a more general game in which parties can repeatedly propose a new list if the previous one is turned down.

The assumption of a particular and fixed cross-cutting cost

In the paper we consider that a citizen i incurs the cross-cutting cost if at least one of i's friends runs for office for the alternative i does not favor and i either runs for office or votes. Thus, this model applies to situations in which the benefit of holding office is high enough, which justifies that the citizens in the lists have incentives to exert pressure on the friends that do not support their "political" careers.

Additionally, our results follow from a particular assumption about the cross-cutting cost: the cross-cutting cost c is independent of the number of friends that prefer the other alternative. We make this assumption for analytical convenience, as it allow us to isolate from pivotal considerations that would complicate and obscure the analysis of the effects of cross-pressure on voting outcomes. To illustrate this complexity, suppose we consider that c is a function of the number (or relative number) of friends with a different ideology. In this case, different citizens may incur different levels of cross-cutting cost. Hence, for a given network, there may be citizens who incur a cross-cutting cost but still get a positive net benefit if they vote. These citizens will find it profitable to vote just if their vote is pivotal. This kind of considerations are analogous to those already observed in the case of a low cross-cutting cost, (*cf.* Section 5.1), and likewise, may imply non-existence of equilibrium.

Nevertheless, we consider that to better understand how cross-pressures affect voting outcomes, it would be helpful to introduce some heterogeneity in the formulation of the cross-cutting cost; for example, to make the cost different depending on whether a voter runs for office or just votes. This and other extensions are left for future work.

Appendix

A Examples

Example 2 (Cont.) Consider the situation presented in Figure 2 (Section 4), where $|N_A| = |N_B| = 8$ and $m_A = m_B = 6$. Let $L_A^* = \{l_A \in L_A \mid |F(l_A)| > 4\}$ and $L_B^* = \{l_B \in L_B \mid |F(l_B)| > 4\}$. By Proposition 3, there is $s \in S^{**}$ where $s_A \in L_A^*$ and, for each $l_A \in L_A$, $s_B(l_A) \in L_B^*$. In Table 1 below, we obtain $s \in S^{**}$, *i.e.*, the (unique) equilibrium outcome in terms of which alternative gets into office.

l_A	$F(l_A)$	l_B	$F(l_B)$	$\|F(l_A)\|$	$\|F(l_B)\|$	$\|C_A\|$	$\|C_B\|$	$\|V_A\|$	$\|V_B\|$	$\|V_A\|-\|V_B\|$
{3,6}	{10,11,12,13,14,15}	{13,16}	{3,4,5,6,7,8}	6	6	2	1	4	3	1
		{9,13}	{1,3,4,5,6}		5	2	1	5	3	2
		{10,11}	{1,2,3,4,5}		5	1	2	4	4	**0**
		{10,13}	{1,3,4,5,6}		5	2	2	5	4	1
		{10,16}	{1,3,4,7,8}		5	1	1	4	3	1
		{11,13}	{2,3,4,5,6}		5	2	2	5	4	1
		{11,16}	{2,3,5,7,8}		5	1	1	4	3	1
{1,3}	{9,10,11,12,13}	{13,16}	{3,4,5,6,7,8}	5	6	1	1	3	4	-1
		{9,13}	{1,3,4,5,6}		5	2	2	5	5	0
		{10,11}	{1,2,3,4,5}		5	2	2	5	5	0
		{10,13}	{1,3,4,5,6}		5	2	2	5	5	0
		{10,16}	{1,3,4,7,8}		5	2	1	5	4	1
		{11,13}	{2,3,4,5,6}		5	1	2	4	5	-1
		{11,16}	{2,3,5,7,8}		5	1	1	4	4	0
{1,6}	{9,10,13,14,15}	{13,16}	{3,4,5,6,7,8}	5	6	1	1	3	4	-1
		{9,13}	{1,3,4,5,6}		5	2	2	5	5	0
		{10,11}	{1,2,3,4,5}		5	1	1	4	4	0
		{10,13}	{1,3,4,5,6}		5	2	2	5	5	0
		{10,16}	{1,3,4,7,8}		5	1	1	4	4	0
		{11,13}	{2,3,4,5,6}		5	1	1	4	4	0
		{11,16}	{2,3,5,7,8}		5	0	0	3	3	0
{3,7}	{10,11,12,13,16}	{13,16}	{3,4,5,6,7,8}	5	6	2	2	4	5	-1
		{9,13}	{1,3,4,5,6}		5	1	1	4	4	0
		{10,11}	{1,2,3,4,5}		5	1	2	4	5	-1
		{10,13}	{1,3,4,5,6}		5	1	2	4	5	-1
		{10,16}	{1,3,4,7,8}		5	2	2	5	5	0
		{11,13}	{2,3,4,5,6}		5	1	2	4	5	-1
		{11,16}	{2,3,5,7,8}		5	2	2	5	5	0
{3,8}	{10,11,12,13,16}	{13,16}	{3,4,5,6,7,8}	5	6	2	2	4	5	-1
		{9,13}	{1,3,4,5,6}		5	1	1	4	4	0
		{10,11}	{1,2,3,4,5}		5	1	2	4	5	-1
		{10,13}	{1,3,4,5,6}		5	1	2	4	5	-1
		{10,16}	{1,3,4,7,8}		5	2	2	5	5	0
		{11,13}	{2,3,4,5,6}		5	1	2	4	5	-1
		{11,16}	{2,3,5,7,8}		5	2	2	5	5	0

Table 1.

There is a row for each $(l_A, l_B) \in L_A^* \times L_B^*$, where we obtain $|V_A(l_A, l_B)| - |V_B(l_A, l_B)|$.[22] Given conditions (1)-(3), we shade: i) for each $l_A \in L_A^*$, the best response(s) $s_B(l_A)$ (3rd column) and the associated $|V_A(l_A, s_B(l_A))| - |V_B(l_A, s_B(l_A))|$ (last column); and ii) the optimal list s_A for party A (1st column). In bold we show the equilibrium path $(s_A, s_B(s_A))$. Since $|V_A(s_A, s_B(s_A))| - |V_B(s_A, s_B(s_A))| = 0$, by Proposition 1, for each $s' \in S^*$, $P_A(s') = 1/2$.

Example 3 (Cont.) Consider Figure 3 (Section 4), with $X = \{1, 2, ..., 8\}$ and $Y = \{9, 10, ..., 16\}$. First consider Game 1, *i.e.*, $X = N_A$ and $Y = N_B$. Note that $m_A = m_B = 6$. In Table 2 we show that, for each $l_A \in L_A$ such that $|F(l_A)| > 4$, there is $l_B \in L_B$ with $|F(l_B)| > 4$ such that $|V_A(l_A, l_B)| < |V_B(l_A, l_B)|$. Hence, by Propositions 1 and 3, $P_A^* = 0$.

l_A	$F(l_A)$	l_B	$F(l_B)$	$\|F(l_A)\|$	$\|F(l_B)\|$	$\|C_A\|$	$\|C_B\|$	$\|V_A\|$	$\|V_B\|$	$\|V_A\|-\|V_B\|$
{4,8}	{10,11,13,14,15,16}	{11,14}	{2,3,4,5,6,7}	6	6	1	2	3	4	-1
{1,4}	{9,10,11,13,14}	{11,14}	{2,3,4,5,6,7}	5	6	1	2	3	5	-2
{2,3}	{9,10,11,12,13}	{9,14}	{1,3,4,5,6,7}	5	6	1	1	3	4	-1
{2,4}	{10,11,12,13,14}	{9,14}	{1,3,4,5,6,7}	5	6	1	1	3	4	-1
{2,8}	{10,11,12,15,16}	{9,14}	{1,3,4,5,6,7}	5	6	0	0	2	3	-1
{3,4}	{9,10,11,13,14}	{9,14}	{1,3,4,5,6,7}	5	6	2	2	4	5	-1
{3,5}	{9,11,12,13,14}	{9,14}	{1,3,4,5,6,7}	5	6	2	2	4	5	-1
{3,8}	{9,11,13,15,16}	{9,14}	{1,3,4,5,6,7}	5	6	1	1	3	4	-1
{4,5}	{10,11,12,13,14}	{11,14}	{2,3,4,5,6,7}	5	6	2	2	4	5	-1

Table 2.

Consider now Game 2, *i.e.*, $X = N_{B'}$ and $Y = N_{A'}$. In Table 3 we show that, for each $l_{A'} \in L_{A'}$ such that $|F(l_{A'})| > 4$, there is $l_{B'} \in L_{B'}$ with $|F(l_{B'})| > 4$ such that $|V_{A'}(l_{A'}, l_{B'})| \leq |V_{B'}(l_{A'}, l_{B'})|$. Hence, by Propositions 1 and 3, $P_{B'}^* > 0$.

$l_{A'}$	$F(l_{A'})$	$l_{B'}$	$F(l_{B'})$	$\|F(l_{A'})\|$	$\|F(l_{B'})\|$	$\|C_{A'}\|$	$\|C_{B'}\|$	$\|V_{A'}\|$	$\|V_{B'}\|$	$\|V_{A'}\|-\|V_{B'}\|$
{9,14}	{1,3,4,5,6,7}	{4,8}	{10,11,13,14,15,16}	6	6	1	1	3	3	0
{11,14}	{2,3,4,5,6,7}	{2,3}	{9,10,11,12,13}	6	5	1	2	4	4	0
{10,14}	{2,4,5,6,7}	{4,8}	{10,11,13,14,15,16}	5	6	2	1	4	4	0
{12,14}	{2,4,5,6,7}	{4,8}	{10,11,13,14,15,16}	5	6	1	1	3	4	-1
{13,14}	{3,4,5,6,7}	{4,8}	{10,11,13,14,15,16}	5	6	2	1	4	4	0
{14,15}	{4,5,6,7,8}	{4,8}	{10,11,13,14,15,16}	5	6	2	2	4	5	-1
{14,16}	{4,5,6,7,8}	{4,8}	{10,11,13,14,15,16}	5	6	2	2	4	5	-1

Table 3.

[22] Note that $|V_A(l_A, l_B)| = |N_A| - |F(l_B)| + |C_A(l_A, l_B)|$ and $|V_B(l_A, l_B)| = |N_B| - |F(l_A)| + |C_B(l_A, l_B)|$.

B Proofs

Proof of Lemma 1.

In equilibrium, for each $i \in N \backslash (V_A(l_A, l_B) \cup V_B(l_A, l_B))$, $v_i(l_A, l_B) = 0$ since, by (A1), $v_i(l_A, l_B) = 1$ is strictly dominated. The outcome of the election is thus determined by the votes of the citizens in $V_A(l_A, l_B)$ and $V_B(l_A, l_B)$. Let us define $D = \{j \in V_A(l_A, l_B) \cup V_B(l_A, l_B) \mid v_j(l_A, l_B) = 0\}$.

i) Consider $|V_A(l_A, l_B)| > |V_B(l_A, l_B)|$. First, suppose $\sum_{a \in N_A} v_a(l_A, l_B) > |V_B(l_A, l_B)|$. In this case, party A wins the election and each $a \in V_A(l_A, l_B)$ obtains a strictly positive payoff (either d, $(d + h)$ or $(d + h) - c$). Hence, no $a \in V_A(l_A, l_B)$ has incentives to be part of a deviating coalition: by deviating, a can neither save the cross-cutting cost, nor increase the probability that A gets into office. Second, suppose $\sum_{a \in N_A} v_a(l_A, l_B) < |V_B(l_A, l_B)|$. In this case, if $\sum_{b \in N_B} v_b(l_A, l_B) \leq \sum_{a \in N_A} v_a(l_A, l_B)$, the coalition $D \cap V_B(l_A, l_B)$ benefits from deviating to vote; otherwise, the coalition $D \cap V_A(l_A, l_B)$ benefits from deviating to vote. Hence, if $|V_A(l_A, l_B)| > |V_B(l_A, l_B)|$, s is an equilibrium if and only $\sum_{a \in N_A} v_a(l_A, l_B) > |V_B(l_A, l_B)|$ and, for each $i \in N \backslash (V_A(l_A, l_B) \cup V_B(l_A, l_B))$, $v_i(l_A, l_B) = 0$.

ii) Analogous to case i).

iii) Consider $|V_A(l_A, l_B)| = |V_B(l_A, l_B)|$. First, suppose that for each $j \in V_A(l_A, l_B) \cup V_B(l_A, l_B)$, $v_j(l_A, l_B) = 1$. The election outcome is a tie and each $j \in V_A(l_A, l_B) \cup V_B(l_A, l_B)$ obtains a strictly positive payoff (either $\frac{1}{2}d$, $\frac{1}{2}(d+h)$ or $\frac{1}{2}(d+h) - c$). Hence, no subset of $V_A(l_A, l_B) \cup V_B(l_A, l_B)$ has incentives to form a deviating coalition: by deviating, a player can neither save the cross-cutting cost, nor increase the probability that his party gets into office. Second, suppose that for some $j \in V_A(l_A, l_B) \cup V_B(l_A, l_B)$, $v_j(l_A, l_B) = 0$. In this case, if $\sum_{b \in N_B} v_b(l_A, l_B) \leq \sum_{a \in N_A} v_a(l_A, l_B)$, the coalition $D \cap V_B(l_A, l_B)$ benefits from deviating to vote; otherwise, the coalition $D \cap V_A(l_A, l_B)$ benefits from deviating to vote. Hence, if $|V_A(l_A, l_B)| = |V_B(l_A, l_B)|$, s is an equilibrium if and only if, for each $j \in V_A(l_A, l_B) \cup V_B(l_A, l_B)$, $v_j(l_A, l_B) = 1$ and, for each $i \in N \backslash (V_A(l_A, l_B) \cup V_B(l_A, l_B))$, $v_i(l_A, l_B) = 0$. □

Proof of Lemma 2.

Part i) Given $(l_A, l_B) \in L_A \times L_B$, first consider $|V_B(l_A, l_B)| \geq |V_A(l_A, l_B)|$ and $r_B(l_A, l_B) = 1$. In this case, by Lemma 1, B either wins or ties the election and each $b \in l_B$ obtains, at least, $\frac{h+d}{2} - c > 0$; whereas if a coalition in l_B deviates, each $b \in l_B$ obtains 0. Now consider $|V_B(l_A, l_B)| \geq |V_A(l_A, l_B)|$ and $r_B(l_A, l_B) = 0$. In this case, l_B does not form and each $b \in l_B$ obtains payoff 0; whereas if each $b \in l_B$ deviates to $r_b(l_A, l_B) = 1$, by Lemma 1, each of them gets a strictly positive payoff of at least $\frac{h+d}{2} - c$. Hence, if $|V_B(l_A, l_B)| \geq |V_A(l_A, l_B)|$, s is an equilibrium (of the subgame where, given $(l_A, l_B) \in L_A \times L_B$, nominees in l_B decide whether to run for office) if and only if $r_B(l_A, l_B) = 1$. Second, consider $|V_B(l_A, l_B)| < |V_A(l_A, l_B)|$, $r_B(l_A, l_B) = 0$ and there is $b' \in l_B$ such that $l_A \cap F(b') \neq \emptyset$. In this case, the payoff to each $b \in l_B$ is 0; whereas if each $b \in l_B$ deviates to $r_b(l_A, l_B) = 1$, by Lemma 1, the payoff to $b' \in l_B$ is $-c$. Now consider $|V_B(l_A, l_B)| < |V_A(l_A, l_B)|$, $r_B(l_A, l_B) = 1$ and there is $b' \in l_B$ such that

$l_A \cap F(b') \neq \emptyset$. In this case, by Lemma 1, b' obtains payoff $-c$; whereas if b' deviates, he obtains 0. Hence, if $|V_B(l_A, l_B)| < |V_A(l_A, l_B)|$ and there is $b' \in l_B$ such that $l_A \cap F(b') \neq \emptyset$, s is an equilibrium (of the subgame where, given $(l_A, l_B) \in L_A \times L_B$, nominees in l_B decide whether to run for office) if and only if $r_B(l_A, l_B) = 0$.

Given $l_A \in L_A$, by Lemma 1 and part i), the proof of part ii) is immediate. $\square$

Proof of Proposition 2.

For each $(l_A, l_B) \in L_A \times L_B$, $V_A(l_A, l_B) = (N_A \backslash F(l_B)) \cup l_A$ and $V_B(l_A, l_B) = (N_B \backslash F(l_A)) \cup l_B$. Therefore, $V_A(l_A, l_B) = |N_A| - |F(l_B)| + |C_A(l_A, l_B)|$ and $|V_B(l_A, l_B)| = |N_B| - |F(l_A)| + |C_B(l_A, l_B)|$. Hence, for each $l_A \in L_A$, $\arg\max_{l_B \in L_B} |V_B(l_A, l_B)| - |V_A(l_A, l_B)|$ coincides with $\arg\max_{l_B \in L_B} |F(l_B)| + |C_B(l_A, l_B)| - |C_A(l_A, l_B)|$. Moreover, for each $l_A \in L_A$, let $s_B(l_A) \in L_B$. Then, $\arg\max_{l_A \in L_A} |V_A(l_A, s_B(l_A))| - |V_B(l_A, s_B(l_A))|$ coincides with $\arg\max_{l_A \in L_A} |F(l_A)| - |F(s_B(l_A))| + |C_A(l_A, l_B)| - |C_B(l_A, l_B)|$. Since, by definition, $s \in S^{**}$ if and only if (1)-(3) hold, the proof follows. $\square$

Proof of Proposition 3.

First, we prove that, if $s \in S^{**}$, for all $l_A \in L_A$, $|F(s_B(l_A))| \geq m_B - 2(k-1)$. Note that, for each $(l_A, l_B) \in L_A \times L_B$, i) both $|C_A(l_A, l_B)|$ and $|C_B(l_A, l_B)|$ are in $\{1, ..., k\}$ and ii) $|C_A(l_A, l_B)| = 0$ if and only if $|C_B(l_A, l_B)| = 0$. Hence $|C_B(l_A, l_B)| - |C_A(l_A, l_B)| \in \{-(k-1), ..., k-1\}$. Let $l_B^m \in L_B^m$. For each $l_A \in L_A$, by Proposition 2, $|F(s_B(l_A))| + |C_B(l_A, s_B(l_A))| - |C_A(l_A, s_B(l_A))| \geq m_B + |C_B(l_A, l_B^m)| - |C_A(l_A, l_B^m)|$. Hence, $(|C_B(l_A, s_B(l_A))| - |C_A(l_A, s_B(l_A))|) - (|C_B(l_A, l_B^m)| - |C_A(l_A, l_B^m)|) \leq 2(k-1)$ and, therefore, $|F(s_B(l_A))| \geq m_B - 2(k-1)$.

Second, we prove that, if $s \in S^{**}$, $|F(s_A)| \geq m_A - 2(k-1)$. Let $l_A^m \in L_A^m$. By Proposition 2, $(|F(s_B(l_A^m))| + |C_B(l_A^m, s_B(l_A^m))| - |C_A(l_A^m, s_B(l_A^m))|) - (|F(s_B(s_A))| + |C_B(s_A, s_B(s_A))| - |C_A(s_A, s_B(s_A))|) \geq m_A - |F(s_A)|$. Since, for each $l_A \in L_A$, $|C_B(l_A, s_B(s_A))| - |C_A(l_A, s_B(s_A))| \in \{-k+1, ..., k-1\}$, $|F(s_B(l_A^m))| + |C_B(l_A^m, s_B(l_A^m))| - |C_A(l_A^m, s_B(l_A^m)| \leq m_B + (k-1)$ and, by Proposition 2, $|F(s_B(s_A))| + |C_B(s_A, s_B(s_A))| - |C_A(s_A, s_B(s_A))| \geq m_B - (k-1)$. Hence, $(|F(s_B(l_A^m))| + |C_B(l_A^m, s_B(l_A^m))| - |C_A(l_A^m, s_B(l_A^m))|) - (|F(s_B(s_A))| + |C_B(s_A, s_B(s_A))| - |C_A(s_A, s_B(s_A))|) \leq 2(k-1)$ and, therefore, $2(k-1) \geq m_A - |F(s_A)|$. This completes the proof of the first part of the proposition.

Now, we prove that there is $s' \in S^{**}$ such that $|F(s'_A)| > m_A - 2(k-1)$ and, for each $l_A \in L_A$, $|F(s'_B(s'_A))| > m_B - 2(k-1)$. Let $s \in S^{**}$. Then, $|F(s_A)| \geq m_A - 2(k-1)$ and, for each $l_A \in L_A$, $|F(s_B(l_A))| \geq m_B - 2(k-1)$. Let $l_A^m \in L_A^m$ and $l_B^m \in L_B^m$. For each $a \in N_A$, $b \in N_B$, $l_A \in L_A$ and $l_B \in L_B$, we define $s' \in S$ as follows:

$$r'_a(l_A) = r_a(l_A) \qquad r'_b(l_A, l_B)) = r_b(l_A, l_B)) \qquad v'_a(l_A, l_B) = v_a(l_A, l_B) \quad v'_b(l_A, l_B) = v_b(l_A, l_B)$$

$$s'_A = \begin{cases} s_A & \text{if } |F(s_A)| > m_A - 2(k-1) \\ l_A^m & \text{if } |F(s_A)| = m_A - 2(k-1). \end{cases}$$

$$s'_B(l_A) = \begin{cases} s_B(l_A) & \text{if } |F(s_B(l_A))| > m_B - 2(k-1) \\ l_B^m & \text{if } |F(s_B(l_A))| = m_B - 2(k-1). \end{cases}$$

Note that s' satisfies (1)-(2). We claim that s' satisfies (3).

Given $l_A \in L_A$, there are two possibilities. i) $|F(s_B(l_A))| > m_B - 2(k-1)$. Then $s'_B(l_A) = s_B(l_A) \in \arg\max_{l_B \in L_B} |V_B(l_A, l_B)| - |V_A(l_A, l_B)|$. ii) $|F(s_B(l_A))| = m_B - 2(k-1)$. Then, by Proposition 2, $s_B(l_A) \in \arg\max_{l_B \in L_B} |F(l_B)| + |C_B(l_A, l_B)| - |C_A(l_A, l_B)|$. Moreover, $(|C_B(l_A, s_B(l_A))| - |C_A(l_A, s_B(l_A))|) - (|C_B(l_A, l_B^m)| - |C_A(l_A, l_B^m)|) \leq 2(k-1)$. Hence, $s'_B(l_A) = l_B^m \in \arg\max_{l_B \in L_B} |F(l_B)| + |C_B(l_A, l_B)| - |C_A(l_A, l_B)|$. Therefore, for each $l_A \in L_A$, $s'_B(l_A)$ satisfies (3).

Regarding s'_A, there are two possibilities. i) $|F(s_A)| > m_A - 2(k-1)$. Then $s'_A = s_A \in \arg\max_{l_A \in L_A} |V_A(l_A, s_B(s_A))| - |V_B(l_A, s_B(s_A))|$. Since, for each $l_A \in L_A$, $|V_B(l_A, s_B(l_A))| - |V_A(l_A, s_B(l_A))| = |V_B(l_A, s'_B(l_A))| - |V_A(l_A, s'_B(l_A))|$, s'_A satisfies (9). ii) $|F(s_A)| = m_A - 2(k-1)$. For each $l_A \in L_A$, let $K(l_A) = |F(s_B(l_A))| + |C_B(l_A, s_B(l_A))| - |C_A(l_A, s_B(l_A))| = |F(s'_B(l_A))| + |C_B(l_A, s'_B(l_A))| - |C_A(l_A, s'_B(l_A))|$.[23] In the first part of the proposition, we proved that $2(k-1) \geq K(l_A^m) - K(s_A) \geq m_A - |F(s_A)|$. Since $2(k-1) = m_A - |F(s_A)|$, $K(l_A^m) - K(s_A) = m_A - |F(s_A)|$, *i.e.*, $m_A - K(l_A^m) = |F(s_A)| - K(s_A)$. Since $s'_A = l_A^m$, the claim follows. □

Proof of Proposition 5.

(Necessity) Assume there is $(l'_A, l'_B) \in L_A \times L_B$ such that neither (I), (II) nor (III) holds. Then $|V_B(l'_A, l'_B)| \neq |V_A(l'_A, l'_B)|$, $|N_B| > |V_A(l'_A, l'_B)|$ and $|N_A| > |V_B(l'_A, l'_B)|$. We claim that there is no strong Nash equilibrium in the subgame where A and B propose l'_A and l'_B, and all agents in l'_A and l'_B accept to run for office. Assume, for a contradiction, that there is $v_i(l'_A, l'_B)_{i \in N}$ that is a strong Nash equilibrium in the subgame. There are 3 possibilities:

i) $\sum_{b \in N_B} v_b(l'_A, l'_B) > \sum_{a \in N_A} v_a(l'_A, l'_B)$. Then B wins the election. We first claim that, for each $b \in N_B \backslash V_B(l'_A, l'_B)$, $v_b(l'_A, l'_B) = 0$. Assume not, *i.e.*, there is $\hat{b} \in N_B \backslash V_B(l'_A, l'_B)$ such that $v_{\hat{b}}(l'_A, l'_B) = 1$. Then, the payoff to b is $d - c$. However, if $\hat{b}$ deviates to abstain, he gets at least $d/2$, which, since $d > c > d/2$, exceeds $d - c$, a contradiction. Hence, $|N_A| > |V_B(l'_A, l'_B)| \geq \sum_{b \in N_B} v_b(l'_A, l'_B)$. Since $d > c$, the coalition formed by all $a \in N_A$ such that $v_a(l'_A, l'_B) = 0$ has incentives to deviate to vote and, therefore, make A win the election, a contradiction.

ii) $\sum_{a \in N_A} v_a(l'_A, l'_B) > \sum_{b \in N_B} v_b(l'_A, l'_B)$. Then A wins the election. Analogously to the previous case, the coalition formed by all $b \in N_B$ such that $v_b(l'_A, l'_B) = 0$ has incentives to deviate, a contradiction.

iii) $\sum_{b \in N_B} v_b(l'_A, l'_B) = \sum_{a \in N_A} v_a(l'_A, l'_B)$. Then there is a tie. We first claim that, for each $i \in N \backslash (V_B(l'_A, l'_B) \cup V_A(l'_A, l'_B))$, $v_i(l'_A, l'_B) = 0$. Assume not, *i.e.*, there is $\hat{\imath} \in N \backslash (V_B(l'_A, l'_B) \cup V_A(l'_A, l'_B))$ such that $v_{\hat{\imath}}(l'_A, l'_B) = 1$. Then, the payoff to $\hat{\imath}$ is $d/2 - c < 0$. However, if $\hat{\imath}$ deviates to abstain, he gets 0, a contradiction. Hence $\min\{|V_B(l'_A, l'_B)|, |V_A(l'_A, l'_B)|\} \geq \sum_{b \in N_B} v_b(l'_A, l'_B) = \sum_{a \in N_A} v_a(l'_A, l'_B)$. Since $|V_B(l'_A, l'_B)| \neq |V_A(l'_A, l'_B)|$, there are two possibilities: First, if $|V_B(l'_A, l'_B)| > |V_A(l'_A, l'_B)|$, $|V_B(l'_A, l'_B)| > \sum_{b \in N_B} v_b(l'_A, l'_B)$. Hence, there is $\hat{b} \in V_B(l'_A, l'_B)$ such that $v_{\hat{b}}(l'_A, l'_B) = 0$. Clearly, $\hat{b}$ has incentives to deviate to vote, since by deviating B wins the election (recall that the voting decision does not imply any cost for citizens in

[23] Recall that, for each $l_A \in L_A$, both $s_B(l_A)$ and $s'_B(l_A)$ are in $\arg\max_{l_B \in L_B} |F(l_B)| + |C_B(l_A, l_B)| - |C_A(l_A, l_B)|$.

$V_B(l'_A, l'_B) \cup V_A(l'_A, l'_B))$, a contradiction. Analogously, if $|V_A(l'_A, l'_B)| > |V_B(l'_A, l'_B)|$, $|V_A(l'_A, l'_B)| > \sum_{a \in N_A} v_a(l'_A, l'_B)$ and there is $\hat{a} \in V_A(l'_A, l'_B)$ such that $v_{\hat{a}}(l'_A, l'_B) = 0$ and, therefore, $\hat{a}$ has incentives to deviate, a contradiction.

This proves that there is no strong Nash equilibrium in the subgame and, therefore, $S^* = \emptyset$.

(Sufficiency) Assume that, for each $(l_A, l_B) \in L_A \times L_B$, either (I), (II) or (III) holds. We claim that if $s \in S$ satisfies (1)-(3), s is a strong perfect equilibrium of the game.[24] We prove it by backward induction.

Stage three. Consider $(l_A, l_B) \in L_A \times L_B$. First, suppose (I) holds: $|V_A(l_A, l_B)| = |V_B(l_A, l_B)|$. By (1), there is a tie. Each $i \in N \setminus (V_A(l_A, l_B) \cup V_B(l_A, l_B))$ obtains a payoff of $\frac{1}{2}d > 0$; and each $j \in V_A(l_A, l_B) \cup V_B(l_A, l_B)$ obtains a strictly positive payoff (either $\frac{1}{2}d$, $\frac{1}{2}(d+h)$ or $\frac{1}{2}(d+h)-c$). Hence, no $i \in N \setminus (V_A(l_A, l_B) \cup V_B(l_A, l_B))$ has incentives to be part of a deviating coalition: by deviating, i obtains, at most, $d - c$, which is strictly lower than $\frac{1}{2}d$ (since $c > \frac{1}{2}d$). Similarly, no $j \in V_A(l_A, l_B) \cup V_B(l_A, l_B)$ has incentives to be part of a deviating coalition: by deviating, j can neither save the cross-cutting cost, nor increase the probability that his party wins office. Second, suppose (II) holds: $|V_A(l_A, l_B)| \geq |N_B|$. By (1), A wins the election. Each $a' \in N \setminus V_A(l_A, l_B)$ obtains a payoff of d; and each $a \in V_A(l_A, l_B)$ obtains a strictly positive payoff (either d, $d + h$ or $d + h - c$). As $\frac{1}{2}d < c$, no $a \in N_A$ has incentives to be part of a deviating coalition. Additionally, no coalition in N_B has incentives to deviate: they are not numerous enough to alter the outcome of the election. Case (III) is analogous to case (II).

Stage two. Part i) Consider $(l_A, l_B) \in L_A \times L_B$. First, suppose $|V_B(l_A, l_B)| \geq |V_A(l_A, l_B)|$. By (2), $r_B(l_A, l_B) = 1$ and, by (1), party B either wins or ties the election. Each $b \in l_B$ obtains, at least, $(h+d)/2 - c > 0$; whereas if a coalition in l_B deviates, each $b \in l_B$ obtains 0. Second, suppose $|V_B(l_A, l_B)| < |V_A(l_A, l_B)|$. By (2), $r_B(l_A, l_B) = 0$ and party A gets into office. Each $b \in l_B$ obtains 0; whereas if all $b \in l_B$ deviate to $r_b(l_A, l_B) = 1$, there is an election and, by (1), party B loses. Part ii) Consider $l_A \in L_A$. Given (1)-(2), it is immediate to see that the behavior that (3) prescribes for party B is optimal.

Stage one. The proof that s is a strong perfect equilibrium for the challenger stage follows analogous arguments to those used in stage two. □

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[24] First note that, by construction, the set of strategies satisfying (1)-(3) is not empty. Second, note that (A1)-(A2) no longer hold. Therefore, we need to prove that s is a (strong perfect) equilibrium.

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